

ON THE SOLVABILITY OF ELLIPTIC EQUATIONS WITH LOGARITHMICALLY SINGULAR SOURCE TERMS

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ABSTRACT. We establish existence results for solutions of singular elliptic Dirichlet problems involving logarithmic singularities, measure data, and solution-dependent operators. By employing fixed-point methods, truncation techniques, and uniform estimates, we prove the existence of solutions in the classical Sobolev space for both bounded and unbounded domains.

Keywords. Existence theory, logarithmic singularity, measure data, nonlinear elliptic equations, unbounded domain.

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1. INTRODUCTION

The study of elliptic partial differential equations with singular nonlinearities has attracted considerable attention in recent decades due to their mathematical richness and their applications in various physical contexts, including chemical reaction kinetics, non-Newtonian fluid mechanics, population dynamics, and semiconductor theory. Among these, problems featuring a singularity of the form $1/[u^\gamma \ln^\lambda(1+u)]$ present particular analytical challenges because the logarithmic factor modifies the blow-up rate near zero compared to pure power singularities. This paper is devoted to the analysis of the nonlinear Dirichlet problem

$$\begin{cases} -\operatorname{div}(a(x, u, \nabla u)) = \frac{\mu}{u^\gamma \ln^\lambda(1+u)} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is an open set (bounded or unbounded), μ is a nonnegative function or measure belonging to the dual Sobolev space $W^{-1,p'}(\Omega)$, and $\gamma, \lambda > 0$ are positive parameters. The operator $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is assumed to be a Carathéodory function satisfying standard Leray–Lions type conditions: coercivity, monotonicity with respect to the gradient variable, and appropriate growth bounds.

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The literature on singular elliptic equations began with the classical model

$$\begin{cases} -\Delta u = \frac{f}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where $\gamma > 0$ and $f \geq 0$. Pioneering works by Crandall, Rabinowitz and Tartar [8], Lazer and McKenna [12], and Stuart [18] established fundamental existence, uniqueness and regularity properties. For smooth data f , they proved the existence of a unique classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$, while also showing that $u \notin C^1(\overline{\Omega})$ in general and that $u \in H_0^1(\Omega)$ if and only if $\gamma < 3$. Later developments extended the theory to more general operators and less regular data. In particular, Boccardo and Orsina [5] studied semilinear equations with singular nonlinearities and L^1 or measure data, proving existence of solutions in $W_0^{1,1}(\Omega)$. The framework of renormalized solutions, introduced by Dal Maso, Murat, Orsina and Prignet [9] and further developed by Murat and Porretta [15], provided a robust setting for elliptic equations with general measure data. De Cave [10], De Cave, Durastanti and Oliva [11], and Oliva and Petitta [16] investigated singular nonlinearities within this framework. Moreover, some recent results on singular elliptic equations can be found in [1, 2, 3, 4].

More recently, Chipot [6] considered problems of the form

$$\langle Au, v \rangle = \langle \mu, H(u)v \rangle, \quad u \in W_0^{1,p}(\Omega),$$

where A is a monotone operator defined by a Carathéodory function $a(x, \nabla u)$ and H is a singular function satisfying certain conditions. However, Chipot's work assumes that a does not depend on u , and treats H as a general singular function, not specifically the logarithmic type considered here. Chipot and De Cave [7] later developed new techniques for certain classes of singular elliptic equations, but still under the assumption that $a = a(x, \nabla u)$.

The present work distinguishes itself from previous literature by simultaneously addressing three challenging features that, to our knowledge, have not been treated together before. First, we consider the logarithmic singularity $h(u) = 1/[u^\gamma \ln^\lambda(1+u)]$, which exhibits a weaker blow-up at $u = 0$ than the pure power $1/u^\gamma$ but introduces new analytical difficulties due to the logarithmic factor. Second, we allow the operator to depend on u itself through $a(x, u, \nabla u)$, making the problem fully nonlinear and preventing direct application of earlier results that assume $a = a(x, \nabla u)$. Third, we permit μ to be a general nonnegative element of $W^{-1,p'}(\Omega)$, which includes measures and functions in negative Sobolev spaces. Additionally, we treat both bounded and unbounded domains, provided suitable Sobolev embeddings hold.

Our main results establish the existence of solutions to (1.1) under various settings. In Section 3, we consider the case where h is bounded and Lipschitz, which serves as a model for the fixed-point method. In Section 4, we address the logarithmic singularity on bounded domains via a careful truncation procedure. In Section 5, we extend the results to unbounded domains using an exhaustion

argument. Section 6 provides concrete examples, including the p -Laplacian with logarithmic singularity, illustrating the applicability of our abstract theory. The work not only generalizes earlier results by Chipot [6] and others, but also provides a unified existence theory for singular elliptic problems with logarithmic nonlinearities, paving the way for further studies on regularity, uniqueness, critical exponents, and parabolic counterparts.

The proofs combine tools from monotone operator theory (Browder–Minty theorem), Schauder’s fixed-point theorem, truncation techniques, compensated compactness via the Minty trick, and refined Sobolev embeddings. A key technical innovation is the use of the transformation $\Phi(s) = \int_0^s h(t) dt$, which converts the singular problem into a nonsingular one, enabling uniform $W^{1,p}$ estimates independent of truncation parameters.

The paper is organized as follows. Section 2 introduces the mathematical assumptions, notation, and functional framework. Section 3 establishes preliminary results and a priori estimates for regularized problems. Section 4 treats the case where h is bounded and Lipschitz, serving as a blueprint for the fixed-point methodology. Section 5 addresses the logarithmic singularity on bounded domains. Section 6 extends results to unbounded domains. Section 7 provides concrete applications with the p -Laplacian.

2. MATHEMATICAL ASSUMPTIONS AND NOTATION

Throughout this paper, $\Omega \subseteq \mathbb{R}^N$ ($N \geq 2$) is an open set (bounded or unbounded). For $p > 1$, we denote $p' = p/(p-1)$ the conjugate exponent. The space $W_0^{1,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in the norm $\|u\|_{W^{1,p}} = (\int_\Omega |\nabla u|^p dx)^{1/p}$, and $W^{-1,p'}(\Omega)$ is its dual.

The operator $A : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is defined by

$$\langle Au, v \rangle = \int_\Omega a(x, u, \nabla u) \cdot \nabla v \, dx \quad \forall u, v \in W_0^{1,p}(\Omega), \quad (2.1)$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function (measurable in x for all (t, ξ) , continuous in (t, ξ) for almost all x) satisfying the following conditions for some constants $\alpha, \beta > 0$ and function $\zeta \in L^{p'}(\Omega)$

$$a(x, t, \xi) \cdot \xi \geq \alpha |\xi|^p \quad \text{a.e. } x \in \Omega, \quad \forall t \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N, \quad (2.2)$$

$$(a(x, t, \xi) - a(x, t, \xi')) \cdot (\xi - \xi') \geq 0 \quad \text{a.e. } x \in \Omega, \quad \forall t \in \mathbb{R}, \quad \forall \xi, \xi' \in \mathbb{R}^N, \quad (2.3)$$

$$|a(x, t, \xi)| \leq \beta(\zeta(x) + |t|^{p-1} + |\xi|^{p-1}) \quad \text{a.e. } x \in \Omega, \quad \forall t \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N. \quad (2.4)$$

For stronger convergence results, we sometimes assume enhanced monotonicity

$$\text{If } p \geq 2 : \quad (a(x, t, \xi) - a(x, t, \xi')) \cdot (\xi - \xi') \geq b_p |\xi - \xi'|^p, \quad (2.5)$$

$$\text{If } 1 < p < 2 : \quad (a(x, t, \xi) - a(x, t, \xi')) \cdot (\xi - \xi') \geq b_p \frac{|\xi - \xi'|^2}{(|\xi| + |\xi'|)^{2-p}}, \quad (2.6)$$

for some $b_p > 0$, almost all $x \in \Omega$, and all $t \in \mathbb{R}$, $\xi, \xi' \in \mathbb{R}^N$.

We assume that the function ζ in (2.4) belongs to $L^{p'}(\Omega) \cap L^{(p^*)'}(\Omega)$ when $p < N$, and to $L^{p'}(\Omega) \cap L^q(\Omega)$ for some $q > 1$ when $p \geq N$, to ensure the needed integrability properties.

The singular nonlinearity of primary interest is

$$h(u) = \frac{1}{u^\gamma \ln^\lambda(1+u)}, \quad \gamma > 0, \lambda > 0. \quad (2.7)$$

We denote by p^* the Sobolev conjugate exponent: $p^* = Np/(N-p)$ if $p < N$, while if $p \geq N$, p^* can be taken as any finite exponent for which the Sobolev embedding holds. The notation $\rightharpoonup$ denotes weak convergence, $\rightarrow$ strong convergence, and $\rightharpoonup^*$ weak-* convergence.

3. PRELIMINARY RESULTS AND A PRIORI ESTIMATES

Throughout this section, we assume Ω is bounded and that either

$$N = 1 \quad \text{or} \quad p > \frac{2N}{N+2}. \quad (3.1)$$

Condition (3.1) ensures the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, which is crucial for our approximation scheme. Indeed, when $p > 2N/(N+2)$, we have $p^* > 2$, so by Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$.

Given $\varepsilon > 0$ and $\hat{\mu} \in W_0^{1,p}(\Omega)$, consider the regularized problem

$$\begin{cases} u_\varepsilon \in W_0^{1,p}(\Omega), \\ \varepsilon Au_\varepsilon + u_\varepsilon = \hat{\mu} \quad \text{in } W^{-1,p'}(\Omega). \end{cases} \quad (3.2)$$

Proposition 3.1. *Under assumptions (2.2)–(2.4) and (3.1), for each $\varepsilon > 0$ there exists a unique solution u_ε to (3.2). Moreover, as $\varepsilon \rightarrow 0$, we have*

$$u_\varepsilon \rightharpoonup \hat{\mu} \quad \text{in } W_0^{1,p}(\Omega), \quad Au_\varepsilon \rightharpoonup A\hat{\mu} \quad \text{in } W^{-1,p'}(\Omega). \quad (3.3)$$

If additionally $A\hat{\mu} \geq 0$ in the sense of distributions (i.e., $\langle A\hat{\mu}, \phi \rangle \geq 0$ for all $\phi \in C_c^\infty(\Omega)$, $\phi \geq 0$), then $Au_\varepsilon \geq 0$ and $u_\varepsilon \leq \hat{\mu}$ almost everywhere in Ω .

Proof. Step 1: Existence and uniqueness for fixed ε . Consider the operator $T_\varepsilon : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ defined by $T_\varepsilon(u) = \varepsilon Au + u$. We verify its properties

(1) Monotonicity: For any $u, v \in W_0^{1,p}(\Omega)$,

$$\langle T_\varepsilon(u) - T_\varepsilon(v), u - v \rangle = \varepsilon \langle Au - Av, u - v \rangle + \|u - v\|_{L^2}^2 \geq \|u - v\|_{L^2}^2 \geq 0,$$

by (2.3) and the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$.

(2) Hemicontinuity: For fixed $u, v, w \in W_0^{1,p}(\Omega)$, the map $t \mapsto \langle T_\varepsilon(u + tv), w \rangle$ is continuous on $\mathbb{R}$ by the Carathéodory condition on a .

(3) Coercivity: Using (2.2),

$$\frac{\langle T_\varepsilon(u), u \rangle}{\|u\|_{W^{1,p}}} \geq \frac{\varepsilon \alpha \|\nabla u\|_{L^p}^p + \|u\|_{L^2}^2}{\|u\|_{W^{1,p}}} \rightarrow \infty \quad \text{as } \|u\|_{W^{1,p}} \rightarrow \infty.$$

(4) Boundedness: By (2.4), for any bounded set $B \subset W_0^{1,p}(\Omega)$, $T_\varepsilon(B)$ is bounded in $W^{-1,p'}(\Omega)$.

By the Browder–Minty theorem [13], there exists a unique $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that $T_\varepsilon(u_\varepsilon) = \hat{\mu}$.

Take $v = u_\varepsilon - \hat{\mu}$ as test function in (3.2)

$$\varepsilon \langle Au_\varepsilon, u_\varepsilon - \hat{\mu} \rangle + \int_\Omega (u_\varepsilon - \hat{\mu})^2 dx = 0. \quad (3.4)$$

Thus $\varepsilon \langle Au_\varepsilon, u_\varepsilon - \hat{\mu} \rangle = -\|u_\varepsilon - \hat{\mu}\|_{L^2}^2 \leq 0$.

Using (2.2) and (2.4), we estimate

$$\begin{aligned} \alpha \varepsilon \int_\Omega |\nabla u_\varepsilon|^p dx &\leq \varepsilon \int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx \\ &= \varepsilon \int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \hat{\mu} dx - \|u_\varepsilon - \hat{\mu}\|_{L^2}^2 \\ &\leq \varepsilon \int_\Omega \beta(\zeta(x) + |u_\varepsilon|^{p-1} + |\nabla u_\varepsilon|^{p-1}) |\nabla \hat{\mu}| dx. \end{aligned}$$

Applying Hölder's inequality with exponents p and p'

$$\begin{aligned} \alpha \varepsilon \|\nabla u_\varepsilon\|_{L^p}^p &\leq \varepsilon \beta (\|\zeta\|_{L^{p'}} \|\nabla \hat{\mu}\|_{L^p} + \|u_\varepsilon\|_{L^p}^{p-1} \|\nabla \hat{\mu}\|_{L^p} + \|\nabla u_\varepsilon\|_{L^p}^{p-1} \|\nabla \hat{\mu}\|_{L^p}) \\ &\leq \varepsilon \beta (C_1 + C_2 \|\nabla u_\varepsilon\|_{L^p}^{p-1}), \end{aligned}$$

where $C_1 = \|\zeta\|_{L^{p'}} \|\nabla \hat{\mu}\|_{L^p} + C \|\nabla \hat{\mu}\|_{L^p}^p$ (using Poincaré's inequality $\|u_\varepsilon\|_{L^p} \leq C \|\nabla u_\varepsilon\|_{L^p}$) and $C_2 = \beta \|\nabla \hat{\mu}\|_{L^p}$. Dividing by $\varepsilon > 0$

$$\alpha \|\nabla u_\varepsilon\|_{L^p}^p \leq \beta (C_1 + C_2 \|\nabla u_\varepsilon\|_{L^p}^{p-1}).$$

By Young's inequality $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$ with $a = \|\nabla u_\varepsilon\|_{L^p}^{p-1}$ and $b = \beta C_2 / \alpha$, we obtain

$$\|\nabla u_\varepsilon\|_{L^p}^p \leq C,$$

where C depends on $\alpha, \beta, \zeta, \hat{\mu}$ but not on ε .

Since $\{u_\varepsilon\}$ is bounded in $W_0^{1,p}(\Omega)$, by reflexivity ($p > 1$) there exists a subsequence (still denoted u_ε) and $u \in W_0^{1,p}(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. From (3.2)

$$u_\varepsilon - \hat{\mu} = -\varepsilon Au_\varepsilon \rightarrow 0 \quad \text{in } W^{-1,p'}(\Omega),$$

since $\{Au_\varepsilon\}$ is bounded in $W^{-1,p'}(\Omega)$ by (2.4) and the boundedness of $\{u_\varepsilon\}$. By uniqueness of the weak limit in $W^{-1,p'}(\Omega)$ (which contains $W_0^{1,p}(\Omega)$ via the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ under (3.1)), we have $u = \hat{\mu}$. The entire sequence converges weakly to $\hat{\mu}$ by uniqueness of the limit.

Since $\{Au_\varepsilon\}$ is bounded in $W^{-1,p'}(\Omega)$, extract a subsequence such that $Au_\varepsilon \rightharpoonup \chi$ in $W^{-1,p'}(\Omega)$ for some χ . From (3.4)

$$\langle Au_\varepsilon, u_\varepsilon - \hat{\mu} \rangle = -\frac{1}{\varepsilon} \|u_\varepsilon - \hat{\mu}\|_{L^2}^2 \leq 0.$$

Taking limsup as $\varepsilon \rightarrow 0$

$$\limsup_{\varepsilon \rightarrow 0} \langle Au_\varepsilon, u_\varepsilon \rangle \leq \langle \chi, \hat{\mu} \rangle.$$

By monotonicity (2.3), for any $v \in W_0^{1,p}(\Omega)$

$$\langle Au_\varepsilon - Av, u_\varepsilon - v \rangle \geq 0 \quad \Rightarrow \quad \langle Au_\varepsilon, u_\varepsilon \rangle \geq \langle Au_\varepsilon, v \rangle + \langle Av, u_\varepsilon - v \rangle.$$

Taking limsup and using the weak convergences

$$\langle \chi, \hat{\mu} \rangle \geq \langle \chi, v \rangle + \langle Av, \hat{\mu} - v \rangle \quad \forall v \in W_0^{1,p}(\Omega).$$

Rearranging: $\langle \chi - Av, \hat{\mu} - v \rangle \geq 0$ for all $v \in W_0^{1,p}(\Omega)$.

Apply Minty's trick: for any $w \in W_0^{1,p}(\Omega)$ and $t > 0$, take $v = \hat{\mu} - tw$

$$\langle \chi - A(\hat{\mu} - tw), tw \rangle \geq 0 \quad \Rightarrow \quad \langle \chi - A(\hat{\mu} - tw), w \rangle \geq 0.$$

Let $t \rightarrow 0^+$. By hemicontinuity (implied by Carathéodory conditions), $A(\hat{\mu} - tw) \rightharpoonup A\hat{\mu}$ in $W^{-1,p'}(\Omega)$, so

$$\langle \chi - A\hat{\mu}, w \rangle \geq 0 \quad \forall w \in W_0^{1,p}(\Omega).$$

Replacing w by $-w$ gives $\langle \chi - A\hat{\mu}, w \rangle = 0$ for all w , hence $\chi = A\hat{\mu}$. The entire sequence converges weakly to $A\hat{\mu}$.

Assume $A\hat{\mu} \geq 0$. Test (3.2) with $(u_\varepsilon - \hat{\mu})^+ = \max(u_\varepsilon - \hat{\mu}, 0)$

$$\varepsilon \langle Au_\varepsilon, (u_\varepsilon - \hat{\mu})^+ \rangle + \int_\Omega [(u_\varepsilon - \hat{\mu})^+]^2 dx = \langle \hat{\mu}, (u_\varepsilon - \hat{\mu})^+ \rangle.$$

But $\langle \hat{\mu}, (u_\varepsilon - \hat{\mu})^+ \rangle = \int_\Omega \hat{\mu}(u_\varepsilon - \hat{\mu})^+ dx \leq 0$ since $\hat{\mu} \geq 0$ and $(u_\varepsilon - \hat{\mu})^+ \geq 0$ only where $u_\varepsilon \geq \hat{\mu}$. Thus

$$\varepsilon \langle Au_\varepsilon, (u_\varepsilon - \hat{\mu})^+ \rangle + \int_\Omega [(u_\varepsilon - \hat{\mu})^+]^2 dx \leq 0.$$

By monotonicity, $\langle Au_\varepsilon, (u_\varepsilon - \hat{\mu})^+ \rangle \geq \langle A\hat{\mu}, (u_\varepsilon - \hat{\mu})^+ \rangle \geq 0$ (since $A\hat{\mu} \geq 0$). Therefore

$$\int_\Omega [(u_\varepsilon - \hat{\mu})^+]^2 dx \leq -\varepsilon \langle Au_\varepsilon, (u_\varepsilon - \hat{\mu})^+ \rangle \leq 0,$$

so $(u_\varepsilon - \hat{\mu})^+ = 0$ a.e., i.e., $u_\varepsilon \leq \hat{\mu}$ a.e. Then from (3.2), $\varepsilon Au_\varepsilon = \hat{\mu} - u_\varepsilon \geq 0$, so $Au_\varepsilon \geq 0$. $\square$

Proposition 3.2. *Assume, in addition to (2.2)–(2.4), the strengthened monotonicity conditions (2.5) or (2.6). Then the convergences in (3.3) are strong*

$$u_\varepsilon \rightarrow \hat{\mu} \quad \text{in } W_0^{1,p}(\Omega), \quad Au_\varepsilon \rightarrow A\hat{\mu} \quad \text{in } W^{-1,p'}(\Omega).$$

Proof. We provide the proof for $p \geq 2$. The case $1 < p < 2$ is similar with appropriate modifications (see [6] for details).

From (2.5) with $\xi = \nabla u_\varepsilon$, $\xi' = \nabla \hat{\mu}$, and $t = u_\varepsilon$

$$b_p \int_\Omega |\nabla u_\varepsilon - \nabla \hat{\mu}|^p dx \leq \int_\Omega (a(x, u_\varepsilon, \nabla u_\varepsilon) - a(x, u_\varepsilon, \nabla \hat{\mu})) \cdot (\nabla u_\varepsilon - \nabla \hat{\mu}) dx.$$

Adding and subtracting $a(x, \hat{\mu}, \nabla \hat{\mu})$

$$\begin{aligned} b_p \|\nabla u_\varepsilon - \nabla \hat{\mu}\|_{L^p}^p &\leq \int_\Omega (a(x, u_\varepsilon, \nabla u_\varepsilon) - a(x, \hat{\mu}, \nabla \hat{\mu})) \cdot (\nabla u_\varepsilon - \nabla \hat{\mu}) dx \\ &\quad - \int_\Omega (a(x, u_\varepsilon, \nabla \hat{\mu}) - a(x, \hat{\mu}, \nabla \hat{\mu})) \cdot (\nabla u_\varepsilon - \nabla \hat{\mu}) dx \\ &= \langle Au_\varepsilon - A\hat{\mu}, u_\varepsilon - \hat{\mu} \rangle - \int_\Omega (a(x, u_\varepsilon, \nabla \hat{\mu}) - a(x, \hat{\mu}, \nabla \hat{\mu})) \cdot (\nabla u_\varepsilon - \nabla \hat{\mu}) dx. \end{aligned}$$

From (3.2)

$$\langle Au_\varepsilon - A\hat{\mu}, u_\varepsilon - \hat{\mu} \rangle = -\frac{1}{\varepsilon} \|u_\varepsilon - \hat{\mu}\|_{L^2}^2 - \langle A\hat{\mu}, u_\varepsilon - \hat{\mu} \rangle.$$

Thus

$$b_p \|\nabla u_\varepsilon - \nabla \hat{\mu}\|_{L^p}^p \leq -\frac{1}{\varepsilon} \|u_\varepsilon - \hat{\mu}\|_{L^2}^2 - \langle A\hat{\mu}, u_\varepsilon - \hat{\mu} \rangle - I_\varepsilon,$$

where $I_\varepsilon = \int_\Omega (a(x, u_\varepsilon, \nabla \hat{\mu}) - a(x, \hat{\mu}, \nabla \hat{\mu})) \cdot (\nabla u_\varepsilon - \nabla \hat{\mu}) dx$.

Since $u_\varepsilon \rightharpoonup \hat{\mu}$ in $W_0^{1,p}(\Omega)$ and $u_\varepsilon \rightarrow \hat{\mu}$ in $L^p(\Omega)$ by compact embedding, and by the growth condition (2.4), $|a(x, u_\varepsilon, \nabla \hat{\mu}) - a(x, \hat{\mu}, \nabla \hat{\mu})| \leq \beta |u_\varepsilon - \hat{\mu}|^{p-1} \rightarrow 0$ in $L^{p'}(\Omega)$, we have $I_\varepsilon \rightarrow 0$. Also $\langle A\hat{\mu}, u_\varepsilon - \hat{\mu} \rangle \rightarrow 0$. Therefore

$$\limsup_{\varepsilon \rightarrow 0} \|\nabla u_\varepsilon - \nabla \hat{\mu}\|_{L^p}^p \leq 0,$$

so $u_\varepsilon \rightarrow \hat{\mu}$ in $W_0^{1,p}(\Omega)$. The convergence $Au_\varepsilon \rightarrow A\hat{\mu}$ in $W^{-1,p'}(\Omega)$ then follows from the continuity of A on bounded sets. $\square$

Remark 3.3. If we assume strict monotonicity in the sense that

$$(a(x, t, \xi) - a(x, t, \xi')) \cdot (\xi - \xi') > 0 \quad \text{a.e. } x \in \Omega, \quad \forall t \in \mathbb{R}, \quad \forall \xi \neq \xi' \in \mathbb{R}^N, \quad (3.5)$$

then the operator A is strictly monotone. Consequently, for any $f \in W^{-1,p'}(\Omega)$, there exists a unique $u \in W_0^{1,p}(\Omega)$ solving

$$\langle Au, v \rangle = \langle f, v \rangle \quad \forall v \in W_0^{1,p}(\Omega). \quad (3.6)$$

3.1. Continuity of the Solution Map. Assume (3.5) holds. Define $T : W^{-1,p'}(\Omega) \rightarrow W_0^{1,p}(\Omega)$ as the solution operator to (3.6), i.e., $T(f) = u$ where u solves $\langle Au, v \rangle = \langle f, v \rangle$ for all $v \in W_0^{1,p}(\Omega)$.

Lemma 3.4. *The operator $T : W^{-1,p'}(\Omega) \rightarrow W_0^{1,p}(\Omega)$ is continuous when $W^{-1,p'}(\Omega)$ is equipped with the strong topology and $W_0^{1,p}(\Omega)$ with the weak topology.*

Proof. Let $f_n \rightarrow f$ strongly in $W^{-1,p'}(\Omega)$. Set $u_n = T(f_n)$, $u = T(f)$. Test the equations for u_n and u with $u_n - u$

$$\begin{aligned} \langle Au_n - Au, u_n - u \rangle &= \langle f_n - f, u_n - u \rangle \\ &\leq \|f_n - f\|_{W^{-1,p'}} \|u_n - u\|_{W^{1,p}}. \end{aligned}$$

By strict monotonicity (3.5) and the inequality $(a(x, t, \xi) - a(x, t, \xi')) \cdot (\xi - \xi') \geq \alpha |\xi - \xi'|^p$ (which follows from (2.2) and (2.3) via standard arguments)

$$\alpha \|\nabla(u_n - u)\|_{L^p}^p \leq \|f_n - f\|_{W^{-1,p'}} \|u_n - u\|_{W^{1,p}}.$$

Thus $\|\nabla(u_n - u)\|_{L^p}^{p-1} \leq \frac{1}{\alpha} \|f_n - f\|_{W^{-1,p'}} \rightarrow 0$, so $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$, hence also weakly. $\square$

4. CASE OF BOUNDED DOMAINS WITH BOUNDED NONLINEARITIES

In this section, we assume Ω is bounded, $p > 2N/(N+2)$ (so that (3.1) holds), and a satisfies (2.2)–(2.4) along with strict monotonicity (3.5) and the strengthened conditions (2.5) or (2.6). We also assume the integrability conditions on ζ stated after (2.4).

Given $\mu \in W^{-1,p'}(\Omega)$, let $\hat{\mu}$ be the unique solution to

$$\langle A\hat{\mu}, v \rangle = \langle \mu, v \rangle \quad \forall v \in W_0^{1,p}(\Omega). \quad (4.1)$$

Such $\hat{\mu}$ exists by Remark 3.3.

Lemma 4.1. *Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded Lipschitz continuous function satisfying*

$$|h(s)| \leq M \quad \forall s \in \mathbb{R}, \quad |h(s) - h(t)| \leq L|s - t| \quad \forall s, t \in \mathbb{R}. \quad (4.2)$$

If additionally $(sh(s))' \in L^\infty(\mathbb{R})$, then the function $\Phi(s) = sh(s)$ is Lipschitz continuous with constant $C_h = \|(sh(s))'\|_{L^\infty}$.

Proof. Since Φ is differentiable almost everywhere with bounded derivative, it is Lipschitz by the mean value theorem. $\square$

Theorem 4.2. *Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded Lipschitz continuous function satisfying (4.2) and such that $(sh(s))' \in L^\infty(\mathbb{R})$, i.e., the derivative of $s \mapsto sh(s)$ is essentially bounded. Then there exists at least one solution $u \in W_0^{1,p}(\Omega)$ to*

$$\langle Au, v \rangle = \langle A\hat{\mu}, h(u)v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \quad (4.3)$$

Proof. Step 1: Approximation scheme. For each $n \in \mathbb{N}$, let $\hat{\mu}_n$ be the solution to

$$\frac{1}{n}A\hat{\mu}_n + \hat{\mu}_n = \hat{\mu} \quad \text{in } W^{-1,p'}(\Omega), \quad (4.4)$$

which exists and is unique by Proposition 3.1. By Proposition 3.2, we have

$$\hat{\mu}_n \rightarrow \hat{\mu} \quad \text{in } W_0^{1,p}(\Omega), \quad A\hat{\mu}_n \rightarrow A\hat{\mu} \quad \text{in } W^{-1,p'}(\Omega). \quad (4.5)$$

Moreover, from (4.4), $A\hat{\mu}_n = n(\hat{\mu} - \hat{\mu}_n) \in W_0^{1,p}(\Omega)$. Under the integrability conditions on ζ , we have $A\hat{\mu}_n \in L^q(\Omega)$ for an appropriate exponent q (see Remark below).

Remark 4.3. When $p < N$, by Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, we have $A\hat{\mu}_n \in L^{p^*}(\Omega)$. For $p \geq N$, $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ for all $r < \infty$, so we can take any $r > p'$ to ensure $A\hat{\mu}_n \in L^r(\Omega)$.

For the fixed-point setup. We define the operator $T_n : L^r(\Omega) \rightarrow L^r(\Omega)$ as follows, where r will be chosen appropriately. For $w \in L^r(\Omega)$, let $T_n(w) = u$ where u solves

$$\langle Au, v \rangle = \int_{\Omega} A\hat{\mu}_n h(w)v \, dx \quad \forall v \in W_0^{1,p}(\Omega). \quad (4.6)$$

We need to verify that the right-hand side defines an element of $W^{-1,p'}(\Omega)$. Consider two cases

For $p < N$. Then $p^* = Np/(N-p) > p$. Since $A\hat{\mu}_n \in L^{p^*}(\Omega)$ and h is bounded, $A\hat{\mu}_n h(w) \in L^{p^*}(\Omega)$. Note that $p^* > (p^*)'$ is equivalent to $p > 2N/(N+2)$, which holds. For any $v \in W_0^{1,p}(\Omega)$

$$\left| \int_{\Omega} A\hat{\mu}_n h(w) v \, dx \right| \leq \|A\hat{\mu}_n h(w)\|_{L^{(p^*)'}} \|v\|_{L^{p^*}} \leq C \|A\hat{\mu}_n\|_{L^{p^*}} \|h\|_{\infty} \|\nabla v\|_{L^p},$$

by Hölder and Sobolev inequalities. Here we take $r = p^*$.

For $p \geq N$. Then $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q < \infty$. Choose $r > p'$ and note that by the integrability conditions, $A\hat{\mu}_n \in L^r(\Omega)$ with r satisfying $1/r + 1/p' \leq 1$ (possible since r can be taken large). Then

$$\left| \int_{\Omega} A\hat{\mu}_n h(w) v \, dx \right| \leq \|A\hat{\mu}_n h(w)\|_{L^{p'}} \|v\|_{L^p} \leq C \|A\hat{\mu}_n\|_{L^r} \|h\|_{\infty} \|\nabla v\|_{L^p},$$

by Hölder with exponents $1/p' = 1/r + 1/s$ where s is chosen appropriately.

In both cases, the right-hand side of (4.6) defines a bounded linear functional on $W_0^{1,p}(\Omega)$, so $u = T_n(w)$ exists and is unique by strict monotonicity.

A priori bounds for T_n . Take $v = u$ in (4.6)

$$\langle Au, u \rangle = \int_{\Omega} A\hat{\mu}_n h(w) u \, dx.$$

By (2.2): $\alpha \|\nabla u\|_{L^p}^p \leq \langle Au, u \rangle$. Thus

$$\alpha \|\nabla u\|_{L^p}^p \leq \|A\hat{\mu}_n h(w)\|_{L^{(p^*)'}} \|u\|_{L^{p^*}} \leq C \|A\hat{\mu}_n\|_{L^{p^*}} \|h\|_{\infty} \|u\|_{L^{p^*}}. \quad (4.7)$$

Using Sobolev inequality $\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p}$

$$\alpha \|\nabla u\|_{L^p}^p \leq C \|A\hat{\mu}_n\|_{L^{p^*}} \|h\|_{\infty} \|\nabla u\|_{L^p},$$

so $\|\nabla u\|_{L^p}^{p-1} \leq \frac{C}{\alpha} \|A\hat{\mu}_n\|_{L^{p^*}} \|h\|_{\infty}$. Hence

$$\|\nabla u\|_{L^p} \leq C_n := \left(\frac{C}{\alpha} \|A\hat{\mu}_n\|_{L^{p^*}} \|h\|_{\infty} \right)^{\frac{1}{p-1}}.$$

Therefore T_n maps $L^r(\Omega)$ into the ball $B_{R_n} = \{v \in L^r(\Omega) : \|v\|_{L^r} \leq C' C_n\}$ for some constant C' (by Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ for appropriate r).

Step 2: Compactness and continuity of T_n . The image $T_n(L^r(\Omega))$ is bounded in $W_0^{1,p}(\Omega)$, hence relatively compact in $L^r(\Omega)$ by the compact Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ for $r < p^*$.

For continuity: let $w_k \rightarrow w$ in $L^r(\Omega)$. Then up to a subsequence, $w_k \rightarrow w$ a.e. By dominated convergence (since h is bounded and continuous, and $|A\hat{\mu}_n h(w_k)| \leq \|A\hat{\mu}_n\|_{L^\infty(\Omega)} \|h\|_{\infty}$ locally), we have $A\hat{\mu}_n h(w_k) \rightarrow A\hat{\mu}_n h(w)$ in $L^{(p^*)'}(\Omega)$ (or $L^{p'}(\Omega)$ if $p \geq N$). Thus the right-hand sides of (4.6) converge in $W^{-1,p'}(\Omega)$. By Lemma 3.4, $T_n(w_k) \rightarrow T_n(w)$ in $W_0^{1,p}(\Omega)$. But by compactness, a subsequence converges strongly in $L^r(\Omega)$, and the limit must be $T_n(w)$ by uniqueness. Hence the entire sequence $T_n(w_k) \rightarrow T_n(w)$ in $L^r(\Omega)$.

Step 3: Existence of approximate solutions u_n . By Schauder's fixed-point theorem applied to $T_n : B_{R_n} \rightarrow B_{R_n}$, there exists $u_n \in B_{R_n}$ such that $T_n(u_n) = u_n$, i.e.

$$\langle Au_n, v \rangle = \langle A\hat{\mu}_n, h(u_n) v \rangle \quad \forall v \in W_0^{1,p}(\Omega). \quad (4.8)$$

We search now uniform estimates for $\{u_n\}$. Take $v = u_n$ in (4.8)

$$\alpha \|\nabla u_n\|_{L^p}^p \leq \langle Au_n, u_n \rangle = \langle A\hat{\mu}_n, h(u_n)u_n \rangle.$$

Write $h(u_n)u_n = (sh(s))|_{s=u_n}$. By Lemma 4.1, the function $s \mapsto sh(s)$ is Lipschitz with constant C_h . Then

$$\begin{aligned} \langle A\hat{\mu}_n, h(u_n)u_n \rangle &= \int_{\Omega} a(x, \hat{\mu}_n, \nabla \hat{\mu}_n) \cdot \nabla(h(u_n)u_n) dx \\ &\leq \int_{\Omega} \beta(\zeta(x) + |\hat{\mu}_n|^{p-1} + |\nabla \hat{\mu}_n|^{p-1}) |\nabla(h(u_n)u_n)| dx \\ &\leq C_h \beta \int_{\Omega} (\zeta(x) + |\hat{\mu}_n|^{p-1} + |\nabla \hat{\mu}_n|^{p-1}) |\nabla u_n| dx. \end{aligned}$$

By Hölder and Young inequalities

$$\alpha \|\nabla u_n\|_{L^p}^p \leq C (\|\zeta\|_{L^{p'}} + \|\hat{\mu}_n\|_{L^p}^{p-1} + \|\nabla \hat{\mu}_n\|_{L^p}^{p-1}) \|\nabla u_n\|_{L^p}.$$

Since $\{\hat{\mu}_n\}$ is bounded in $W_0^{1,p}(\Omega)$ by (4.5), we obtain $\|\nabla u_n\|_{L^p} \leq C$ independently of n .

We now pass to the limit. Extract a subsequence with $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u_n \rightarrow u$ a.e. Fix $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $k > 0$. Consider $(v - u_n)_k = \max(\min(v - u_n, k), -k)$. Test (4.8) with $(v - u_n)_k$

$$\int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla(v - u_n)_k dx = \int_{\Omega} a(x, \hat{\mu}_n, \nabla \hat{\mu}_n) \cdot \nabla(h(u_n)(v - u_n)_k) dx.$$

By monotonicity of a in ξ

$$\int_{\Omega} a(x, v, \nabla v) \cdot \nabla(v - u_n)_k dx \geq \int_{\Omega} a(x, \hat{\mu}_n, \nabla \hat{\mu}_n) \cdot \nabla(h(u_n)(v - u_n)_k) dx.$$

As $n \rightarrow \infty$, $(v - u_n)_k \rightharpoonup (v - u)_k$ in $W_0^{1,p}(\Omega)$. For the right-hand side, $\nabla(h(u_n)(v - u_n)_k) = h(u_n)\nabla(v - u_n)_k + h'(u_n)(v - u_n)_k \nabla u_n$. Since h is Lipschitz, $|h'(u_n)| \leq L$, and $|(v - u_n)_k| \leq k$, the second term is bounded in $L^p(\Omega)$. Also $h(u_n) \rightarrow h(u)$ a.e. and boundedly. Thus $\nabla(h(u_n)(v - u_n)_k) \rightharpoonup \nabla(h(u)(v - u)_k)$ in $L^p(\Omega)$. Moreover, $a(x, \hat{\mu}_n, \nabla \hat{\mu}_n) \rightarrow a(x, \hat{\mu}, \nabla \hat{\mu})$ in $L^{p'}(\Omega)$ by Proposition 3.2.

Passing to the limit

$$\int_{\Omega} a(x, v, \nabla v) \cdot \nabla(v - u)_k dx \geq \int_{\Omega} a(x, \hat{\mu}, \nabla \hat{\mu}) \cdot \nabla(h(u)(v - u)_k) dx.$$

Let $k \rightarrow \infty$ to obtain

$$\int_{\Omega} a(x, v, \nabla v) \cdot \nabla(v - u) dx \geq \int_{\Omega} a(x, \hat{\mu}, \nabla \hat{\mu}) \cdot \nabla(h(u)(v - u)) dx.$$

Take $v = u + tw$ with $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $t > 0$

$$t \int_{\Omega} a(x, u + tw, \nabla(u + tw)) \cdot \nabla w dx \geq t \int_{\Omega} a(x, \hat{\mu}, \nabla \hat{\mu}) \cdot \nabla(h(u)w) dx.$$

Divide by t and let $t \rightarrow 0^+$. By hemicontinuity

$$\int_{\Omega} a(x, u, \nabla u) \cdot \nabla w dx \geq \int_{\Omega} a(x, \hat{\mu}, \nabla \hat{\mu}) \cdot \nabla(h(u)w) dx.$$

Replacing w by $-w$ gives equality, so u satisfies (4.3). $\square$

Theorem 4.4. *Under the assumptions of Theorem 4.2, if additionally $\mu \geq 0$ (i.e., $\langle \mu, \phi \rangle \geq 0$ for all $\phi \in C_c^\infty(\Omega)$, $\phi \geq 0$) and $h(s) \geq 0$ for all $s \leq 0$, then any solution u of (4.3) satisfies $u \geq 0$ almost everywhere in Ω .*

Proof. Let u be a solution. For $k > 0$, test (4.3) with $v = u^- \wedge k = \min(u^-, k)$, where $u^- = \max(-u, 0)$. Note that $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Observe that $h(u)(u^- \wedge k) = 0$ on $\{u \geq 0\}$, and on $\{u < 0\}$, we have $u^- = -u > 0$, so $h(u) = h(-u^-) \geq 0$ by the assumption $h(s) \geq 0$ for $s \leq 0$. Thus $h(u)(u^- \wedge k) \geq 0$ everywhere.

From (4.3)

$$\langle Au, u^- \wedge k \rangle = \langle A\hat{\mu}, h(u)(u^- \wedge k) \rangle = \langle \mu, h(u)(u^- \wedge k) \rangle \geq 0,$$

since $\mu \geq 0$ and $h(u)(u^- \wedge k) \geq 0$.

Now compute $\langle Au, u^- \wedge k \rangle$

$$\langle Au, u^- \wedge k \rangle = \int_{\Omega} a(x, u, \nabla u) \cdot \nabla(u^- \wedge k) dx.$$

On $\{u \geq 0\}$, $u^- \wedge k = 0$, so the integral vanishes. On $\{u < 0\}$, $\nabla u = -\nabla u^-$, and $\nabla(u^- \wedge k) = \chi_{\{0 < u^- < k\}} \nabla u^-$. Thus

$$\langle Au, u^- \wedge k \rangle = \int_{\{0 < u^- < k\}} a(x, -u^-, -\nabla u^-) \cdot \nabla u^- dx.$$

By (2.2)

$$a(x, -u^-, -\nabla u^-) \cdot \nabla u^- = -a(x, -u^-, -\nabla u^-) \cdot (-\nabla u^-) \leq -\alpha |\nabla u^-|^p.$$

Hence

$$0 \leq \langle Au, u^- \wedge k \rangle \leq -\alpha \int_{\{0 < u^- < k\}} |\nabla u^-|^p dx = -\alpha \|\nabla(u^- \wedge k)\|_{L^p}^p.$$

Therefore $\|\nabla(u^- \wedge k)\|_{L^p} = 0$ for all $k > 0$, so $u^- \wedge k = 0$ a.e. Letting $k \rightarrow \infty$ gives $u^- = 0$ a.e., i.e., $u \geq 0$. $\square$

5. LOGARITHMIC SINGULARITIES ON BOUNDED DOMAINS

We consider the logarithmic singularity

$$h(u) = \frac{1}{u^\gamma \ln^\lambda(1+u)}, \quad \gamma, \lambda > 0,$$

on a bounded domain $\Omega \subset \mathbb{R}^N$, with $p > \frac{2N}{N+2}$. Let $\mu \in W^{-1,p'}(\Omega)$ be a nonnegative measure, and let $\hat{\mu} \in W_0^{1,p}(\Omega)$ denote the unique solution of

$$\langle A\hat{\mu}, v \rangle = \langle \mu, v \rangle \quad \forall v \in W_0^{1,p}(\Omega).$$

Theorem 5.1. *Assume Ω is bounded, $p > 2N/(N+2)$, and $\mu \in W^{-1,p'}(\Omega)$ is a nonnegative bounded measure. Let a satisfy (2.2)–(2.4), (3.5), and (2.5) or (2.6). Then there exists $u \in W_0^{1,p}(\Omega)$, $u > 0$ almost everywhere, such that*

$$\langle Au, v \rangle = \left\langle A\hat{\mu}, \frac{v}{u^\gamma \ln^\lambda(1+u)} \right\rangle \quad \forall v \in C_c^1(\Omega). \quad (5.1)$$

Proof. The proof is carried out in several steps.

Step 1: The transformation Φ . Define the function $\Phi : (0, \infty) \rightarrow (0, \infty)$ by

$$\Phi(s) = \int_0^s h(t) dt = \int_0^s \frac{dt}{t^\gamma \ln^\lambda(1+t)}. \quad (5.2)$$

Then Φ is strictly increasing and C^1 on $(0, \infty)$, with $\Phi'(s) = h(s) > 0$. We denote by $\Psi = \Phi^{-1}$ its inverse, defined on the range of Φ .

Step 2: Approximation. For each $n \in \mathbb{N}$, define the truncation

$$h_n(s) = \min\{h(s), n\}, \quad s > 0,$$

extended by $h_n(s) = n$ for $s \leq 0$. Then h_n is bounded and Lipschitz continuous on $\mathbb{R}$.

By Theorems 4.2 and 4.4, there exists $u_n \in W_0^{1,p}(\Omega)$, $u_n \geq 0$, such that

$$\langle Au_n, v \rangle = \langle A\hat{\mu}, h_n(u_n)v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \quad (5.3)$$

Step 3: Change of variables. Set

$$v_n = \Phi(u_n).$$

Since Φ is increasing and $\Phi(0) = 0$, we have $v_n \geq 0$. Moreover,

$$\nabla v_n = h(u_n) \nabla u_n \quad \text{a.e. in } \Omega.$$

Lemma 5.2. *The sequence $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$.*

Proof. Take $v = v_n$ in (5.3). Then

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla v_n dx = \langle A\hat{\mu}, h_n(u_n)v_n \rangle.$$

Using $\nabla v_n = h(u_n) \nabla u_n$ and the coercivity condition (2.2), we obtain

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla v_n dx = \int_\Omega a(x, u_n, \nabla u_n) \cdot (h(u_n) \nabla u_n) dx \geq \alpha \int_\Omega h(u_n) |\nabla u_n|^p dx.$$

Hence,

$$\alpha \int_\Omega h(u_n) |\nabla u_n|^p dx \leq \langle A\hat{\mu}, h_n(u_n)v_n \rangle.$$

Since $h_n(u_n)$ is bounded and $v_n = \Phi(u_n)$, we estimate

$$\langle A\hat{\mu}, h_n(u_n)v_n \rangle \leq \|A\hat{\mu}\|_{W^{-1,p'}} \|\nabla(h_n(u_n)v_n)\|_{L^p}.$$

Using the chain rule,

$$\nabla(h_n(u_n)v_n) = h'_n(u_n)v_n \nabla u_n + h_n(u_n) \nabla v_n.$$

Since h_n is bounded and Lipschitz, both terms are controlled, and we obtain

$$\|\nabla(h_n(u_n)v_n)\|_{L^p} \leq C \|\nabla v_n\|_{L^p}.$$

Consequently,

$$\|\nabla v_n\|_{L^p} \leq C,$$

which proves the claim. $\square$

Step 4: Compactness. From Lemma 5.2, there exist a subsequence (still denoted v_n) and $v \in W_0^{1,p}(\Omega)$ such that

$$v_n \rightharpoonup v \quad \text{in } W_0^{1,p}(\Omega), \quad v_n \rightarrow v \quad \text{a.e. in } \Omega.$$

Define

$$u = \Psi(v).$$

Then, up to a subsequence, $u_n \rightarrow u$ almost everywhere in Ω .

Step 5: Positivity of the limit.

Lemma 5.3. *The limit function v satisfies $v > 0$ almost everywhere in Ω . Consequently, $u = \Psi(v)$ satisfies $u > 0$ a.e. in Ω .*

Proof. From (5.3) and the fact that $h_n(u_n) \geq 0$, we have

$$-\operatorname{div}(a(x, u_n, \nabla u_n)) = A\hat{\mu} h_n(u_n) \geq 0$$

in the sense of distributions. Thus u_n is a supersolution.

By the weak Harnack inequality for monotone operators (see [6], Theorem 2.1), for any $\omega \subset\subset \Omega$, there exists $c_\omega > 0$ such that

$$u_n(x) \geq c_\omega > 0 \quad \text{for a.e. } x \in \omega,$$

for all sufficiently large n . Passing to the limit, we obtain

$$u(x) \geq c_\omega > 0 \quad \text{a.e. in } \omega.$$

Since Φ is strictly increasing and $\Phi(0) = 0$, it follows that

$$v = \Phi(u) \geq \Phi(c_\omega) > 0 \quad \text{a.e. in } \omega.$$

As ω is arbitrary, $v > 0$ a.e. in Ω . □

Step 6: Passage to the limit. Let $\varphi \in C_c^1(\Omega)$. From (5.3),

$$\int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla \varphi \, dx = \int_{\Omega} a(x, \hat{\mu}, \nabla \hat{\mu}) \cdot \nabla (h_n(u_n) \varphi) \, dx.$$

Since $u_n \rightarrow u$ a.e. in Ω and $u_n \geq c_\omega > 0$ locally, we have

$$h_n(u_n) \rightarrow h(u) \quad \text{a.e. in } \Omega,$$

and the convergence is locally bounded. Moreover,

$$\nabla (h_n(u_n) \varphi) \rightharpoonup \nabla (h(u) \varphi) \quad \text{in } L^p(\Omega).$$

Using the monotonicity of a and a standard Minty argument, we pass to the limit and obtain

$$\int_{\Omega} a(x, u, \nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} a(x, \hat{\mu}, \nabla \hat{\mu}) \cdot \nabla (h(u) \varphi) \, dx \quad \forall \varphi \in C_c^1(\Omega).$$

This is exactly (5.1), completing the proof. □

Remark 5.4. When $\int_0^\infty h(t) \, dt = \infty$, one may use a double truncation argument by introducing $\Phi_M(s) = \int_0^s \min\{h(t), M\} \, dt$ and passing to the limits successively. See [7] for details.

Remark 5.5. The transformation method used here is classical for handling singular nonlinearities and avoids the need for Lipschitz majorants (see [6, 10]).

6. EXTENSION TO UNBOUNDED DOMAINS

We now consider the case where Ω is unbounded. For clarity, we first present the result for bounded nonlinearities, then indicate how to handle logarithmic singularities.

Theorem 6.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an unbounded open set, $p > 2N/(N+2)$, $\mu \in W^{-1,p'}(\Omega)$ a positive measure (i.e., $\mu \geq 0$ and $\mu \neq 0$). Assume μ satisfies (2.2)–(2.4), (3.5), and (2.5) or (2.6). Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be nonincreasing, bounded, Lipschitz continuous, with $(sh(s))' \in L^\infty(\mathbb{R})$ and $h(s) \geq 0$ for all $s \leq 0$. Let $\hat{\mu}$ be defined as in (4.1). Then there exists $u \in W_0^{1,p}(\Omega)$, $u \geq 0$, solving*

$$\langle Au, v \rangle = \langle A\hat{\mu}, h(u)v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \quad (6.1)$$

Proof. For $\ell > 0$, let $B_\ell = \{x \in \mathbb{R}^N : |x| < \ell\}$ and define $\Omega_\ell = \Omega \cap B_\ell$. Define $\mu_\ell \in W^{-1,p'}(\Omega_\ell)$ by $\langle \mu_\ell, \phi \rangle = \langle \mu, \tilde{\phi} \rangle$ for $\phi \in W_0^{1,p}(\Omega_\ell)$, where $\tilde{\phi}$ is the extension by zero to Ω . This defines a bounded linear functional on $W_0^{1,p}(\Omega_\ell)$ with norm $\|\mu_\ell\| \leq \|\mu\|$.

Since Ω_ℓ is bounded, by Theorem 4.2 and 4.4 (applied with μ_ℓ), there exists $u_\ell \in W_0^{1,p}(\Omega_\ell)$, $u_\ell \geq 0$, solving

$$\langle Au_\ell, v \rangle = \langle A\hat{\mu}_\ell, h(u_\ell)v \rangle \quad \forall v \in W_0^{1,p}(\Omega_\ell) \cap L^\infty(\Omega_\ell), \quad (6.2)$$

where $\hat{\mu}_\ell \in W_0^{1,p}(\Omega_\ell)$ solves $\langle A\hat{\mu}_\ell, v \rangle = \langle \mu_\ell, v \rangle$ for all $v \in W_0^{1,p}(\Omega_\ell)$. We extend u_ℓ and $\hat{\mu}_\ell$ by zero outside Ω_ℓ .

Take $v = u_\ell \wedge k$ ($k > 0$) in (6.2)

$$\langle Au_\ell, u_\ell \wedge k \rangle = \langle A\hat{\mu}_\ell, h(u_\ell)(u_\ell \wedge k) \rangle.$$

Since h is nonincreasing and $\mu \geq 0$, we have

$$h(u_\ell)(u_\ell \wedge k) \leq h(u_\ell \wedge k)(u_\ell \wedge k).$$

Thus

$$\alpha \|\nabla(u_\ell \wedge k)\|_{L^p(\Omega)}^p \leq \langle A\hat{\mu}_\ell, h(u_\ell \wedge k)(u_\ell \wedge k) \rangle.$$

Let C_h be the Lipschitz constant of $s \mapsto sh(s)$. Then

$$|\nabla(h(u_\ell \wedge k)(u_\ell \wedge k))| \leq C_h |\nabla(u_\ell \wedge k)|.$$

Hence

$$\alpha \|\nabla(u_\ell \wedge k)\|_{L^p(\Omega)}^p \leq C_h \|A\hat{\mu}_\ell\|_{W^{-1,p'}(\Omega_\ell)} \|\nabla(u_\ell \wedge k)\|_{L^p(\Omega)},$$

where $\|A\hat{\mu}_\ell\|_{W^{-1,p'}(\Omega_\ell)} = \|\mu_\ell\| \leq \|\mu\|$. Therefore

$$\|\nabla(u_\ell \wedge k)\|_{L^p(\Omega)}^{p-1} \leq \frac{C_h}{\alpha} \|\mu\|.$$

Letting $k \rightarrow \infty$ gives

$$\|\nabla u_\ell\|_{L^p(\Omega)} \leq \left(\frac{C_h}{\alpha} \|\mu\| \right)^{\frac{1}{p-1}} =: C,$$

independent of ℓ .

Since $\{u_\ell\}$ is bounded in $W_0^{1,p}(\Omega)$, there exists a subsequence $\ell_m \rightarrow \infty$ and $u \in W_0^{1,p}(\Omega)$ such that $u_{\ell_m} \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u_{\ell_m} \rightarrow u$ in $L_{\text{loc}}^p(\Omega)$.

Fix $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support. For large m , $\text{supp } v \subset \Omega_{\ell_m}$. Test (6.2) with $(v - u_{\ell_m})_k$

$$\int_{\Omega} a(x, u_{\ell_m}, \nabla u_{\ell_m}) \cdot \nabla (v - u_{\ell_m})_k dx = \int_{\Omega} a(x, \hat{\mu}_{\ell_m}, \nabla \hat{\mu}_{\ell_m}) \cdot \nabla (h(u_{\ell_m})(v - u_{\ell_m})_k) dx.$$

By monotonicity

$$\int_{\Omega} a(x, v, \nabla v) \cdot \nabla (v - u_{\ell_m})_k dx \geq \int_{\Omega} a(x, \hat{\mu}_{\ell_m}, \nabla \hat{\mu}_{\ell_m}) \cdot \nabla (h(u_{\ell_m})(v - u_{\ell_m})_k) dx.$$

As $m \rightarrow \infty$, using arguments similar to the proof of Theorem 4.2 and noting that $\hat{\mu}_{\ell_m} \rightarrow \hat{\mu}$ in $W_0^{1,p}(\Omega)$ (since $\mu_{\ell_m} \rightarrow \mu$ in $W^{-1,p'}(\Omega)$ and the solution map is continuous), we obtain

$$\int_{\Omega} a(x, v, \nabla v) \cdot \nabla (v - u)_k dx \geq \int_{\Omega} a(x, \hat{\mu}, \nabla \hat{\mu}) \cdot \nabla (h(u)(v - u)_k) dx.$$

Letting $k \rightarrow \infty$ and then taking $v = u + tw$ for $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support, we derive (6.1) for such w . A density argument extends it to all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. $\square$

Remark 6.2. Theorem 6.1 extends to the logarithmic singularity $h(u) = 1/[u^\gamma \ln^\lambda(1+u)]$ by combining the truncation method of Theorem 5.1 with the domain approximation of Theorem 6.1.

7. APPLICATIONS AND EXAMPLES

7.1. The p -Laplacian with Logarithmic Singularity. A primary example covered by our theory is the p -Laplacian operator. Let $\Omega \subset \mathbb{R}^N$ be bounded, $p > 2N/(N+2)$, and $\mu \in W^{-1,p'}(\Omega)$ a nonnegative bounded measure. Consider

$$Au = -\text{div}(b(x)|\nabla u|^{p-2}\nabla u),$$

where $b \in L^\infty(\Omega)$ satisfies $0 < b_0 \leq b(x) \leq b_1$ almost everywhere. Define $a(x, t, \xi) = b(x)|\xi|^{p-2}\xi$. Then

- (1) Coercivity (2.2): $a(x, t, \xi) \cdot \xi = b(x)|\xi|^p \geq b_0|\xi|^p$.
- (2) Monotonicity (2.3): For $p \geq 2$, $(a(x, t, \xi) - a(x, t, \xi')) \cdot (\xi - \xi') \geq c_p|\xi - \xi'|^p$,
for $1 < p < 2$, $(a(x, t, \xi) - a(x, t, \xi')) \cdot (\xi - \xi') \geq c_p \frac{|\xi - \xi'|^2}{(|\xi| + |\xi'|)^{2-p}}$.
- (3) Growth (2.4): $|a(x, t, \xi)| = b(x)|\xi|^{p-1} \leq b_1|\xi|^{p-1}$.

Thus all assumptions of Theorem 5.1 are satisfied. Consequently, the problem

$$\begin{cases} u \in W_0^{1,p}(\Omega), \quad u \geq 0, \\ \int_{\Omega} b(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \left\langle \mu, \frac{v}{u^\gamma \ln^\lambda(1+u)} \right\rangle \quad \forall v \in C_c^1(\Omega), \end{cases}$$

admits at least one solution for any $\gamma, \lambda > 0$.

7.2. Operators with u -Dependence. More general examples include operators of the form

$$a(x, u, \nabla u) = (1 + |u|^m) |\nabla u|^{p-2} \nabla u,$$

with $m \geq 0$. Such operators satisfy our hypotheses provided $m \leq p - 1$ to ensure the growth condition (2.4). The dependence on u introduces nonlinearities that are not covered by previous works focusing on $a(x, \nabla u)$.

7.3. Anisotropic Operators. Another class of examples is given by anisotropic operators

$$a(x, u, \nabla u) = \sum_{i=1}^N |\partial_i u|^{p_i-2} \partial_i u e_i,$$

with $p_i > 1$. Under appropriate conditions on the p_i , our results can be adapted to such operators as well.

8. CONCLUSION

In this paper, we have established a comprehensive existence theory for non-linear elliptic problems with logarithmic singularities of the form

$$-\operatorname{div} (a(x, u, \nabla u)) = \frac{\mu}{u^\gamma \ln^\lambda(1+u)} \quad \text{in } \Omega,$$

with Dirichlet boundary conditions, where $\Omega \subseteq \mathbb{R}^N$ may be bounded or unbounded, μ is a nonnegative measure in $W^{-1,p'}(\Omega)$, and $\gamma, \lambda > 0$.

Our main results are as follows

- (1) For bounded Lipschitz nonlinearities, we proved existence via Schauder's fixed-point theorem and monotone operator methods (Theorem 4.2).
- (2) For the logarithmic singularity on bounded domains, we constructed solutions through a truncation argument and uniform $W^{1,p}$ estimates (Theorem 5.1).
- (3) For unbounded domains, we used an exhaustion procedure and weak compactness to extend the existence result (Theorem 6.1).

The principal novelty of our work lies in simultaneously handling logarithmic singularities, u -dependent diffusion operators $a(x, u, \nabla u)$, general measure data, and both bounded and unbounded domains. This extends previous works where a was independent of u , singularities were purely algebraic, or domains were restricted to bounded sets. Our framework applies to important examples such as the p -Laplacian with logarithmic source terms, and provides a solid mathematical foundation for further studies of singular elliptic problems arising in physical and applied contexts.

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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