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PROPERTIES OF A SUBCLASS OF HARMONIC MAPPINGS WITH RESTRICTED ANALYTIC PART

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Abstract. In this article, a new subclass of harmonic mappings with restricted analytic part is introduced. Coefficient bounds, Fekete-Szegö inequality, and integral representation for functions in this subclass are established. Further closure of this class under convex combination is proved. Finally we obtain an upper bound for the Bloch's constant for functions in this subclass.

1. Introduction and preliminaries

Harmonic univalent functions were introduced by Clunie and Shiel-Small in their seminal paper [3]. Denote by $\mathcal{H}$ the classs of all harmonic mappings f defined on the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with values in $\mathbb{C}$, normalised by $f(0) = 0, f'(0) = 1$. The functions f in the class $\mathcal{H}$ can be represented as $f = h + \overline{g}$. where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1.1}$$

Both h and g are analytic in the unit disk and are called "analytic" and "co-analytic" parts of f respectively. If $g \equiv 0$ then the class $\mathcal{H}$ reduces to the class $\mathcal{A}$ of normalized analytic functions on the unit disk. For $f = h+\overline{g} \in \mathcal{H}$, the Jacobian is defined as $|h'(z)|^2 - |g'(z)|^2$ and is denoted by $J_f(z)$. The harmonic function $f = h+\overline{g}$ is locally univalent in a domain if and only if $J_f(z) \neq 0$ [7]. In particular, it is called sense-preserving if $J_f(z) > 0$ and sense-reversing if $J_f(z) < 0$ for each $z \in \Delta$. The subclass of $\mathcal{H}$ consisting of univalent sense-preserving harmonic functions in the unit disk Δ is denoted by $\mathcal{S}_{\mathcal{H}}$. The classical family of normalized, analytic and univalent functions on the unit disk is $\mathcal{S} = \{f \in \mathcal{S}_{\mathcal{H}} : g \equiv 0\}$. Let

$$\mathcal{H}^0 = \{f \in \mathcal{H} : f_{\overline{z}} = 0\}.$$

Then the functions in $\mathcal{H}^0$ are those functions in $\mathcal{H}$ with series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} b_n z^n}. \tag{1.2}$$

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The subclass $S_{\mathcal{H}}^0$ consists of those functions f in $\mathcal{S}_H$ with $f_{\bar{z}} = 0$. A more detailed account about harmonic univalent mappings can be found in the monograph by Duren [5]. A domain D in the complex plane is said to be close-to-convex if the complement of D can be written as union of non-intersecting half lines. A function $f \in \mathcal{H}$ is called close-to-convex in Δ if $f(\Delta)$ is close-to-convex. The class of harmonic close-to-convex mappings in Δ is denoted by $\mathcal{C}_{\mathcal{H}}$. If $f = h + \overline{g} \in \mathcal{H}^0$, then the corresponding subclass is denoted by ${\mathcal{C}_{\mathcal{H}}}^0$ These subclasses were studied by Bshouty *et.al.* in [1, 2]. A new subclass of harmonic univalent functions using modified Salagean differential operator was defined and studied by Fadipe *et.al.* [6]. In [10], a new subclass of harmonic univalent functions defined by fractional calculus was introduced and investigated.

In the recent past, researchers have defined and studied the subclasses of harmonic mappings by restricting their analytic part to some of the known subclasses of analytic functions. For more details, one may refer to [4, 11, 12]. Motivated by these works, in the present paper, we define a new subclass of harmonic mappings whose analytic parts are restricted to vary over a subclass of normalized analytic functions whose second derivative satisfies a differential inequality.

Following [9] denote by $\mathcal{B}(M)$ the subclass of all analytic functions h defined on the unit disk with normalization $h(0) = h'(0) - 1 = 0$ satisfying

$$|zh''(z)| \leq M, \quad M > 0, \ z \in \Delta.$$

The harmonic analogue of mappings related to the above class was introduced and studied by Ghosh *et.al.* in [8].

We now introduce a new subclass $\Upsilon_{\mathcal{H}}(M)$ of harmonic functions defined in the open unit disk whose analytic part satisfies the above differential inequality.

Definition 1.1. For $M > 0$, the subclass $\Upsilon_{\mathcal{H}}(M)$ is defined as the set of all harmonic functions $f = h + \overline{g}$ where h and g are given by (1) satisfying

$$g'(z) = zh'(z), \text{ and } |zh''(z)| < M, \ z \in \Delta.$$

The major results are discussed in three sections. In section 2, coefficient estimates for both the analytic and co-analytic parts are computed together with the Fekete-Szegö inequality. In section 3 we ontain an expression for function in this class and prove the growth and covering theorem for functions in this subclass. We also obtain lower and upper bound for area of a disk of radius r inside the unit disk. In the last section, we prove that the subclass introduced is closed under convex combination and obtain an estimate for the Bloch's constant.

2. Coefficient Estimates

We now obtain the estimate for the coefficients of analytic and co-analytic parts of functions in the subclass $\Upsilon_{\mathcal{H}}(M)$.

Theorem 2.1. *If $f = h + \overline{g}$ given by (2) is in $\Upsilon_{\mathcal{H}}(M)$, then*

$$|a_n| \leq \frac{M}{n(n-1)}, \ n \geq 2,$$

$$|b_2| = \frac{1}{2} \text{ and } |b_n| \leq \frac{M}{n^2-1},\ n \geq 3.$$

The result is sharp for the function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{M}{n(n-1)} z^n + \overline{\frac{1}{2}z^2 + \sum_{n=3}^{\infty} \frac{M}{n^2-1} z^n}.$$

Proof. By Cauchy's theorem,

$$n(n-1)|a_n| = \frac{1}{2\pi} \int_{|z|=r} \frac{zh''(z)}{z^n} dz,\ 0 < r < 1.$$

Therefore

$$\begin{aligned} n(n-1)|a_n| &= \left| \frac{1}{2\pi} \int_{|z|=r} \frac{zh''(z)}{z^n} dz \right| \\ &\leq \frac{1}{2\pi} \int_{|z|=r} \frac{|zh''(z)|}{|z|^n} |dz| \\ &\leq \frac{1}{2\pi} \int_{|z|=r} \frac{M}{|z|^n} |dz| \\ &= \frac{M}{2\pi} \int_0^{2\pi} \frac{d\theta}{r^{n-1}} \\ \Longrightarrow n(n-1)|a_n| &\leq \frac{M}{r^{n-1}} \end{aligned}$$

Allowing $r \to 1^-$, we get $|a_n| \leq \frac{M}{n(n-1)},\ n = 2, 3, 4, \cdots$. From $g'(z) = zh'(z)$, we find

$$(n+1)b_{n+1} = na_n, n \geq 2.$$

Using $|a_1| = 1$ and the bounds for $|a_n|,\ n \geq 2$ we get

$$|b_2| = \frac{1}{2} \text{ and } |b_{n+1}| \leq \frac{M}{n^2-1},\ n \geq 3.$$

It is easy to verify that the bounds obtained are sharp for the function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{M}{n(n-1)} z^n + \overline{\frac{1}{2}z^2 + \sum_{n=3}^{\infty} \frac{M}{n^2-1} z^n}.$$

□

We now derive the Fekete-Szegö inequality for the coefficents of both analytic and co-analytic parts of functions $f = h + \overline{g}$ in the class $\Upsilon_{\mathcal{H}}(M)$.

Theorem 2.2. *For a complex number λ and $f = h + \overline{g} \in \Upsilon_{\mathcal{H}}(M)$, where h and g have series expansions as in* (1),

$$|a_3 - \lambda a_2^2| \leq \frac{M}{12}(14 + 3|\lambda|M)$$

and

$$|b_3 - \lambda b_2^2| \leq \frac{1}{12}(4M + 3|\lambda|).$$

Proof. Since $f = h + \overline{g} \in \Upsilon_{\mathcal{H}}(M)$,

$$|zh''(z)| \leq M,\ 0 < M < 3$$

which implies that there is a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$zh''(z) = M\omega(z).$$

Using the relation between the Schwarz function and the Caratheodory function, we have

$$zh''(z) = M\left(\frac{p(z)-1}{p(z)+1}\right) \tag{2.1}$$

where $p(z) = 1 + c_1 z + c_2 z^2 + \cdot + c_n(z)^n + \cdots$ and $|c_i| \leq 2$ for each $i = 1, 2, \cdots$ Equating the coefficients on both sides of we obtain

$$a_2 = \frac{Mc_1}{4}, \text{and } a_3 = \frac{Mc_2 - 3Mc_1^2}{12}.$$

Then

$$\begin{aligned}
&|a_3 - \lambda a_2^2| \\
&= \left|\frac{Mc_2 - 3Mc_1^2}{12} - \lambda\left(\frac{Mc_1}{4}\right)^2\right| \\
&= \left|\frac{Mc_2}{12} - \left(\frac{M}{4} + \frac{\lambda M^2}{16}\right)c_1^2\right| \\
&\leq \frac{M}{12}|c_2| + \left(M + |\lambda|\frac{M^2}{16}\right)|c_1|^2 \\
&\leq \frac{M}{6} + M + |\lambda|\frac{M^2}{4},\ \text{since } |c_2| \leq 2 \text{ and } |c_1| \leq 2. \\
&= \frac{M}{12}(14 + 3|\lambda|M)
\end{aligned}$$

Now, $g'(z) = zh'(z)$ gives $b_2 = \frac{1}{2}$ and $b_3 = \frac{2}{3}a_2$. Then,

$$|b_3 - \lambda b_2^2| = \left|\frac{2}{3}a_2 - \lambda\frac{1}{4}\right| \leq \frac{2}{3}|a_2| + \frac{|\lambda|}{4} \leq \frac{M}{3} + \frac{|\lambda|}{4} = \frac{1}{12}(4M + 3|\lambda|).$$

□

3. Geometric Properties

Lemma 3.1. [8] *If $h \in \mathcal{A}$ satisfies $|zh''(z)| \leq M$ for some $M > 0$, then*

$$1 - M|z| \leq |h'(z)| \leq 1 + M|z|. \tag{3.1}$$

Theorem 3.2. ***Growth theorem:*** *Let $f = h + \overline{g}$ belong to the class $\Upsilon_{\mathcal{H}}(M)$. Then,*

$$\left(1 + \frac{Mr^2}{3}\right)r - \frac{M+1}{2}r^2 \leq |f(z)| \leq \left(1 + \frac{Mr^2}{3}\right)r + \frac{M+1}{2}r^2$$

Proof. Since $f = h + \overline{g} \in \Upsilon_{\mathcal{H}}(M)$, $\;g'(z) = zh'(z),\; z \in \Delta$ and therefore

$$|g'(z)| = |z||h'(z)|,\; \forall z \in \Delta. \tag{3.2}$$

Let Γ be the radial segment from 0 to z. Then,

$$\begin{aligned}
|f(z)| &= \left| \int_\Gamma \frac{\partial f}{\partial \zeta} d\zeta + \frac{\partial f}{\partial \overline{\zeta}} d\overline{\zeta} \right| \\
&\le \int_\Gamma (|h'(\zeta)| + |g'(\zeta)|)|d\zeta| \\
&= \int_\Gamma (1 + |\zeta|)|h'(\zeta)||d\zeta|, \quad \text{(Using Definition 1.1)} \\
&\le \int_\Gamma (1 + |\zeta|)(1 + M|\zeta|)|d\zeta| \quad \text{(Using (3.1))} \\
&= \int_0^r (1+\rho)(1+M\rho)d\rho \\
&= \left(1 + \frac{Mr^2}{3}\right)r + \frac{M+1}{2}r^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|f(z)| &= \left| \int_\Gamma \frac{\partial f}{\partial \zeta} d\zeta + \frac{\partial f}{\partial \overline{\zeta}} d\overline{\zeta} \right| \\
&\ge \int_\Gamma (|h'(\zeta)| - |g'(\zeta)|)|d\zeta| \\
&= \int_\Gamma (1 - |\zeta|)|h'(\zeta)||d\zeta|, \quad \text{(Using Definition 1.1)} \\
&\ge \int_\Gamma (1 - |\zeta|)(1 - M|\zeta|)|d\zeta|, \; \text{(Using (3.1))} \\
&= \int_0^r (1-\rho)(1-M\rho)d\rho \\
&= \left(1 + \frac{Mr^2}{3}\right)r - \frac{M+1}{2}r^2.
\end{aligned}$$

$\square$

Theorem 3.3. *If $f = h + \overline{g} \in \Upsilon_{\mathcal{H}}(M)$, then*

$$|z| - M|z|^2 \le |g'(z)| \le |z| + M|z|^2.$$

Proof. Since $f = h + \overline{g}$ is in the subclass $\Upsilon_{\mathcal{H}}(M)$,

$$g'(z) = zh'(z) \implies |g(z)| = |z||h'(z)|$$

$$\implies |g'(z)| \le |z|(1 + M|z|) \text{ and } |g'(z)| \ge |z|(1 - M|z|).$$

$$\implies |z| - M|z|^2 \le |g'(z)| \le |z| + M|z|^2.$$

$\square$

Theorem 3.4. *Let $f = h + \overline{g} \in \Upsilon_{\mathcal{H}}(M)$. Then*

$$f(z) = z + M \int_0^z \int_0^\zeta \frac{\omega(t)}{t} dt d\zeta + \overline{\frac{z^2}{2} + M \int_0^z \zeta \int_0^\zeta \frac{\omega(t)}{t} dt d\zeta}.$$

Proof. $f = h + \overline{g} \in \Upsilon_{\mathcal{H}}(M)$ implies

$$|zh''(z)| \leq M, \; 0 < M < 3, \; z \in \Delta.$$

Then there exists an analytic function $\omega : \Delta \to \mathbb{C}$ such that $\omega(0) = 0$ and $|\omega(z)| \leq 1$ such that

$$zh''(z) = M\omega(z) \tag{3.3}$$

On integration, we have

$$h'(z) = 1 + M \int_0^z \frac{\omega(t)}{t} dt.$$

Integrating again, we get

$$h(z) = z + M \int_0^z \int_0^\zeta \frac{\omega(t)}{t} dt d\zeta \tag{3.4}$$

Since $g'(z) = zh'(z)$, on substituting for $h'(z)$ and integrating, we have

$$g(z) = \frac{z^2}{2} + M \int_0^z \zeta \int_0^\zeta \frac{\omega(t)}{t} dt d\zeta. \tag{3.5}$$

Substituting (3.4) and (3.5) in $f = h + \overline{g}$, we get the representation for f. □

Theorem 3.5. *If $f \in \Upsilon_{\mathcal{H}}(M)$, then the largest disk centered at the origin and contained by $f(\Delta)$ is $|\omega| < \frac{2-M}{2}$.*

Proof. Let $f \in \Upsilon_{\mathcal{H}}(M)$. Then by Theorem 3.1, we have

$$\left(1 + \frac{Mr^2}{3}\right) r - \frac{M+1}{2} r^2 \leq |f(z)|$$

Letting $r \to 1^-$, we get the desired result. □

In the next theorem, we determine the area $A(f(\Delta_r))$ of the image of the disk $\Delta_r = \{z \in \mathbb{C} : |z| < r\}, \; 0 < r < 1\}$ under $f \in \Upsilon_{\mathcal{H}}(M)$.

Theorem 3.6. *Let $f \in \Upsilon_{\mathcal{H}}(M)$. Then*

$$\frac{\pi}{30}[(30r^2 - 15r^4) + (20r^3 - 12r^5)M] \leq A(f(\Delta_r)) \leq \frac{\pi}{30}[(30r^2 - 15r^4) - (20r^3 - 12r^5)M].$$

Proof. If $f = h + \overline{g} \in \Upsilon_{\mathcal{H}}(M)$, then

$$\begin{aligned}
A(f(\Delta_r)) &= \int\int_{\Delta_r} (|h(z)|^2 - |g'(z)|^2) dxdy \\
&= \int\int_{\Delta_r} (|h(z)|^2 - |z|^2|h'(z)|^2) dxdy, \text{ since } g'(z) = zh'(z) \\
&= \int\int_{\Delta_r} (1 - |z|^2)|h'(z)|^2 dxdy \\
&\leq \int\int_{\Delta_r} (1 - |z|^2)(1 + M|z|) dxdy \\
&= \int_0^{2\pi}\int_0^r (1 - \rho^2)(1 + M\rho)\rho d\rho d\theta
\end{aligned}$$

which on evaluation gives the upper bound for the area of $f(\Delta_r)$.
Similarly,

$$\begin{aligned}
A(f(\Delta_r)) &\geq \int\int_{\Delta_r} (1 - |z|^2)(1 - M|z|) dxdy \\
&= \int_0^{2\pi}\int_0^r (1 - \rho^2)(1 - M\rho)\rho d\rho d\theta
\end{aligned}$$

gives the lower bound. $\square$

4. Convex Combination and Bloch's Constant

Theorem 4.1. *The class $\Upsilon_{\mathcal{H}}(M)$ is closed under convex combination.*

Proof. Let $f_1 = h_1 + \overline{g_1}, f_2 = h_2 + \overline{g_2}, \cdots, f_n = h_n + \overline{g_n} \in \Upsilon_{\mathcal{H}}(M)$ and $t_1, t_2, \cdots, t_n$ be real numbers such that $0 \leq t_i \leq 1,\ i = 1, 2, \cdots, n$ with $\sum_{i=1}^n t_i = 1$. Then,

$$g_i'(z) = zh_i'(z), \text{ and } |zh_i''(z)| \leq M,\ 0 < M < 3,\ i = 1, 2, \cdots, n. \tag{4.1}$$

Define $h(z) = \sum_{i=1}^n t_i h_i$ and $g(z) = \sum_{i=1}^n t_i g_i(z)$. Then $h(0) = 0 = h'(0) - 1 = g(0)$. Then the convex combination of $f_1, f_2 \cdots, f_n$ is

$$f(z) = \sum_{i=1}^n f_i(z) = \sum_{i=1}^n (h_i(z) + g_i(z)) = h(z) + \overline{g(z)}.$$

Using (6),

$$g'(z) = \sum_{i=1}^n t_i g_i'(z) = \sum_{i=1}^n t_i z h_i'(z) = z\sum_{i=1}^n t_i h_i'(z) = zh'(z) \tag{4.2}$$

and

$$|zh''(z)| = \left|\sum_{i=1}^n z t_i h_i''(z)\right| \leq \sum_{i=1}^n t_i |zh_i''(z)| \leq \sum_{i=1}^n t_i M = M \tag{4.3}$$

Equations (4.2) and (4.3) implies $f = \sum_{i=1}^n t_i f_i \in \Upsilon_{\mathcal{H}}(M)$ and hence the subclass $\Upsilon_{\mathcal{H}}(M)$ is closed under convex combinations. $\square$

We recall the following concept of Bloch mapping and Bloch constant from [4]. The Bloch's constant for a harmonic function $f = h + \overline{g}$ is defined as

$$B_f = sup_{z\in\Delta}(1 - |z|^2)(|h'(z)| + |g'(z)|).$$

In the next theorem, we estimate the Bloch's constant for functions in the class $\Upsilon_{\mathcal{H}}(M)$.

Theorem 4.2. *Let $f = h + \overline{g} \in \Upsilon_{\mathcal{H}}(M)$. Then*

$$B_f \leq (1 + r_0)^2(1 - r_0)(1 + Mr_0)$$

where r_0 is the root of the equation

$$1 + M + 2(M - 1)r - 3(M + 1)r^2 - 4Mr^3.$$

Proof. Since $f = h + \overline{g} \in \Upsilon_{\mathcal{H}}(M)$, $g'(z) = zh'(z)$ and hence, the dilatation is $\omega(z) = z$. Then,

$$\begin{aligned}
B_f &= sup_{z\in\Delta}(1 - |z|^2)(|h'(z)| + |g'(z)|)\\
&= sup_{z\in\Delta}(1 - |z|^2)(|h'(z)| + |zh'(z)|)\\
&= sup_{z\in\Delta}(1 - |z|^2)|h'(z)|(1 + |z|)\\
&\leq sup_{z\in\Delta}(1 - |z|^2)(1 + M|z|)(1 + |z|), \quad \text{by}(4)\\
&= sup_{0<r<1}(1 - r^2)(1 + Mr)(1 + r), \ \text{where } |z| = r.
\end{aligned}$$

Let $S(r) = (1 - r^2)(1 + Mr)(1 + r)$.
Then, $T(r) = S'(r) = 1+M+2(M-1)r-3(M+1)r^2-4Mr^3$. Since $T(0)T(1) < 0$, there is a real number $r_0 \in (0, 1)$ such that $T(r_0) = 0$.
Now $H'(r) = 2(M - 1) - 6(M + 1)r - 12Mr^2$. For a fixed r, this can be viewed as a function of M, and hence

$$L(M) = H'(r) = 2(M - 1) - 6(M + 1)r - 12Mr^2 = -2 - 6r + (2 - 6r - 12r^2)M.$$

Note that $L(0) = -2(1 + 3r) < 0$ and $L(1) = -12r(1 + r) < 0$. The invariance of sign change of the coefficients of $L(M)$ shows that $L(M) < 0$ for each $M \in (0, 3)$. Hence $H'(r) < 0$ for each $r \in (0, 1)$. Thus $r_0 \in (0, 1)$ is unique. □

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