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REGULARITY OF ENTROPY SOLUTIONS FOR THE 1D RELATIVISTIC HEAT EQUATION

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Abstract. This paper investigates the one-dimensional Cauchy problem for the relativistic heat equation with initial data $u_0 \in W^{1,1}(\mathbb{R}) \cap BV(\mathbb{R})$. We prove that the associated entropy solution preserves bounded variation for all positive times, that is, $u(t) \in BV(\mathbb{R})$ for every $t > 0$, and propagates with finite speed, so that its support remains confined within an interval of the form $[a - ct, b + ct]$. Furthermore, we establish that the time derivative u_t is a Radon measure on $(0, T) \times \mathbb{R}$, providing a precise characterization of the temporal regularity of entropy solutions. These results contribute to a deeper understanding of the fine regularity properties of solutions to flux-limited diffusion equations.

Keywords. Entropy solutions, bounded variation, finite speed of propagation, Radon measures.
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1. Introduction

The modeling of diffusion phenomena in physical media has traditionally relied on the classical heat equation, which implicitly assumes an infinite speed of propagation. Such an assumption is not compatible with the principles of special relativity, according to which the speed of light represents a universal bound for the transmission of information. This limitation has motivated the development of alternative diffusion models incorporating a finite propagation speed. In this context, Rosenau [30] introduced a modified diffusion framework involving flux limitation, which was later revisited and extended by Brenier [13] using tools from optimal transport theory. These approaches gave rise to what is now known as the Relativistic Heat Equation (RHE), a nonlinear diffusion model characterized by a saturation mechanism on the flux, ensuring that the propagation speed remains bounded by a finite constant c. Related flux-limited diffusion models have been extensively studied in the literature (see for instance [14, 16, 27]). In this paper, we investigate the Cauchy problem associated with the one-dimensional

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Relativistic Heat Equation:

$$\begin{cases} u_t = (F(u, u_x))_x & (t, x) \in (0, T) \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{P}$$

where F denotes the nonlinear flux defined by

$$F(u, u_x) = \alpha \frac{u\, u_x}{\sqrt{u^2 + \left(\frac{\alpha}{c}\right)^2 |u_x|^2}},$$

with $\alpha > 0$ a diffusion parameter and $c > 0$ a characteristic maximal propagation speed. Throughout this work, the initial datum is assumed to satisfy

$$u_0 \in W^{1,1}(\mathbb{R}) \cap BV(\mathbb{R}).$$

Equation $(\mathrm{P})_1$ belongs to the class of nonlinear degenerate parabolic equations with saturated flux, for which the finite speed of propagation plays a central role. Such equations naturally arise in various applications, including radiation hydrodynamics [24], relativistic diffusion processes [11], and optimal transport theory [19]. The mathematical analysis of such nonlinear diffusion equations has been widely developed in the framework of entropy solutions. In particular, Carrillo, Caselles, and Moll [15] established the existence and uniqueness of entropy solutions for the Cauchy problem associated with the RHE, together with qualitative properties such as comparison principles and finite propagation speed. These results rely on the general theory of entropy solutions for degenerate parabolic equations developed in [9] and further extended in [3, 4, 5, 6]. More generally, nonlinear evolution equations with possibly irregular solutions have been studied using a combination of compactness methods, monotonicity arguments, and nonlinear semigroup theory (see [10, 18, 17]). Additional contributions on stability and qualitative behavior can be found in [29, 28, 22], while related nonlinear problems in generalized functional frameworks have been investigated in [20, 21]. We also refer to [2, 23] for recent developments concerning degenerate parabolic equations with boundary conditions. In a previous work (see [7]), the Dirichlet problem associated with the Relativistic Heat Equation on bounded domains was studied in detail. That work was devoted to the entropy formulation of the problem and established the equivalence between two entropy frameworks, namely the formulation introduced by Bénilan–Touré [8] and the one developed by Mazón–Caselles–Moll [25]. Within this setting, existence and uniqueness of entropy solutions under Dirichlet boundary conditions were obtained. The present work extends this analysis to the Cauchy problem posed on the whole real line. The main objective is to investigate the regularity properties of entropy solutions, with a particular focus on bounded variation (BV) estimates and the measure-valued structure of the time derivative. The methods developed in the Dirichlet framework, especially those based on entropy inequalities and compactness arguments, play a fundamental role in the analysis carried out in this paper. Throughout this work, we assume that the initial datum u_0 satisfies

$$u_0 \in W^{1,1}([a, b]) \cap BV([a, b]),\ u_0(x) \geq \beta > 0 \text{ on } [a, b],\ u_0(x) = 0 \text{ for } x \in \mathbb{R} \setminus [a, b].$$

The paper is organized as follows. In Section 2, we introduce the necessary preliminaries, including the functional framework and auxiliary results. The main regularity results for entropy solutions of (P) are presented in Section 3.

2. Preliminaries

2.1. **Functional spaces and notations.** We denote by $\mathbb{R}$ the real line and by $\mathbb{R}^+ := [0, +\infty)$. For $1 \leq p \leq \infty$, $L^p(\mathbb{R})$ denotes the Lebesgue space endowed with the norm

$$\|u\|_{L^p(\mathbb{R})} := \begin{cases} \left(\displaystyle\int_{\mathbb{R}} |u(x)|^p \, dx \right)^{1/p}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x \in \mathbb{R}} |u(x)|, & p = \infty. \end{cases}$$

We also use the space $L^1_{\text{loc}}(\mathbb{R})$, consisting of measurable functions integrable on every compact subset of $\mathbb{R}$. Given a Banach space X, we denote by $C([0,T]; X)$ the space of continuous functions from $[0,T]$ into X, endowed with the norm

$$\|u\|_{C([0,T];X)} := \sup_{t \in [0,T]} \|u(t)\|_X.$$

In particular, entropy solutions of (P) belong to

$$C([0,T]; L^1(\mathbb{R})).$$

The space of functions of bounded variation on $\mathbb{R}$, denoted by $BV(\mathbb{R})$, is defined as

$$BV(\mathbb{R}) := \left\{ u \in L^1(\mathbb{R}) : |Du|(\mathbb{R}) < +\infty \right\},$$

where Du is the distributional derivative of u, understood as a finite Radon measure. The total variation is

$$|Du|(\mathbb{R}) := \sup \left\{ \int_{\mathbb{R}} u(x) \varphi'(x) \, dx : \varphi \in C^1_c(\mathbb{R}), \ \|\varphi\|_{L^\infty} \leq 1 \right\}.$$

The norm in $BV(\mathbb{R})$ is

$$\|u\|_{BV(\mathbb{R})} := \|u\|_{L^1(\mathbb{R})} + |Du|(\mathbb{R}).$$

For $u \in BV(\mathbb{R})$, the Radon measure Du admits the decomposition

$$Du = D^a u + D^s u,$$

where $D^a u = u_x \mathcal{L}^1$ is the absolutely continuous part and $D^s u$ the singular part. Moreover,

$$D^s u = D^c u + D^j u,$$

where $D^c u$ is the Cantor part and $D^j u$ the jump part concentrated on the jump set J_u. For $x \in J_u$, the approximate limits $u^-(x)$ and $u^+(x)$ exist and

$$[u](x) := u^+(x) - u^-(x).$$

In one dimension,

$$D^j u = [u] \, \nu_u \, \mathcal{H}^0 \llcorner J_u.$$

We define truncations

$$\tau_r := \{\mathcal{T}^{[a,b]} : 0 < a < b\}, \qquad \tau^+ := \{\mathcal{T}^{[a,b]} : 0 < a < b, \ \mathcal{T}^{[a,b]} \geq 0\},$$

with

$$\mathcal{T}^{[a,b]}(r) := \begin{cases} a & r < a, \\ r & a \le r \le b, \\ b & r > b, \end{cases} \qquad \mathcal{T}^k := \mathcal{T}^{[-k,k]}.$$

We consider the space

$$TBV^+(\mathbb{R}^+) := \left\{ u \in L^1(\mathbb{R}^+) \ : \ \mathcal{T}^{[a,b]}(u) \in BV(\mathbb{R}) \right\}.$$

It follows from the chain rule in BV (see [1]) that there exists a measurable function $v : \mathbb{R} \to \mathbb{R}$ satisfying

$$\mathcal{T}_x^{[a,b]}(u) = v\,\chi_{\{a<u<b\}} \quad \text{a.e. in } \mathbb{R}.$$

We now define the notion of entropy solution for (P).

Definition 2.1. Assume that $u_0 \in (L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))_+$. A measurable function u is an entropy solution of (P) in $(0,T) \times \mathbb{R}$ if $u \in C([0,T]; L^1(\mathbb{R}))$, and for all $0 < a < b$, $\mathcal{T}^{[a,b]}(u(\cdot)) - a \in L^1_{\rm loc}(0,T; BV(\mathbb{R}))$, and the entropy inequality holds:

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \phi\, h_S(u, D\mathcal{T}(u)) + \int_0^T \int_{\mathbb{R}} \phi\, h_T(u, DS(u)) \\ &\le \int_0^T \int_{\mathbb{R}} \Big(J_{\mathcal{T}S}(u(t))\phi'(t) - F(u,u_x) \cdot \phi_x \mathcal{T}(u) S(u) \Big), \end{aligned}$$

for all truncations $S, \mathcal{T} \in \tau^+$ and all test functions $\phi \in \mathcal{D}((0,T) \times \mathbb{R})$.

2.2. **Generalized Green's formula.** We recall some standard results concerning the generalized Green formula and normal traces, which can be found in [9, 24]. These results provide a rigorous framework to interpret the flux integral appearing in the Cauchy problem. Let

$$\Omega(t) = [a - ct,\, b + ct] \subset \mathbb{R},$$

where $a < b$, $c > 0$, and $t > 0$. We consider the space

$$E^p(\Omega(t)) := \{ v \in L^\infty(\Omega(t)) \ : \ v_x \in L^p(\Omega(t)),\ p \ge 1 \}.$$

For $v \in E^p(\Omega(t))$ and $w \in BV(\Omega(t)) \cap L^p(\Omega(t))$, one can define the pairing (v, Dw) in the sense of distributions (see [9]). This defines a Radon measure on $\Omega(t)$ and extends the classical product $v\,w_x$ when w is smooth. Moreover, it is shown in [24] that the normal trace operator

$$\mathcal{L} : E^p(\Omega(t)) \to L^\infty(\partial\Omega(t))$$

is well defined and satisfies

$$\|\mathcal{L}(v)\|_{L^\infty} \le \|v\|_{L^\infty}.$$

The corresponding Green formula reads

$$\int_{\Omega(t)} v\, w_x\, dx + \int_{\Omega(t)} (v, Dw) = \int_{\partial\Omega(t)} \mathcal{L}(v)\, w\, d\mathcal{H}^0. \tag{2.1}$$

2.3. **Properties of functionals defined on BV.** Let $l : \Omega(t) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ be a Borel function such that

$$I(x)|z| - J(x) \leq l(x, y, z) \leq K(x) + M|z|,$$

for $(x, y, z) \in \Omega(t) \times \mathbb{R} \times \mathbb{R}$, with $|y| \leq R$, $M \geq 0$, and $I, J, K \in L^1(\Omega(t))$ nonnegative. For $u \in BV(\Omega(t)) \cap L^\infty(\Omega(t))$ define

$$\mu_l(u) = \int_{\Omega(t)} l(x, u, u_x)\, dx + \int_{\Omega(t)} l^0\left(x, \tilde{u}, \frac{Du}{|Du|}\right) |D^c u| + \int_{J_u} \left(\int_{u^-}^{u^+} l^0(x, y, \nu_u)\, dy \right) d\mathcal{H}^0, \tag{2.2}$$

where the recession function is

$$l^0(x, y, z) = \lim_{t \to 0^+} t\, l\left(x, y, \frac{z}{t}\right).$$

Under the assumptions of [1], $\mu_l(u)$ is sequentially lower semicontinuous in $L^1(\Omega(t))$. If l does not depend on $\Omega(t)$, then

$$I|z| - J \leq l(y, z) \leq K(1 + |z|).$$

For $u \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the Radon measure $\langle l(u, Du) \rangle$ is defined by

$$\langle l(u, Du), \varphi \rangle := \mu_{l\varphi}(u), \quad \varphi \in C_c^+(\mathbb{R}).$$

We introduce the flux function

$$h(u, u_x) = F(u, u_x) \cdot u_x = \alpha \frac{u|u_x|^2}{\sqrt{u^2 + \left(\frac{\alpha}{c}\right)^2 |u_x|^2}},$$

which plays a central role in the entropy inequality. For $T, S \in \tau^+$, the Radon measures $h_S(u, DT(u))$ and $h_T(u, DS(u))$ are defined accordingly and satisfy the properties described in [1].
Entropy solutions to (P) are well-posed in $C([0, T]; L^1(\mathbb{R}))$ for $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ [15], and they satisfy finite speed of propagation, mass conservation, and preservation of BV regularity [26].

3. Main Regularity Result

The following result summarizes the principal regularity features of entropy solutions associated with problem (P).

Theorem 3.1. *Let $u_0 \in W^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $u_0(x) \geq \beta > 0$ for $x \in (a, b)$ and $u_0(x) = 0$ for $x \notin (a, b)$. Let u denote the entropy solution of (P). Then $u \in C([0, T]; L^1(\mathbb{R}))$ and the following properties hold:*

(i) *There exists a function $\beta : (0, T) \to \mathbb{R}^+ \setminus \{0\}$ such that*

$$u(t, x) \geq \beta(t) > 0 \quad \text{for } x \in (a - ct,\, b + ct),$$

and

$$u(t, x) = 0 \quad \text{for } x \notin [a - ct,\, b + ct], \quad t \in (0, T).$$

(ii) *For almost every* $t \in (0,T)$*, one has*

$$u(t) \in BV(\mathbb{R}) \quad \textit{and} \quad u(t) \in W^{1,1}(a-ct,\, b+ct),$$

and, in addition, $u(t)$ *is smooth in the interior of its support.*

(iii) *If, moreover,* $u_0 \in W^{2,1}(a,b)$*, then the time derivative* u_t *is a Radon measure on* $(0,T)\times\mathbb{R}$*.*

Property (i) follows from the results established in [25], whereas property (ii) is a direct consequence of the next statement.

Proposition 3.2. *Let* u *be the entropy solution of* $(\mathrm{P})_1$*. Then* $u \in \mathcal{C}([0,T];L^1(\mathbb{R}))$ *and the following assertions hold:*

(j) *For every* $t \in (0,T)$*,*

$$u(t) \in BV(\mathbb{R}) \cap W^{1,1}_{\mathrm{loc}}(a-t, b+t),$$

and $u(t)$ *enjoys additional regularity inside its support;*

(jj) *The equation is satisfied in the sense of distributions,*

$$u_t = \partial_x F \quad \textit{in } \mathcal{D}'((0,T)\times\mathbb{R}),$$

where

$$F(t) = \frac{u(t)\, u_x(t)}{\sqrt{u(t)^2 + u_x(t)^2}}.$$

The proof of Proposition 3.2 can be deduced from Proposition 2.5 in [15]. The regularity result stated in point (iii) of Theorem 3.1 is established in the following proposition.

Proposition 3.3. *Let* $u_0 \in W^{1,1}(\mathbb{R}) \cap BV(\mathbb{R})$ *such that*

$$u_0(x) \geq \beta > 0 \quad \textit{for all } x \in \Omega_0, \qquad u_0(x) = 0 \quad \textit{for } x \in \mathbb{R}\setminus\Omega_0.$$

Let u *be the entropy solution of* (P)*. If, in addition,* $u_0 \in W^{2,1}(\Omega_0)$ *and* $u_{0,x} \in L^\infty(\Omega_0)$*, then*

$$\partial_t u \in \mathcal{M}((0,T)\times\mathbb{R}).$$

The proof of Proposition 3.3 is carried out through several steps, which will be developed in the subsequent subsections.

3.1. **Part I: Truncated elliptic problem.**

Lemma 3.4. *Let* $R>0$*,* $\varepsilon>0$*, and* $\lambda>0$*. We consider the truncated elliptic problem given by*

$$\begin{cases} u + \lambda A^{R,\varepsilon}u - \varepsilon u_{xx} = f, & \textit{in } \Omega_R = (-R,R), \\ F(u)(\pm R) = 0, \end{cases} \qquad (P_{R,\varepsilon})$$

where $u \in V_R := H^1(\Omega_R)$*,* $A^{R,\varepsilon}u := -F^{R,\varepsilon}_x(u)$ *and* $f \in L^2(\Omega_R)$*. If the operator*

$$B^{R,\varepsilon}u := u + \lambda A^{R,\varepsilon}u - \varepsilon u_{xx}$$

is bounded, hemicontinuous, strictly monotone, and coercive, then problem $(P_{R,\varepsilon})$ *admits a unique entropy solution* $u \in V_R$*.*

Proof. Indeed, from $(P_{R,\varepsilon})_1$ we define the domain of the operator $A^{R,\varepsilon}$ by

$$D(A^{R,\varepsilon}) = \Big\{u \in V_R : F^{R,\varepsilon}(u) \in L^2(\Omega_R)\Big\}.$$

We have

$$\int_{\Omega_R} A^{R,\varepsilon} u \, dx = -\int_{\Omega_R} F_x^{R,\varepsilon}(u)(x)\, dx = -\big[F^{R,\varepsilon}(u)(x)\big]_{-R}^{R} = 0.$$

Hence,

$$A^{R,\varepsilon} u \in L^1(\Omega_R).$$

For a fixed $a > 0$, the function $b \mapsto F^{R,\varepsilon}(a, b)$ is nondecreasing, that is,

$$\langle F^{R,\varepsilon}(a, b) - F^{R,\varepsilon}(a, c), b - c\rangle \geq 0, \quad \forall\, b, c \in \mathbb{R}.$$

Let S_ε be a smooth convex approximation of the absolute value, defined by

$$S_\varepsilon(r) = \sqrt{r^2 + \varepsilon^2}, \quad \text{so that} \quad S_\varepsilon'(r) = \frac{r}{\sqrt{r^2+\varepsilon^2}} = \operatorname{sign}_\varepsilon(r),$$

and

$$S_\varepsilon''(r) = \frac{(r^2+\varepsilon^2)^{1/2} - r^2(r^2+\varepsilon^2)^{-1/2}}{(r^2+\varepsilon^2)} = \frac{\varepsilon^2}{(r^2+\varepsilon^2)^{3/2}} \geq 0.$$

Let $u, v \in D(A^{R,\varepsilon})$. Then

$$\int_{\Omega_R} S_\varepsilon'(u-v)\big(A^{R,\varepsilon}u - A^{R,\varepsilon}v\big)\, dx = \int_{\Omega_R} S_\varepsilon''(u-v)(u-v)_x\big(F^{R,\varepsilon}(u) - F^{R,\varepsilon}(v)\big)\, dx,$$

where $S_\varepsilon' = \operatorname{sign}_\varepsilon$. We decompose $F^{R,\varepsilon}(u) - F^{R,\varepsilon}(v)$ as follows:

$$F^{R,\varepsilon}(u) - F^{R,\varepsilon}(v) = \big(F^{R,\varepsilon}(u, u_x) - F^{R,\varepsilon}(u, v_x)\big) + \big(F^{R,\varepsilon}(u, v_x) - F^{R,\varepsilon}(v, v_x)\big).$$

Let

$$F_1 := F^{R,\varepsilon}(u, u_x) - F^{R,\varepsilon}(u, v_x), \qquad F_2 := F^{R,\varepsilon}(u, v_x) - F^{R,\varepsilon}(v, v_x).$$

The term F_1 is monotone with respect to u_x, and therefore

$$\langle F_1, u_x - v_x\rangle \geq 0. \tag{3.1}$$

Moreover, F_2 is Lipschitz continuous with respect to u, and satisfies

$$|F_2| \leq |u - v| \quad \Rightarrow \quad F_2 \geq -|u-v|,$$

which yields

$$|(u-v)_x|\,|F_2| \geq -|u-v|\,|(u-v)_x|. \tag{3.2}$$

By adding (3.1) and (3.2), we obtain

$$\langle F_1 + F_2, u_x - v_x\rangle \geq -|u-v|\,|(u-v)_x|.$$

Consequently,

$$\int_{\mathbb{R}} \operatorname{sign}_\varepsilon(u-v)\big(A^{R,\varepsilon}u - A^{R,\varepsilon}v\big)\, dx \geq -\int_{\mathbb{R}} S_\varepsilon''(u-v)\,|u-v|\,|(u-v)_x|\, dx.$$

By convexity and the fact that $|u-v| \leq \varepsilon$, we obtain

$$\underbrace{\int_{\mathbb{R}} S_\varepsilon''(u-v)\,|u-v|\,|(u-v)_x|\, dx}_{I} \leq \varepsilon \int_{\{|u-v|\leq C(\varepsilon)\}} S_\varepsilon''(u-v)\,|(u-v)_x|\, dx \leq \varepsilon C.$$

Hence,

$$I \geq -\varepsilon C \quad \Longrightarrow \quad I \geq 0 \quad \text{(after passing to the limit as } \varepsilon \to 0).$$

Therefore,

$$\int_{\mathbb{R}} \operatorname{sign}(u-v)\big(A^{R,\varepsilon}u - A^{R,\varepsilon}v\big)\,dx \geq 0, \quad \text{since } S'_\varepsilon \to \operatorname{sign}. \tag{3.3}$$

Inequality (3.3) is the Kato inequality, which is equivalent to

$$\|u-v\|_{L^1} \leq \|u-v+\lambda(A^{R,\varepsilon}u - A^{R,\varepsilon}v)\|_{L^1}, \quad \forall \lambda > 0.$$

Therefore, $A^{R,\varepsilon}$ is accretive in L^1. The weak formulation of ($P_{R,\varepsilon}$) is given by

$$\langle B^{R,\varepsilon}u, \varphi\rangle = \int_{\Omega_R} \Big(u\varphi + \lambda F^{R,\varepsilon}(u)\,\varphi_x + \varepsilon u_x \varphi_x\Big)\,dx = \int_{\Omega_R} f\varphi\,dx, \qquad \forall \varphi \in V_R.$$

Step 1: Continuity of $F^{R,\varepsilon}$ and boundedness of $B^{R,\varepsilon}u$. We have

$$F^{R,\varepsilon}(a,b) = \begin{cases} \dfrac{ab}{\sqrt{a^2+b^2}} & \text{if } (a,b) \in (\mathbb{R}^*)^2, \\ 0 & \text{if } (a,b) = (0,0). \end{cases}$$

This function is of class C^∞ on $(\mathbb{R}^*)^2$ and continuous on $\mathbb{R}^2$. Moreover,

$$\big|F^{R,\varepsilon}(a,b)\big| \leq \frac{|a||b|}{\sqrt{a^2+b^2}} \leq |a|.$$

We have

$$\begin{aligned} \|B^{R,\varepsilon}u\|_{V'_R} &= \sup_{\|\varphi\|_{V_R}=1} \big|\langle B^{R,\varepsilon}u, \varphi\rangle\big| \\ &\leq \int_{\Omega_R} |u|\,|\varphi|\,dx + \lambda \int_{\Omega_R} |F^{R,\varepsilon}(u)|\,|\varphi_x|\,dx + \varepsilon \int_{\Omega_R} |u_x|\,|\varphi_x|\,dx. \end{aligned} \tag{3.4}$$

By the Cauchy–Schwarz inequality,

$$\int_{\Omega_R} |u|\,|\varphi|\,dx \leq \|u\|_{L^2}\|\varphi\|_{L^2} \leq \|u\|_{L^2}\|\varphi\|_{H^1}, \tag{3.5}$$

since $\|\varphi\|_{L^2} \leq \|\varphi\|_{H^1}$. For the relativistic flux term, we have

$$\begin{aligned} \left|\int_{\Omega_R} F^{R,\varepsilon}(u)\,\varphi_x\,dx\right| &\leq \int_{\Omega_R} |F^{R,\varepsilon}(u)|\,|\varphi_x|\,dx \\ &\leq \int_{\Omega_R} |u|\,|\varphi_x|\,dx \leq \|u\|_{L^2}\|\varphi_x\|_{L^2} \\ &\leq (\|u\|_{L^2} + \|u_x\|_{L^2})(\|\varphi\|_{L^2} + \|\varphi_x\|_{L^2}) = \|u\|_{H^1}\|\varphi\|_{H^1}. \end{aligned} \tag{3.6}$$

For the viscosity term, we similarly obtain:

$$\begin{aligned} \left|\int_{\Omega_R} u_x\varphi_x\,dx\right| &\leq \|u_x\|_{L^2}\|\varphi_x\|_{L^2} \\ &\leq \|u\|_{H^1}\|\varphi\|_{H^1}. \end{aligned} \tag{3.7}$$

By inserting (3.5), (3.6), and (3.7) into (3.4), we obtain

$$\|B^{R,\varepsilon}u\|_{V_R'} \;\leq\; (1+\lambda+\varepsilon)\,\|u\|_{H^1}\,\|\varphi\|_{H^1}. \tag{3.8}$$

Since φ is arbitrary, we choose $\|\varphi\|_{H^1} \equiv 1$, and therefore

$$\|B^{R,\varepsilon}u\|_{V_R'} \leq C\,\|u\|_{H^1}, \quad \text{where } C = C(\lambda,\varepsilon) = 1+\lambda+\varepsilon.$$

Hence, $B^{R,\varepsilon}u$ is bounded on V_R.

Step 2: Hemicontinuity of $B^{R,\varepsilon}$. In this step, we show that for all $u, v, w \in V_R$, the function H defined by

$$H(\theta) = \langle B^{R,\varepsilon}(u+\theta v), w\rangle$$

is continuous on $\mathbb{R}$. Indeed, for $\theta, \theta_0 \in \mathbb{R}$, we have

$$\begin{aligned}
H(\theta) - H(\theta_0) &= \langle B^{R,\varepsilon}(u+\theta v), w\rangle - \langle B^{R,\varepsilon}(u+\theta_0 v), w\rangle \\
&= \int_{\Omega_R} \big((u+\theta v)w - (u+\theta_0 v)w\big)\,dx \\
&\quad + \lambda \int_{\Omega_R} \Big(F^{R,\varepsilon}(u+\theta v) - F^{R,\varepsilon}(u+\theta_0 v)\Big) w_x\,dx \\
&\quad + \varepsilon \int_{\Omega_R} \big((u+\theta v)_x - (u+\theta_0 v)_x\big) w_x\,dx \\
&= (\theta-\theta_0)\int_{\Omega_R} vw\,dx + \lambda \int_{\Omega_R} \Big(F^{R,\varepsilon}(u+\theta v) - F^{R,\varepsilon}(u+\theta_0 v)\Big) w_x\,dx \\
&\quad + \varepsilon(\theta-\theta_0)\int_{\Omega_R} v_x w_x\,dx =: I_1 + I_2 + I_3.
\end{aligned} \tag{3.9}$$

Consequently,

$$|H(\theta) - H(\theta_0)| \leq |I_1| + |I_2| + |I_3|. \tag{3.10}$$

– For I_1, we have

$$|I_1| = \left|(\theta-\theta_0)\int_{\Omega_R} vw\,dx\right| \leq |\theta-\theta_0| \int_{\Omega_R} |vw|\,dx \leq |\theta-\theta_0| \|v\|_{L^2}\|w\|_{L^2}.$$

– The same estimate holds for I_3:

$$|I_3| \leq \varepsilon|\theta-\theta_0|\,\|v_x\|_{L^2}\|w_x\|_{L^2}.$$

– For I_2, we use the global Lipschitz continuity:

$$\big|F^{R,\varepsilon}(u+\theta v) - F^{R,\varepsilon}(u+\theta_0 v)\big| \leq |\theta-\theta_0|\,(|v| + |v_x|).$$

Therefore,

$$|I_2| \leq \lambda|\theta-\theta_0| \left(\int_{\Omega_R} |v|\,|w_x|\,dx + \int_{\Omega_R} |v_x|\,|w_x|\,dx\right).$$

Hence,

$$|I_2| \leq \lambda|\theta-\theta_0|\Big(\|v\|_{L^2}\|w_x\|_{L^2} + \|v_x\|_{L^2}\|w_x\|_{L^2}\Big).$$

By inserting the estimates for $|I_1|$, $|I_2|$, and $|I_3|$ into (3.10), we obtain

$$\begin{aligned}|H(\theta)-H(\theta_0)| &\leq |\theta-\theta_0|\Big(\|v\|_{L^2}\|w\|_{L^2}+\lambda(\|v\|_{L^2}+\|v_x\|_{L^2})\|w_x\|_{L^2}+\varepsilon\|v_x\|_{L^2}\|w_x\|_{L^2}\Big)\\ &\leq |\theta-\theta_0|\underbrace{\Big[(\lambda+\varepsilon)\|v_x\|_{L^2}\|w_x\|_{L^2}+(\|w\|_{L^2}+\lambda\|w_x\|_{L^2})\|v\|_{L^2}\Big]}_{=:C}.\end{aligned} \tag{3.11}$$

Consequently, (3.11) can be rewritten as

$$|H(\theta)-H(\theta_0)| \leq C\,|\theta-\theta_0|, \qquad C \geq 0. \tag{3.12}$$

Hence, by (3.12), the function H is Lipschitz continuous and therefore continuous on $\mathbb{R}$. As a consequence, $B^{R,\varepsilon}$ is hemicontinuous.

Step 3: Monotonicity of $B^{R,\varepsilon}$. Let $u, v \in V_R$. We have

$$\begin{aligned}\langle B^{R,\varepsilon}u - B^{R,\varepsilon}v, u-v\rangle &= \int_{\Omega_R}(u-v)^2\,dx + \lambda\int_{\Omega_R}\big(F^{R,\varepsilon}(u)-F^{R,\varepsilon}(v)\big)(u-v)_x\,dx\\ &\quad+\varepsilon\int_{\Omega_R}|(u-v)_x|^2\,dx.\end{aligned}$$

The first and the third terms are nonnegative. For the flux term, we decompose it as follows:

$$F^{R,\varepsilon}(u)-F^{R,\varepsilon}(v) = \underbrace{F(u,u_x)-F(u,v_x)}_{\text{difference in } u_x} + \underbrace{F(u,v_x)-F(v,v_x)}_{\text{difference in } u}.$$

Let

$$F(a,b)=\frac{ab}{r}, \qquad r=\sqrt{a^2+b^2}, \quad a\geq 0,\ b\in\mathbb{R}.$$

Then

$$\partial_b F(a,b) = \frac{a}{r}-\frac{ab}{r^2}\frac{b}{r} = \frac{a}{r}-\frac{ab^2}{r^3} = \frac{a^3}{r^3}.$$

Since

$$a^3 \leq (a^2+b^2)^{3/2} = r^3,$$

we obtain

$$0 \leq \partial_b F(a,b) \leq 1.$$

Therefore, for $a=u$, $b=u_x$, and $c=v_x$, we have

$$(F(u,u_x)-F(u,v_x))(u_x-v_x) = \left(\int_0^1 \partial_{u_x}F\big(u, v_x+\theta(u_x-v_x)\big)\,d\theta\right)(u_x-v_x)^2 \geq \text{ a.e.}$$

Thus,

$$\int_{\Omega_R}\big(F(u,u_x)-F(u,v_x)\big)(u_x-v_x)\,dx \geq 0. \tag{3.13}$$

Similarly,

$$\partial_a F(a,b) = \frac{b^3}{r^3}, \qquad r=\sqrt{a^2+b^2}.$$

Since $|b|^3 \leq (a^2+b^2)^{3/2} = r^3$, we obtain

$$|\partial_a F(a,b)| \leq 1.$$

Therefore,

$$|F(u,v_x)-F(v,v_x)| \le \sup_\theta |\partial_a F(\theta,v_x)|\,|u-v| \le |u-v|.$$

Hence,

$$\big|(F(u,v_x)-F(v,v_x))(u_x-v_x)\big| \le |u-v|\,|u_x-v_x|.$$

By Young's inequality, for $\alpha>0$, we have

$$|u-v|\,|u_x-v_x| \le \frac{\alpha}{2}|u_x-v_x|^2+\frac{1}{2\alpha}|u-v|^2.$$

Consequently,

$$(F(u,v_x)-F(v,v_x))(u_x-v_x) \ge -\frac{\alpha}{2}|u_x-v_x|^2-\frac{1}{2\alpha}|u-v|^2.$$

Integrating over Ω_R, we obtain

$$\int_{\Omega_R}(F(u,v_x)-F(v,v_x))(u_x-v_x)\,dx \ge -\frac{\alpha}{2}\int_{\Omega_R}|u_x-v_x|^2\,dx-\frac{1}{2\alpha}\int_{\Omega_R}|u-v|^2\,dx. \tag{3.14}$$

By summing (3.13) and (3.14), we obtain

$$\int_{\Omega_R}\big(F^{R,\varepsilon}(u)-F^{R,\varepsilon}(v)\big)(u-v)_x\,dx \ge -\frac{\alpha}{2}\int_{\Omega_R}|u_x-v_x|^2\,dx-\frac{1}{2\alpha}\int_{\Omega_R}|u-v|^2\,dx.$$

Therefore,

$$\begin{aligned}\langle B^{R,\varepsilon}u-B^{R,\varepsilon}v,u-v\rangle &\ge \int_{\Omega_R}|u-v|^2\,dx+\varepsilon\int_{\Omega_R}|(u-v)_x|^2\,dx\\ &\qquad -\frac{\alpha\lambda}{2}\int_{\Omega_R}|(u-v)_x|^2\,dx-\frac{\lambda}{2\alpha}\int_{\Omega_R}|u-v|^2\,dx\\ &=\Big(1-\frac{\lambda}{2\alpha}\Big)\|u-v\|_{L^2}^2+\Big(\varepsilon-\frac{\alpha\lambda}{2}\Big)\|(u-v)_x\|_{L^2}^2. \end{aligned}\tag{3.15}$$

Choosing $\alpha=\varepsilon>0$, we obtain

$$\langle B^{R,\varepsilon}u-B^{R,\varepsilon}v,u-v\rangle \ge \Big(1-\frac{\lambda}{2\varepsilon}\Big)\|u-v\|_{L^2}^2+\varepsilon\Big(1-\frac{\lambda}{2}\Big)\|(u-v)_x\|_{L^2}^2.$$

For $\varepsilon>\frac{\lambda}{2}$, the right-hand side is strictly positive whenever $u-v\not\equiv 0$. Hence, $B^{R,\varepsilon}$ is strictly monotone.

Step 4: Coercivity of $B^{R,\varepsilon}$. We have

$$\langle B^{R,\varepsilon}u,u\rangle=\int_{\Omega_R}u^2\,dx+\lambda\int_{\Omega_R}F^{R,\varepsilon}(u)\,u_x\,dx+\varepsilon\int_{\Omega_R}u_x^2\,dx.$$

Since $|F^{R,\varepsilon}(u)|\le|u|$, we have $F^{R,\varepsilon}(u)\,u_x\ge-|u|\,|u_x|$. Thus,

$$\langle B^{R,\varepsilon}u,u\rangle \ge \int_{\Omega_R}u^2\,dx-\lambda\int_{\Omega_R}|u|\,|u_x|\,dx+\varepsilon\int_{\Omega_R}u_x^2\,dx.$$

Similarly, applying Young's inequality for any $\alpha > 0$, we obtain

$$\begin{aligned}
\langle B^{R,\varepsilon} u, u\rangle &\geq \int_{\Omega_R} |u|^2\,dx + \varepsilon \int_{\Omega_R} |u_x|^2\,dx \\
&\quad - \frac{\alpha\lambda}{2}\int_{\Omega_R} |u_x|^2\,dx - \frac{\lambda}{2\alpha}\int_{\Omega_R} |u|^2\,dx \\
&= \left(1-\frac{\lambda}{2\alpha}\right)\|u\|_{L^2}^2 + \left(\varepsilon - \frac{\lambda\alpha}{2}\right)\|u_x\|_{L^2}^2 \\
&\geq \min\left\{1-\frac{\lambda}{2\alpha}, \varepsilon - \frac{\lambda\alpha}{2}\right\}\left(\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2\right) = C'\|u\|_{H^1}^2,
\end{aligned} \tag{3.16}$$

where

$$C' = \min\left\{1-\frac{\lambda}{2\alpha}, \varepsilon - \frac{\lambda\alpha}{2}\right\}.$$

Therefore,

$$\lim_{\|u\|_{H^1}\to+\infty} \frac{\langle B^{R,\varepsilon}(u), u\rangle}{\|u\|_{H^1}} = +\infty,$$

and hence $B^{R,\varepsilon}$ is coercive. Consequently, the operator $B^{R,\varepsilon}$ satisfies the assumptions of the Browder–Minty theorem; it is therefore surjective from V_R into V_R'. Moreover, its strict monotonicity implies injectivity, and thus $B^{R,\varepsilon}$ is bijective. Thus, there exists a unique entropy solution $u^{R,\varepsilon} \in L^2(\Omega_R)$ to ($P_{R,\varepsilon}$). □

3.2. **Part II: Passage to the limits $\varepsilon \downarrow 0$ and $R \uparrow \infty$.**

Lemma 3.5. *Let $u^{R,\varepsilon}$ be the solution to problem ($P_{R,\varepsilon}$). Then there exist a function $u \in L^2_{\text{loc}}(\mathbb{R})$ and $G \in L^2_{\text{loc}}(\mathbb{R})$ such that, up to a subsequence,*

$$\begin{cases}
F\big(u^{R,\varepsilon}, (u^{R,\varepsilon})_x\big) \longrightarrow G & \text{in } L^2_{\text{loc}}(\mathbb{R}), \\
u^{R,\varepsilon} \rightharpoonup u & \text{weakly in } L^2_{\text{loc}}(\mathbb{R}), \\
\partial_x F\big(u^{R,\varepsilon}, (u^{R,\varepsilon})_x\big) \rightharpoonup \partial_x G & \text{in } L^2_{\text{loc}}(\mathbb{R}),
\end{cases}$$

where $\partial_x G = \frac{1}{\lambda}(u-f)$ in the sense of distributions.

Proof. Fix $R > 0$ and choose $\varphi = u^{R,\varepsilon}$ in ($P_{R,\varepsilon}$). We obtain

$$\int_{\Omega_R} |u^{R,\varepsilon}|^2 dx + \lambda \int_{\Omega_R} |u^{R,\varepsilon}||(u^{R,\varepsilon})_x| dx + \varepsilon \int_{\Omega_R} |(u^{R,\varepsilon})_x|^2 dx = \int_{\Omega_R} |f||u^{R,\varepsilon}| dx. \tag{3.17}$$

Applying Young and Hölder inequalities gives

$$\left(1+\frac{\lambda}{2\alpha}\right)\|u^{R,\varepsilon}\|_{L^2}^2 + \left(\varepsilon + \frac{\lambda\alpha}{2}\right)\|(u^{R,\varepsilon})_x\|_{L^2}^2 \leq \|f\|_{L^2}\|u^{R,\varepsilon}\|_{L^2}.$$

Hence

$$\begin{cases}
\|u^{R,\varepsilon}\|_{L^2} \leq \dfrac{1}{1+\frac{\lambda}{2\alpha}}\|f\|_{L^2}, \\
\|(u^{R,\varepsilon})_x\|_{L^2} \leq \dfrac{1}{\sqrt{\left(\varepsilon + \frac{\lambda\alpha}{2}\right)\left(1+\frac{\lambda}{2\alpha}\right)}}\|f\|_{L^2}.
\end{cases} \tag{3.18}$$

Since $|F^{R,\varepsilon}(u^{R,\varepsilon})| \leq |u^{R,\varepsilon}|$,

$$\|F^{R,\varepsilon}(u^{R,\varepsilon})\|_{L^2}^2 \leq \frac{1}{\left(1+\frac{\lambda}{2\alpha}\right)^2}\|f\|_{L^2}^2. \tag{3.19}$$

From (3.18)–(3.19), there exists a subsequence $\varepsilon_k \to 0$ such that

$$u^{R,\varepsilon_k} \rightharpoonup u^R \quad \text{in } L^2(\Omega_R), \qquad F^{R,\varepsilon_k} \rightharpoonup F^R \quad \text{in } L^2(\Omega_R).$$

Passing to the limit in $(P_{R,\varepsilon})$ yields

$$\int_{\Omega_R} u^R \varphi dx + \lambda \int_{\Omega_R} F^R \varphi_x dx = \int_{\Omega_R} f\varphi dx. \tag{FR}$$

Using monotonicity of B^{R,ε_k} in (3.15) and Testing with $v = u^R \pm \theta\varphi$ and letting $\theta \to 0$ gives

$$\lambda \int_{\Omega_R} \big(F^R - F(u^R,(u^R)_x)\big)\varphi_x dx \geq 0, \tag{3.20}$$

and

$$\lambda \int_{\Omega_R} \big(F^R - F(u^R,(u^R)_x)\big)\varphi_x dx \leq 0. \tag{3.21}$$

Hence

$$F^R = F(u^R,(u^R)_x) \quad \text{a.e. in } \Omega_R.$$

From (FR),

$$\partial_x F(u^R,(u^R)_x) = \frac{1}{\lambda}(f - u^R).$$

Hence for any bounded $\Omega \subset \mathbb{R}$,

$$\|\partial_x F(u^R,(u^R)_x)\|_{L^2(\Omega)} \leq C(\Omega, f, \lambda, \alpha). \tag{3.22}$$

Thus $F(u^R,(u^R)_x)$ is bounded in $H^1(\Omega)$. By Rellich–Kondrachov, $F(u^R,(u^R)_x)$ is relatively compact in $L^2(\Omega)$. Hence there exist $R_n \to \infty$ and $G \in L^2_{\text{loc}}(\mathbb{R})$ such that

$$F(u^{R_n},(u^{R_n})_x) \rightharpoonup G, \qquad u^{R_n} \rightharpoonup u \quad \text{in } L^2_{\text{loc}}(\mathbb{R}).$$

A diagonal extraction ensures convergence on every compact subset. Since

$$\partial_x F(u^{R_n},(u^{R_n})_x) = \frac{1}{\lambda}(u^{R_n} - f),$$

passing to the limit gives

$$\partial_x G = \frac{1}{\lambda}(u - f) \quad \text{in the sense of distributions.}$$

Therefore,

$$\begin{cases} F(u^{R_n},(u^{R_n})_x) \rightharpoonup G & \text{in } L^2_{\text{loc}}(\mathbb{R}), \\ u^{R_n} \rightharpoonup u & \text{in } L^2_{\text{loc}}(\mathbb{R}), \\ \partial_x F(u^{R_n},(u^{R_n})_x) \rightharpoonup \partial_x G & \text{in } L^2_{\text{loc}}(\mathbb{R}). \end{cases}$$

□

3.3. **Part III: Approximation of the Initial Data.** In this part, we construct a sequence of smooth functions $u_0^n \in C_c^\infty(\mathbb{R})$ approximating the initial datum u_0 in $L^1(\mathbb{R})$. We recall that any function $u_0 \in W^{1,1}(a,b)$ admits an absolutely continuous representative on $[a,b]$. In particular, for all $x, y \in [a,b]$,

$$u_0(x) = u_0(y) + \int_y^x u_0'(s)\, ds,$$

and u_0 is continuous on $[a,b]$ with well-defined traces at the endpoints (see, e.g., [1]).
We recall that if $u_0 \in W^{2,1}(a,b)$, then $u_0' \in W^{1,1}(a,b)$ and, in particular, $u_0 \in C^1(a,b)$ with well-defined traces $u_0(a)$, $u_0(b)$, $u_0'(a)$ and $u_0'(b)$ (see [12]). We recall that for $u_0 \in W^{1,1}(a,b)$, the traces at the endpoints satisfy the estimates

$$|u_0(a)| + |u_0(b)| \le C \left(\frac{1}{b-a} \int_a^b |u_0(x)|\, dx + \int_a^b |u_0'(x)|\, dx \right),$$

for some constant $C > 0$.

Similarly, if $u_0 \in W^{2,1}(a,b)$, the traces of the derivative satisfy

$$|u_0'(a)| + |u_0'(b)| \le C \left(\frac{1}{b-a} \int_a^b |u_0'(x)|\, dx + \int_a^b |u_0''(x)|\, dx \right).$$

We have

$$u_0(x) = C_a + \int_a^x u_0'(y)\, dy \Longrightarrow |u_0(x)| \le |C_a| + \int_a^b |u_0'(y)|\, dy,$$

and therefore

$$\|u_0\|_{L^\infty(a,b)} \le \|u_0\|_{L^1(a,b)} + 2\|u_0'\|_{L^1(a,b)}.$$

Lemma 3.6 (Hermite extension with cut-off)**.** *Fix $r \in \left(0, \frac{b-a}{2}\right]$. There exists a function $v_0 \in W^{2,1}(\mathbb{R}) \cap C^1(\mathbb{R})$ such that:*

(i) *$v_0 = u_0$ on (a,b),*
(ii) *$\mathrm{supp}(v_0) \subset (a-r, b+r)$,*
(iii) *v_0 matches the boundary values and derivatives, i.e.*

$$v_0(a) = u_0(a), \quad v_0'(a) = u_0'(a), \quad v_0(b) = u_0(b), \quad v_0'(b) = u_0'(b),$$

(iv) *there exists a constant $C = C(a,b,r) > 0$ such that*

$$\|v_0\|_{W^{2,1}(\mathbb{R})} \le C \left(\|u_0\|_{L^1} + \|u_0'\|_{L^1} + \|u_0''\|_{L^1} \right),$$
$$\|v_0\|_{L^\infty(\mathbb{R})} \le C \left(\|u_0\|_{L^\infty} + \|u_0''\|_{L^1} \right).$$

Such a function can be constructed using standard cubic Hermite interpolation.

Proof. The result follows from a standard cubic Hermite interpolation argument (see, e.g., [12]). More precisely, one constructs two C^1 functions on $[a-r, a]$ and $[b, b+r]$ which connect $(0,0)$ to $(u_0(a), u_0'(a))$ and $(u_0(b), u_0'(b))$ to $(0,0)$, respectively, while matching both values and first derivatives at the endpoints. This ensures property (iii). Defining v_0 by gluing these connectors with u_0 on (a,b), we obtain a function $v_0 \in C^1(\mathbb{R})$ such that $v_0 = u_0$ on (a,b) and $\mathrm{supp}(v_0) \subset (a-r, b+r)$, which gives (i) and (ii). Since the construction is polynomial on $[a-r, a] \cup [b, b+r]$ and $u_0 \in W^{2,1}(a,b)$, it follows that $v_0 \in W^{2,1}(\mathbb{R})$. Finally, the estimates in (iv) follow from standard bounds on Hermite interpolation and the trace estimates for u_0 and u_0'. □

Proposition 3.7. *Let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap W^{2,1}(\mathbb{R})$ be a nonnegative function with compact support. Assume that there exists an open set $\Omega \Subset \mathbb{R}$ and $\beta > 0$ such that*

$$u_0 \geq \beta \text{ a.e. on } \Omega, \qquad u_0 = 0 \text{ a.e. on } \mathbb{R} \setminus \Omega.$$

Then there exists a sequence $\{u_0^n\} \subset C_c^\infty(\mathbb{R}) \cap D(A)$ such that

- *$u_0^n \to u_0$ in $L^1(\mathbb{R})$ and a.e.;*
- *$\mathrm{supp}(u_0^n) \subset K \Subset \mathbb{R}$ for some compact K independent of n;*
- *there exist $\varepsilon_n \downarrow 0$ and $\beta_n \downarrow 0$ with*

$$u_0^n \geq \beta_n \quad \text{on } \Omega_{-\varepsilon_n};$$

- *the relativistic flux satisfies the uniform bound*

$$\sup_n \|\partial_x F(u_0^n)\|_{L^1(\mathbb{R})} \leq C,$$

where C is a constant depending only on the initial data.

Such an approximation follows from standard mollification and truncation arguments preserving positivity and compact support (see [26, 15]).

Theorem 3.8. *Let $\{u_0^n\}$ be given by Proposition 3.7, and let $u^n(t) = S(t)u_0^n$ denote the entropy solution of*

$$u_t + Au = 0 \quad \text{in } (0, \infty) \times \mathbb{R}, \qquad u(0) = u_0^n,$$

with $Au = -\partial_x F(u)$. Then for every $t > 0$:

- *$u^n(t) \in D(A)$ and $F(u^n(t)) \in W^{1,1}(\mathbb{R})$;*
- *the time derivative satisfies*

$$\|u_t^n(t)\|_{L^1(\mathbb{R})} = \|Au^n(t)\|_{L^1(\mathbb{R})} \leq C;$$

- *$\{u^n\}$ is uniformly Lipschitz continuous in time with values in $L^1(\mathbb{R})$.*

These properties follow from the m-accretive structure of A in $L^1(\mathbb{R})$ and the nonlinear semigroup theory of Crandall–Liggett (see [9, 15]).

3.4. **Part IV: Semigroup approach and uniform control of the time derivative.** We consider the nonlinear operator

$$Au := -\partial_x F(u), \qquad F(u) = \frac{uu_x}{\sqrt{u^2 + u_x^2}},$$

which is m-accretive in $L^1(\mathbb{R})$. We denote by $\{S(t)\}_{t\geq 0}$ the contraction semigroup generated by A in $L^1(\mathbb{R})$, given by the Crandall–Liggett theorem.
For $\lambda > 0$, we introduce the resolvent and the Yosida approximation

$$J_\lambda = (I + \lambda A)^{-1}, \qquad A_\lambda = \frac{I - J_\lambda}{\lambda}.$$

The resolvent J_λ is a contraction in $L^1(\mathbb{R})$, and the implicit Euler scheme

$$u^0 = v, \qquad u^{k+1} = J_\lambda u^k,$$

is well defined for any $v \in L^1(\mathbb{R})$. Moreover, the discrete increments

$$\|u^{k+1} - u^k\|_{L^1}$$

are non-increasing, yielding uniform bounds on $\|A_\lambda u^k\|_{L^1}$.
Let $u_\lambda(t) = u^N$ for $t \in [(N-1)\lambda, N\lambda)$. Then, as $\lambda \to 0$, one has

$$u_\lambda(t) \longrightarrow S(t)v \quad \text{in } L^1(\mathbb{R}), \qquad \forall t > 0.$$

If $v \in D(A)$, then

$$A_\lambda u_\lambda(t) \longrightarrow AS(t)v \quad \text{in } L^1(\mathbb{R}),$$

and the slope satisfies the decay property

$$\|AS(t)v\|_{L^1(\mathbb{R})} \leq \|Av\|_{L^1(\mathbb{R})}, \qquad \forall t > 0.$$

Let $\{u_0^n\} \subset D(A)$ be the regularized initial data constructed previously, satisfying

$$\sup_n \|Au_0^n\|_{L^1(\mathbb{R})} \leq C,$$

and define

$$u^n(t) := S(t)u_0^n.$$

Then, for all $t > 0$ and all $n \in \mathbb{N}$,

$$\|u_t^n(t)\|_{L^1(\mathbb{R})} = \|Au^n(t)\|_{L^1(\mathbb{R})} \leq C.$$

In particular, for any $0 \leq t_0 < t$,

$$\|u^n(t) - u^n(t_0)\|_{L^1(\mathbb{R})} \leq C|t - t_0|,$$

so that the family $\{u^n\}_n$ is equicontinuous in time in $L^1(\mathbb{R})$. By the Ascoli–Arzelà theorem, there exists

$$u \in C([0,\infty); L^1_{\text{loc}}(\mathbb{R}))$$

such that, for every $T > 0$,

$$\sup_{t\in[0,T]} \|u^n(t) - u(t)\|_{L^1(\mathbb{R})} \longrightarrow 0.$$

Moreover, for each $t > 0$, the sequence $\{u_t^n(t)\}_n$ is bounded in $L^1(\mathbb{R})$, hence converges (up to subsequences) weakly-* to a Radon measure $\mathcal{R}_t$ with

$$\|\mathcal{R}_t\|_{\mathcal{M}(\mathbb{R})} \leq C.$$

Passing to the limit in the weak formulation yields

$$\frac{d}{dt}\langle u(t), \varphi\rangle = \langle \mathcal{R}_t, \varphi\rangle \quad \text{in } \mathcal{D}'(\mathbb{R}^+),$$

and we identify

$$\mathcal{R}_t = \partial_x F(u(t)) \quad \text{in } \mathcal{M}(\mathbb{R}).$$

Consequently,

$$u_t(t,\cdot) = \partial_x F(u(t,\cdot)), \qquad \|u_t(t)\|_{\mathcal{M}(\mathbb{R})} \leq C. \quad \forall t > 0.$$

References

[1] Ambrosio, L., Fusco, N., & Pallara, D. (2000). *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Mathematical Monographs, Oxford University Press. https://doi.org/10.1093/oso/9780198502456.001.0001

[2] Andreianov, B., & Gazibo, M. K. (2013). Entropy formulation of degenerate parabolic equations with zero-flux boundary conditions. *Zeitschrift für Angewandte Mathematik und Physik*, 64, 1471–1491. https://doi.org/10.1007/s00033-012-0297-6

[3] Andreu, F., Caselles, V., & Mazón, J. M. (2002). Existence and uniqueness of solutions for a parabolic quasilinear problem for linear growth functionals with L^1 data. *Mathematische Annalen*, 322, 139–206. https://doi.org/10.1007/s002080100270

[4] Andreu, F., Caselles, V., & Mazón, J. M. (2005). The Cauchy problem for a strongly degenerate quasilinear equation. *Journal of the European Mathematical Society*, 7, 361–393. https://doi.org/10.4171/jems/32

[5] Andreu, F., Caselles, V., & Mazón, J. M. (2005). A strongly degenerate quasilinear elliptic equation. *Nonlinear Analysis*, 61(4), 637–669. https://doi.org/10.1016/j.na.2004.11.020

[6] Andreu, F., Caselles, V., Mazón, J. M., & Moll, S. (2006). Finite propagation speed for limited flux diffusion equations. *Archive for Rational Mechanics and Analysis*, 182(2), 269–297. https://doi.org/10.1007/s00205-006-0428-3

[7] Bassolé, R., Gazibo, M. K., & Ouédraogo, A. (2026). Relativistic Heat Equation in bounded domain with Dirichlet boundary condition. *Gulf Journal of Mathematics, to appear*.

[8] Bénilan, P., & Touré, H. (1995). Sur l'équation générale $u_t = a(\cdot, u, \phi(\cdot, u)_x)_x + v$ dans L^1. In *Evolution Equations*, Lecture Notes in Pure and Applied Mathematics, Vol. 168, pp. 35–62.

[9] Bénilan, P., Crandall, M. G., & Wittbold, P. (1993). Entropy solutions of degenerate elliptic–parabolic problems. *Proceedings of the Royal Society of Edinburgh Section A*, **123**(3), 445–467.

[10] Bertsch, M., & Dal Passo, R. (1992). Hyperbolic phenomena in a strongly degenerate parabolic equation. *Archive for Rational Mechanics and Analysis*, 117(4), 349–387. https://doi.org/10.1007/bf00376188

[11] Bouchut, F. (2004). *Nonlinear stability of finite volume methods for hyperbolic conservation laws and well-balanced schemes for sources*. Frontiers in Mathematics, Birkhäuser. https://doi.org/10.1007/b93802

[12] Brezis, H. (2011). *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer. https://doi.org/10.1007/978-0-387-70914-7

[13] Brenier, Y. (2003). Extended Monge–Kantorovich theory. In *Optimal Transportation and Applications*, Lecture Notes in Mathematics, Vol. 1813, pp. 91–121. Springer. https://doi.org/10.1007/978-3-540-44857-0_4

[14] Carrillo, J. A. (1999). Entropy solutions for nonlinear degenerate problems. *Archive for Rational Mechanics and Analysis*, 147(4), 269–361. https://doi.org/10.1007/s002050050152

[15] Carrillo, J. A., Caselles, V., & Moll, S. (2012). On the relativistic heat equation in one space dimension. *Proceedings of the London Mathematical Society*, **105**(1), 59–90. https://doi.org/10.1112/plms/pdt015

[16] Caselles, V. (2007). Convergence of the relativistic heat equation to the heat equation as $c \to \infty$. *Publicacions Matemàtiques*, 51(1), 121–142. https://doi.org/10.5565/publmat_51107_06

[17] Chertock, A., Kurganov, A., & Rosenau, P. (2003). Formation of discontinuities in flux-saturated degenerate parabolic equations. *Nonlinearity*, 16(5), 1875–1898. https://doi.org/10.1088/0951-7715/16/6/301

[18] Dal Passo, R. (1993). Uniqueness of the entropy solution of a strongly degenerate parabolic equation. *Communications in Partial Differential Equations*, 18(1-2), 265–279. https://doi.org/10.1080/03605309308820930

[19] Evans, L. C. (1997). Partial differential equations and Monge–Kantorovich mass transfer. In *Current Developments in Mathematics*. International Press. https://doi.org/10.4310/cdm.1997.v1997.n1.a2

[20] Konate, I., & Ouaro, S. (2018). Problèmes multivaleurs non linéaires avec des données de mesure à exposant variable. *Gulf Journal of Mathematics*, 6(2). https://doi.org/10.56947/gjom.v6i2.130

[21] El Haji, B., El Moumni, M., & Talha, A. (2020). Entropy solutions for nonlinear parabolic equations in Musielak Orlicz spaces without Delta_2-conditions. *Gulf Journal of Mathematics*, 9(1), 1–26. https://doi.org/10.56947/gjom.v9i1.448

[22] Ouaro, S., & Sawadogo, N. (2021). Structural stability for nonlinear p(u)-Laplacian problem. *Gulf Journal of Mathematics*, 11(1), 1–37. https://doi.org/10.56947/gjom.v11i1.665

[23] Gazibo, M. K. (2024). Degenerate Parabolic Equation with Zero-flux Boundary Condition and its Approximations. *WSEAS Transactions on Mathematics*, 23, 682–705. https://doi.org/10.37394/23206.2024.23.71

[24] Glimm, J., Grove, J. W., Li, X. L., & Zhao, N. (1999). Simple front tracking. In Contemporary Mathematics. https://doi.org/10.1090/conm/238/03544

[25] Mazón, J., Moll, S., & Carrillo, J. A. (2014). The Dirichlet problem for the relativistic heat equation. *Journal de Mathématiques Pures et Appliquées*, **102**(5), 889–931.

[26] Moll, S. (2007). BV solutions for flux-limited diffusion equations. *Journal de Mathématiques Pures et Appliquées*, **87**(5), 453–475.

[27] Caselles, V. (2011). On the entropy conditions for some flux-limited diffusion equations. *Journal of Differential Equations*, 250(8), 3311–3348. https://doi.org/10.1016/j.jde.2011.01.027

[28] Ouaro, S., & Soma, S. (2011). Weak and entropy solutions to nonlinear Neumann boundary value problems with variable exponents. *Complex Variables and Elliptic Equations*, 56(7-9), 829–851. https://doi.org/10.1080/17476933.2010.504840

[29] Ouaro, S. (2009). Uniqueness of Entropy Solutions of Nonlinear Elliptic-Parabolic-Hyperbolic Problems in One Dimension Space. *Revista Matemática Complutense*, 22(1), 7–36. https://doi.org/10.5209/rev_REMA.2009.v22.n1.16303

[30] Rosenau, P. (1992). Tempered diffusion: a transport process with propagating front and inertial delay. *Physical Review A*, **46**(12), 7371–7374. https://doi.org/10.1103/physreva.46.r7371

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