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# SOME REMARKS ON ANALYTIC FUNCTIONS CONCERNED WITH JACK'S LEMMA

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Abstract. In this paper, we give estimates of the Schwarz lemma in a novel class $\mathcal{M}(b)$ of analytical functions in the unit disc. The sharpness of some of the presented in equalities has also been shown.

## 1. Introduction and preliminaries

Let $f$ be an analytic function in the unit disc $U = \{z : |z| < 1\}$, $f(0) = 0$ and $f : U \to U$ . In accordance with the classical Schwarz lemma, for any point $z$ in the unit disc $U$, we have $|f(z)| \leq |z|$ for all $z \in U$ and $|f'(0)| \leq 1$. In addition, if the equality $|f(z)| = |z|$ holds for any $z \neq 0$, or $|f'(0)| = 1$, then $f$ is a rotation; that is $f(z) = ze^{i\theta}$, $\theta$ real ([5], p.329). In electrical and electronics engineering, it is possible to encounter with applications of Schwarz lemma. As an example, the driving point impedance functions obtained as a result of boundary analysis of Schwarz lemma can be possibly used for circuit synthesis. Also, transfer functions in control theory and multi-notched filters in signals and systems can be considered as topics under the title of Schwarz lemma's applications [13, 14, 15].

Furthermore let $f$ and $h$ be analytic functions in $U$. A function is said to be *subordinate* to $h$, written as $f(z) \prec h(z)$, if there exists a Schwarz function $w(z)$, analytic in $U$ with $w(0) = 0$, $|w(z)| < 1$ such that $f(z) = h(w(z))$ [5].

In order to prove our main results, we recall the following lemma [6].

**Lemma 1.1** (Jack's lemma)**.** *Let $f(z)$ be a non-constant anaytic function in $E$ with $f(0) = 0$. If*

$$|f(z_0)| = \max\{|f(z)| : |z| \leq |z_0|\},$$

*then there exists a real number $k \geq 1$ such that*

$$\frac{z_0 f'(z_0)}{f(z_0)} = k.$$

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Let $\mathcal{A}$ denote the class of functions $g(z) = z+b_2z^2+b_3z^3+...$ that are analytic in $U$. Also, let $\mathcal{M}\,(b)$ be the subclass of $\mathcal{A}$ consisting of all functions $g(z)$ satisfying

$$\frac{1}{b}\left(\frac{zg''(z)}{g'(z)}\right) \prec z,\ \ z \in U. \tag{1.1}$$

The certain anaytic functions which is in the class of $\mathcal{M}\,(b)$ on the unit disc $U$ are considered in this paper. The subject of the present paper is to discuss some properties of the function $g(z)$ which belongs to the class of $\mathcal{M}\,(b)$ by applying Jack's Lemma.

Let $g \in \mathcal{M}\,(b)$ and consider the following function

$$\phi(z) = \frac{\ln g'(z)}{b}, \tag{1.2}$$

where $b \in \mathbb{C}$.

It is an analytic function in $U$ and $\phi(0) = 0$. Now, let us show that $|\phi(z)| < 1$ in $U$. From (1.2), we have

$$g'(z) = e^{b\phi(z)}$$

and

$$g''(z) = b\phi'(z)e^{b\phi(z)}.$$

We assume that there exists a $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|} |\phi(z)| = |\phi(z_0)| = 1.$$

From Jack's lemma, we obtain

$$\phi(z_0) = e^{i\theta} \text{ and } \frac{z_0\phi'(z_0)}{\phi(z_0)} = k.$$

Thus, we have that

$$\begin{aligned} \frac{1}{b}\left(\frac{zg''(z)}{g'(z)}\right) &= \frac{1}{b}\left(\frac{z_0g''(z_0)}{g'(z_0)}\right) \\ &= \frac{1}{b}\left(\frac{z_0b\phi'(z_0)e^{b\phi(z_0)}}{e^{b\phi(z_0)}}\right) \\ &= \frac{1}{b}\left(bk\phi(z_0)\right) \\ &= ke^{i\theta}. \end{aligned}$$

This contradicts the $g \in \mathcal{M}\,(b)$. This means that there is no point $z_0 \in U$ such that $\max_{|z|\leq|z_0|} |\phi(z)| = |\phi(z_0)| = 1$. Hence, we take $|\phi(z)| < 1$ in $E$. From the Schwarz lemma, we obtain

$$\begin{aligned} |\phi'(0)| &\leq 1, \\ \frac{1}{|b|}\left|\frac{g''(0)}{g'(0)}\right| &\leq 1 \end{aligned}$$

and

$$|g''(0)| \leq |b|\,.$$

We thus obtain the following lemma.

**Lemma 1.2.** *If $g \in \mathcal{M}(b)$, then we have the inequality*

$$|g''(0)| \leq |b| . \tag{1.3}$$

*The results is sharp with the function*

$$g(z) = \frac{1}{b}\left(e^{bz} - 1\right).$$

Since the area of applicability of Schwarz Lemma is quite wide, there exist many studies about it. Some of these studies, which are called the boundary version of Schwarz Lemma, are about being estimated from below the modulus of the derivative of the function at some boundary point of the unit disc. The boundary version of Schwarz Lemma is given as follows [11]:

**Lemma 1.3.** *If $f(z)$ extends continuously to some boundary point $c$ with $|c| = 1$, and if $|f(c)| = 1$ and $f'(c)$ exists, then*

$$|f'(c)| \geq \frac{2}{1 + |f'(0)|} \tag{1.4}$$

*and*

$$|f'(c)| \geq 1. \tag{1.5}$$

Inequality (1.5) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1, 2, 3, 4, 7, 8, 9, 10, 11, 12].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [16]).

**Lemma 1.4** (Julia-Wolff lemma)**.** *Let $f$ be an analytic function in $U$, $f(0) = 0$ and $f(U) \subset U$. If, in addition, the function $f$ has an angular limit $f(c)$ at $c \in \partial U$, $|f(c)| = 1$, then the angular derivative $f'(c)$ exists and $1 \leq |f'(c)| \leq \infty$.*

**Corollary 1.5.** *The analytic function $f$ has a finite angular derivative $f'(c)$ if and only if $f'$ has the finite angular limit $f'(c)$ at $c \in \partial U$.*

## 2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma for $\mathcal{M}(b)$ class.

**Theorem 2.1.** *Let $g \in \mathcal{M}(b)$. Assume that, for some $c \in \partial U$, $g$ has an angular limit $g(c)$ at $c$, $g'(c) = e^{b}$. Then we have the inequality*

$$|g''(c)| \geq |b|\, e^{\Re b}. \tag{2.1}$$

*The equality in (2.1) occurs for the function*

$$g(z) = \frac{1}{b}\left(e^{bz} - 1\right).$$

*Proof.* Consider the function

$$\phi(z) = \frac{\ln g'(z)}{b}.$$

In addition, since $g'(c) = e^b$, we have

$$\phi(c) = \frac{\ln g'(c)}{b} = \frac{\ln e^b}{b} = 1$$

So, from (1.5), we obtain

$$1 \leq |\phi'(c)| = \frac{1}{|b|} \frac{|g''(c)|}{|g'(c)|}.$$

and

$$|g''(c)| \geq |b| \, e^{\Re b}.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$g(z) = \frac{1}{b}\left(e^{bz} - 1\right).$$

Then

$$g''(z) = be^{bz}$$

and

$$|g''(1)| = \left|be^b\right| = |b| \, e^{\Re b}.$$

□

**Theorem 2.2.** *Under the same assumptions as in Theorem 2.1, we have*

$$|g''(c)| \geq \frac{2\,|b|^2 \, e^{\Re b}}{|b| + |g''(0)|} \tag{2.2}$$

*The inequality* (2.2) *is sharp, with equality for the function*

$$g(z) = \frac{1}{b}\left(e^{bz} - 1\right).$$

*Proof.* Let $\phi(z)$ function be the same as (1.2). So, from (1.4), we obtain

$$\frac{2}{1 + |\phi'(0)|} \leq |\phi'(c)| = \frac{1}{|b|} \frac{|g''(c)|}{|g'(c)|},$$

Since

$$|\phi'(0)| = \frac{1}{|b|} \left|\frac{g''(0)}{g'(0)}\right| = \frac{|g''(0)|}{|b|},$$

we take

$$\frac{2}{1 + \frac{|g''(0)|}{|b|}} \leq \frac{1}{|b|} \frac{|g''(c)|}{|g'(c)|}$$

and

$$|g''(c)| \geq \frac{2\,|b|^2 \, e^{\Re b}}{|b| + |g''(0)|}.$$

Now, we shall show that the inequality (2.2) is sharp. Let

$$g(z) = \frac{1}{b}\left(e^{bz} - 1\right).$$

Then

$$|g''(1)| = \left|be^b\right| = |b| \, e^{\Re b}.$$

Also, since

$$\begin{aligned} z+b_2z^2+b_3z^3+... &= \frac{1}{b}\left(e^{bz}-1\right)=\frac{1}{b}\left(1+bz+\frac{b^2z^2}{2}+...-1\right), \\ b_2z^2+b_3z^3+... &= z+\frac{bz^2}{2}+....... \end{aligned}$$

$$|b_2|=\frac{|b|}{2}$$

and

$$|g''(0)|=|b|,$$

we obtain

$$\frac{2|b|^2 e^{\Re b}}{|b|+|g''(0)|}=\frac{2|b|^2 e^{\Re b}}{|b|+|b|}=|b|\, e^{\Re b}.$$

□

The inequality (2.2) can be strengthened as below by taking into account $b_3$ which is the coefficient in the expansion of the function $g(z)=z+b_2z^2+b_3z^3+....$

**Theorem 2.3.** *Let $g\in\mathcal{M}(b)$. Assume that, for some $c\in\partial U$, $g$ has an angular limit $g(c)$ at $c$, $g'(c)=e^b$. Then we have the inequality*

$$|g''(c)|\geq|b|\, e^{\Re b}\left(1+\frac{2\left(|b|-|g''(0)|\right)^2}{\alpha|b|^2-|g''(0)|^2+|b|\,|g'''(0)|}\right). \tag{2.3}$$

*Proof.* Let $\phi(z)$ be the same as in the proof of Theorem 2.1 and $B(z)=z$. By the maximum principle, for each $z\in U$, we have the inequality $|\phi(z)|\leq|B(z)|$. So,

$$H(z)=\frac{\phi(z)}{B(z)}=\frac{\ln g'(z)}{bz}$$

is analytic function in $U$ and $|H(z)|\leq 1$ for $z\in U$. In particular, we have

$$|H(0)|=\frac{|g''(0)|}{|b|} \tag{2.4}$$

and

$$|H'(0)|=\frac{|g'''(0)|}{|b|}.$$

Furthermore, the geometric meaning of the derivative and the inequality $|\phi(z)|\leq|B(z)|$ imply the inequality

$$\frac{c\phi'(c)}{\phi(c)}=|\phi'(c)|\geq|B'(c)|=\frac{cB'(c)}{B(c)}.$$

The auxiliary function

$$r(z)=\frac{H(z)-H(0)}{1-\overline{H(0)}H(z)}$$

is analytic in $U$, $r(0)=0$, $|r(z)|<1$ for $|z|<1$ and $|r(c)|=1$ for $c\in\partial U$. From (1.4), we obtain

$$\begin{aligned}
\frac{2}{1+|r'(0)|} &\leq |r'(c)| = \frac{1-|H(0)|^2}{\left|1-\overline{H(0)}H(c)\right|^2}|H'(c)| \\
&\leq \frac{1+|H(0)|}{1-|H(0)|}|H'(c)| \\
&= \frac{1+\frac{|g''(0)|}{|b|}}{1-\frac{|g''(0)|}{|b|}}\left\{|\phi'(c)|-|B'(c)|\right\} \\
&= \frac{|b|+|g''(0)|}{|b|-|g''(0)|}\left(\frac{1}{|b|}\frac{|g''(c)|}{|g'(c)|}-1\right).
\end{aligned}$$

Since

$$r'(z) = \frac{1-|H(0)|^2}{\left(1-\overline{H(0)}H(z)\right)^2}H'(z)$$

and

$$|r'(0)| = \frac{|H'(0)|}{1-|H(0)|^2} = \frac{\frac{|g'''(0)|}{|b|}}{1-\left(\frac{|g''(0)|}{|b|}\right)^2} = \frac{|b|\,|g'''(0)|}{|b|^2-|g''(0)|^2},$$

we obtain

$$\frac{2}{1+\frac{|b||g'''(0)|}{|b|^2-|g''(0)|^2}} \leq \frac{|b|+|g''(0)|}{|b|-|g''(0)|}\left(\frac{1}{|b|}\frac{|g''(c)|}{|g'(c)|}-1\right),$$

and

$$|g''(c)| \geq |b|\,e^{\Re b}\left(1+\frac{2\left(|b|-|g''(0)|\right)^2}{\alpha\,|b|^2-|g''(0)|^2+|b|\,|g'''(0)|}\right).$$

$\square$

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