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NONSTANDARD ESTIMATION METHODS IN ONE-DIMENSIONAL AND SPATIAL FIRST-ORDER AUTOREGRESSION MODELS

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Abstract. The article proposes alternative parameter estimators for autoregression that differ from the least squares estimates. In unstable (critical) cases, where the characteristic equation's roots lie on the unit circle, least squares estimators generally exhibit a complex asymptotic distribution. In contrast, the proposed nonstandard estimators tend to have a simpler asymptotic distribution in most critical cases.

Keywords. Nonstandard parameter estimation, spatial autoregressive models, asymptotic properties of estimators, time series and spatial dependence
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Introduction

The report presents nonstandard approaches to constructing estimators in various autoregressive models. The estimation equations are formulated using the recursive relationships that define the original process, with each case considered separately. A similar approach can also be applied to derive classical least squares estimators, without relying on the conventional method of minimizing the corresponding sum of squares with respect to the original values. The necessity of developing such estimators arises from the fact that least squares estimators in critical (unstable) cases exhibit complex asymptotic distributions, which are generally expressed in terms of functionals of the standard Wiener process. From an applied perspective, critical cases are of particular interest. For example, in the case of a first-order autoregressive process, i.e., when $Y_k = \alpha Y_{k-1} + \varepsilon_k$, significant attention has been devoted to tabulating the complex asymptotic distribution when $\alpha = 1$. In this article, we consider such distributions and study their asymptotic estimators.

While ordinary autoregressive models are of great interest to economists (see [11],[15],[1]), spatial autoregressive models, characterized by $U_{k,q} = \alpha U_{k-1,q} +$

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$\beta U_{k,q-1} + \varepsilon_{k,q}$ are relevant in fields such as geology, agriculture, and satellite image processing of the Earth's surface for predicting anomalous events (see [14]). It is also worth noting that intensive research into spatial autoregression models has only gained momentum in recent decades. A more detailed historical overview of each autoregressive model under consideration can be found in the cited literature. Although arising in different mathematical domains, recent works on nonclassical iterative schemes and refined coefficient-based analytical methods suggest a broader shift toward more flexible estimation and approximation frameworks [2]. This methodological perspective motivates the present study, in which nonstandard estimation methods are developed for one-dimensional and spatial first-order autoregressive models [10].

Problem Formulation

We consider the following autoregressive models.

Definition 0.1 ([6])**.** An autoregressive scheme of order p (denoted as $AR(p)$) is a relationship of the form

$$X_k = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{k-j} + \varepsilon_k, \tag{1}$$

where α_j $(j = 0, 1, \dots, p)$ are constants, and ε_k are random variables called *autoregressive noise* or simply *noise*.

In the case of $\alpha_0 = X_0 = 0$ a first-order autoregressive model, it will take the following form

$$X_k = \alpha X_{k-1} + \varepsilon_k, \quad k = 1, 2, \dots, \tag{2}$$

where ε_k- are independent identically distributed random variables (i.i.d.r.v.) with $E\varepsilon_1 = 0, E\varepsilon_1^2 = 1, X_0-$ an initial state.

The known estimate of the parameter α, obtained from n observations using the least squares method has the form

$$\widehat{\alpha}_n = \sum_{k=1}^{n} X_{k-1}X_k / \sum_{k=1}^{n} X_{k-1}^2 \tag{3}$$

In the work this estimate is given as the serial correlation coefficient.
If $\{\varepsilon_i\}$ normally distributed, then estimate (3) coincides with the maximum likelihood estimate.
The complex structure of the estimate $\widehat{\alpha}_n$ makes it difficult to find the limit distribution even in the case of normally distributed noise. $\{\varepsilon_i\}$. It is known that $\sqrt{n}\,(\widehat{\alpha}_n - \alpha)$ has a normal limit distribution when $\alpha \in (-1, 1)$, and for $|\alpha| \geq 1$ the limit distribution is complex and, moreover, when $|\alpha| > 1$ it becomes dependent on the noise distribution. $\{\varepsilon_i\}$. For example, when $\alpha = 1$ the statement holds. at $n \to \infty$

$$n\,(\widehat{\alpha}_n - 1) \Rightarrow \frac{1}{2}\left(w^2(1) - 1\right) / \int_0^1 w^2(x)dx \tag{4}$$

where is $w(x)-$ a standard Wiener process, and the symbol $\Rightarrow$ denotes weak convergence of the corresponding distributions.

In [4] another, simpler in structure, estimate of the parameter was proposed. α. Simple summation of (2) over t from k to n leads to the following estimate [9]

$$\alpha^*_{n,k} - \alpha = \frac{\varepsilon_k + \ldots + \varepsilon_n}{X_k + \ldots + X_n} \tag{5}$$

It has been shown (see [4]) that when $n \to \infty$

$$P\left(n\beta(c)\left(\alpha^*_{n,k} - 1\right) < x\right) \Rightarrow \frac{1}{2} + \frac{1}{\pi}\operatorname{arctg}\left[(x - \rho_c)/\sqrt{1-\rho_c^2}\right]$$

Where $c = \lim_{n\to\infty} \frac{k}{n} < 1, \quad \beta(c) = ((1-c)(1+2c)/3)^{1/2}$,
$\rho_c = \frac{1}{2}\left(\frac{3(1-c)}{1+2c}\right)^{1/2}$.
If $c = 1$, so then

$$P\left((k(n-k))^{1/2}\left(\alpha^*_{n,k} - 1\right) < x\right) \Rightarrow \frac{1}{\pi}\int_{-\infty}^{x} \frac{1}{1+u^2} du$$

Note that in case $\alpha = -1$ there are no results. The invariance principle used in the study of estimate (3) in $\alpha = -1$ does not give any results, and as for estimate (5), it is untenable in this case, which is easy to show.

Spatial autoregressive models have begun to be intensively studied relatively recently and have not yet entered the monographic and educational literature. We will give some overview of the results following the work.

The study of spatial patterns has attracted considerable attention in a wide range of disciplines, including geography, geology, biology, and agriculture. For related discussions, see [...]. In those studies, the so-called one-sided $AR(p)$ model was examined, and it is given by

$$X_{k,l} = \sum_{i=0}^{p_1}\sum_{j=0}^{p_2} \alpha_{i,j} X_{k-i,l-j} + \varepsilon_{k,l}, \qquad \alpha_{0,0} = 0.$$

A particular version of this model was investigated in the case where $p_1 = p_2 = 1$, $\alpha_{0,1} = \alpha_{1,0} =: \alpha$, and $\alpha_{1,1} = 0$, and explicit results were derived for the asymptotic behavior of the estimator α in the unstable setting. Only a limited number of results of this kind are available in the existing literature on spatial models. From a broader perspective, however, it is natural to study models in which $X_{k,l}$ is expressed as a linear combination of all neighboring lattice points. In particular, one may consider an extension of the Sandor Baran model in which $X_{k-1,l}$ and $X_{k,l-1}$ are assigned distinct weights, say α and β. Even in the symmetric case $\alpha = \beta$, such models give rise to mathematically challenging questions and may lead to rather nonstandard results.

In this paper, we propose an estimate of a simpler structure and using only a portion of the observations located along the diagonal in a rectangle. $R_{m,n}$. This approach with a significant reduction in the number of observations is relevant in geology problems and some other areas of application of spatial autoregressive models. A simpler estimate structure also allows us to reduce moment restrictions on noise.

Solution of the Problem and Results

First-order ordinary univariate autoregressive model. Near-critical case ($\alpha = \alpha(n) \to 1$)**.**

In this paper, a first-order autoregressive model is considered. Process (2) is stable in the case when $|\alpha| < 1$, it is unstable at $|\alpha| = 1$ and is of the explosive type at $|\alpha| > 1$.

Note that in the case $\alpha(n) \to 1$ of $n \to \infty$ there are no results. In this paper, we study a case close to critical, when $\alpha(n) \to 1$ for $n \to \infty$ a particular type of estimate (5) ($k = 1$).

Construction of the estimate and its limit behavior.

We will construct the estimation equation [12] by summing the relation (2) t from 1 to n

$$\sum_{k=1}^{n} X_k = \alpha \sum_{k=1}^{n} X_{k-1} + \sum_{k=1}^{n} \varepsilon_k$$

or in abbreviated form

$$Y_n = \alpha Y_{n-1} + E_n \tag{6}$$

Solving equation (6) without taking into account E_n, we obtain the estimate

$$\alpha^*_{n,1} = \frac{Y_n}{Y_{n-1}}$$

Now from (6) we find the deviation

$$\alpha^*_{n,1} - \alpha = \frac{E_n}{Y_{n-1}} \tag{7}$$

Now let us note that from (2) we have $X_{t-1} = \frac{1}{\alpha}\left(X_t - \varepsilon_t\right)$ and therefore

$$Y_{n-1} = \frac{1}{\alpha}\left(Y_n - E_n\right) \tag{8}$$

Theorem 0.2. *If ε_k are i.i.d. random variables with $E(\varepsilon_1) = 0$, $E(\varepsilon_1^2) = 1$, and $X_0 = 0$, then the following hold:*

(1) *As $n \to \infty$, when $\alpha = 1 - \dfrac{c}{n}$, $|c| < \infty$, we have*

$$P\Big(n\sigma_c\big(\alpha^*_{n,1} - \alpha\big) < x\Big) \Rightarrow \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x + \delta_c}{\sqrt{1 - \delta_c^2}}\right),$$

where

$$\sigma_c^2 = \frac{1}{2|c|^3}\left(2|c| - 3 + 4e^{-|c|} - e^{-2|c|}\right), \quad \delta_c = \frac{1}{c^2\sigma_c}\left(e^{-|c|} + |c| - 1\right).$$

(2) *If* $\alpha(n) = 1 - \dfrac{c}{\gamma_n}$, $\gamma_n \to \infty$, *and* $\gamma_n = o(n)$, *then as* $n \to \infty$ *we have*

$$\frac{\sqrt{2}\,\gamma_n}{|c|}\left(\alpha^*_{n,1} - \alpha\right) \Rightarrow \frac{\xi_1}{\xi_2},$$

where (ξ_1, ξ_2) *is a normal random vector with zero mean and covariance matrix*

$$\begin{pmatrix} 1 & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & 1 \end{pmatrix}.$$

(3) *If* $\alpha(n) = 1 - \dfrac{c}{\gamma_n}$, $\gamma_n \to \infty$, *and* $\dfrac{n}{\gamma_n} \to 0$, *then as* $n \to \infty$ *we have*

$$\frac{2n}{\sqrt{3}}\left(\alpha^*_{n,1} - \alpha\right) \Rightarrow \frac{\xi_1}{\xi_2},$$

where (ξ_1, ξ_2) *is a normal random vector with zero mean and covariance matrix*

$$\begin{pmatrix} 1 & \dfrac{\sqrt{3}}{2\sqrt{2}} \\ \dfrac{\sqrt{3}}{2\sqrt{2}} & 1 \end{pmatrix}.$$

Proof. **Of Theorem 1.**
1-case. Let $\alpha(n) = 1 \pm \dfrac{c}{n}$, $0 < c < \infty$. We introduce the following notations:

$$Y_n = \sum_{k=1}^{n} X_k, E_n = \sum_{k=1}^{n} \varepsilon_k$$

To find the limit distributions, we now find the expression Y_n through $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. In the model under consideration (2), we can easily find by a recursive method [8]

$$X_1 = \varepsilon_1, \ldots, X_n = \alpha^{n-1}\varepsilon_1 + \alpha^{n-2}\varepsilon_2 + \alpha^{n-3}\varepsilon_3 + \ldots + \alpha\varepsilon_{n-1} + \varepsilon_n$$

From here $Y_n = \sum_{k=0}^{n} \frac{1-\alpha^{k+1}}{1-\alpha}\varepsilon_{n-k}$.
The dispersion Y_n in this case has the form

$$DY_n \sim \frac{n^3}{2c^3}\left[2c \pm 3 \mp 4e^{\pm c} \pm e^{\pm 2c}\right] = n^3\sigma_c^2$$

From this we conclude that in the general case

$$\sigma_c^2 = n^{-3} DY_n \sim \frac{1}{2|c|^3}\left[2|c| - 3 + 4e^{-|c|} - e^{-2|c|}\right] \qquad (9)$$

Note that

$$D\left(E_n\right) = n$$

Further, from (7) taking into account (8) we have

$$\alpha_{n,1}^{*}-\alpha=\frac{E_n}{\frac{1}{\alpha}\left(Y_n-E_n\right)}. \tag{10}$$

By supplying an asterisk to the values normalized by the root of the variance in (10), we obtain

$$n\sigma_c\left(\alpha_{n,1}^{*}-\alpha\right)=\frac{E_n^{*}}{Y_n^{*}-o_p(1)}$$

where $o_p(1)-$ the quantity tends to zero in probability at $n\to\infty$. The latter follows from the fact that $E\left(E_n^2\right)=n=o\left(E\left(Y_n^2\right)\right)$.
Now for normalized quantities

$$Y_n^{*}=\frac{Y_n}{\alpha\sqrt{DY_n}} \text{ And } E_n^{*}=\frac{E_n}{\sqrt{DE_n}}$$

Let's consider the joint characteristic function

$$f_n\left(t_1,t_2\right)=E\left[\exp\left(it_1Y_n^{*}+it_2E_n^{*}\right)\right]$$

Note that the weak convergence of a vector $\left(Y_n^{*},E_n^{*}\right)$ to the two-dimensional normal law of equivalence is weaker than the convergence of one-dimensional random variables

$$Z_n=t_1Y_n^{*}+t_2E_n^{*}$$

to the one-dimensional normal law for any t_1 and t_2 (Cramer-Wald theorem). The quantity Z_n can be represented in the form

$$Z_n=\sum_{k=1}^{n}\gamma_{n,k}\left(t_1,t_2\right)\varepsilon_{n-k+1} \tag{11}$$

where $\gamma_{n,k}\left(t_1,t_2\right)=\frac{t_1}{\alpha\sqrt{DY_n}}\frac{1-\alpha^k}{1-\alpha}+\frac{t_2}{\sqrt{n}}$.

From here it is easy to show, using (11), that

$$E\left(Z_n^2\right)\sim\sigma^2=t_1^2+2\delta_ct_1t_2+t_2^2$$

where $\delta_c=\frac{1}{c^2\sigma_c}\left(e^{-|c|}+|c|-1\right)$.

Really,

$$DZ_n=\sum_{k=1}^{n}\gamma_{n,k}^2\left(t_1,t_2\right)=\frac{t_1^2}{DY_n}\sum_{k=1}^{n}\left(\frac{1-\alpha^k}{1-\alpha}\right)^2+\frac{2t_1t_2}{\sqrt{nDY_n}}\sum_{k=1}^{n}\frac{1-\alpha^k}{1-\alpha}+\sum_{k=1}^{n}\frac{t_2^2}{n}$$

and if $\alpha=1\pm\frac{c}{n},0<c<\infty$, that

$$\delta_c=\frac{1}{\sqrt{nDY_n}}\sum_{k=1}^{n}\frac{\alpha^k-1}{\alpha-1}\sim\frac{1}{c^2\sigma_c}\left(e^{\pm c}\mp c-1\right). \tag{12}$$

Now let's check the Lindeberg condition

$$L_n=\frac{1}{\sigma^2}\sum_{k=1}^{n}\int_{|x|\geq\frac{\varepsilon\sigma_c}{\left|\gamma_{n,k}\left(t_1,t_2\right)\right|}}\gamma_{n,k}^2\left(t_1,t_2\right)x^2dF(x)\leq$$

$$\leq \frac{1}{\sigma^2}\sum_{k=1}^{n}\int_{|x|\geq\frac{\varepsilon\sigma}{\left|\gamma_{n,k}(t_1,t_2)\right|}} x^2 dF_{\varepsilon_k}(x) \leq \int_{|x|\geq\frac{\varepsilon\sigma}{\left|\gamma_{n,k}(t_1,t_2)\right|}} x^2 dF(x)$$

where

$$|\gamma_{n,k}(t_1,t_2)| \leq \frac{t_1 n\left(1-e^{-c}\right)}{\alpha c\sqrt{DY_n}} + \frac{t_2}{\sqrt{n}} \sim 0$$

from here

$$L_n \leq \int_{|x|\geq\frac{\varepsilon\sigma}{\left|\gamma_{n,k}(t_1,t_2)\right|}} x^2 dF(x) \sim 0$$

Therefore, by virtue of the central limit theorem $f_n(t_1,t_2) = Ee^{iZ_n} \to e^{-\frac{\sigma^2}{2}}$. If we recall (11), it will be clear that the vector (Y_n^*, E_n^*) has as its limit a two-dimensional normal distribution with zero mean and the following matrix of second moments $A = \begin{pmatrix} 1 & \delta_c \\ \delta_c & 1 \end{pmatrix}$, where $\delta_c = \frac{1}{c^2\sigma_c}\left(e^{-|c|}+|c|-1\right)$.

$$f_n(t_1,t_2) = Ee^{iZ_n} \to e^{-\frac{\sigma^2}{2}} = e^{-\frac{1}{2}\left(t_1^2+2\delta_c t_1 t_2+t_2^2\right)}$$

which is equivalent to the statement of the theorem. If $c=0$, then $\alpha_n^* = \alpha_{n1}^*$ and

$$P\left(n\sigma\left(\alpha_{n,1}^*-\alpha\right)<x\right) \Rightarrow \frac{\sqrt{1-\rho^2}}{\pi}\int_{-\infty}^{x}\frac{du}{u^2+2\rho u+1} = \frac{1}{2}+\frac{1}{\pi}\operatorname{arctg}\frac{x+\rho}{\sqrt{1-\rho^2}}$$

where $\sigma^2 = 1/3$ and $\rho = \sqrt{3}/2$.

Indeed, when $c=0$ the parameter estimate α has the following form:

$$\alpha_{n,1}^* = \frac{Y_n}{Y_{n-1}}$$

from here we get the deviations

$$\alpha_{n,1}^* - \alpha = \frac{E_n}{Y_{n-1}}.$$

To find the limit distributions, we find the expression Y_n through $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. In the model under consideration, we can easily find by a recurrent method

$$X_1 = \varepsilon_1, \ldots, X_n = \varepsilon_1+\varepsilon_2+\varepsilon_3+\ldots+\varepsilon_{n-1}+\varepsilon_n$$

From this we have

$$Y_n = \sum_{j=1}^{n}(n-j+1)\varepsilon_j$$

The dispersion Y_n in this case has the form

$$DY_n = \sum_{j=1}^{n}(n-j+1)^2 \sim \frac{n^3}{3}$$

So $\sigma^2 = \frac{1}{n^3}DY_n \sim \frac{1}{3}$.

Further, from (7) taking into account (8) we have

$$\alpha_{n,1}^{*} - \alpha = \frac{E_n}{(Y_n - E_n)} \tag{13}$$

By supplying an asterisk to the normalization by the root of the variance of the quantity in (13), we obtain

$$n\sigma_c\left(\alpha_{n,1}^{*} - \alpha\right) = \frac{E_n^*}{Y_n^* - o_p(1)}$$

where $o_p(1)-$ the quantity tends to zero in probability at $n \to \infty$. The latter follows from the fact that

$$E\left(E_n^2\right) = n = o\left(E\left(Y_n^2\right)\right).$$

$$Y_n^* = \frac{Y_n}{\sqrt{DY_n}} \text{ And } E_n^* = \frac{E_n}{\sqrt{DE_n}},$$

Let's consider the characteristic function

$$f_n\left(t_1, t_2\right) = E\left[\exp\left(it_1 Y_n^* + it_2 E_n^*\right)\right]$$

Weak convergence of a vector (Y_n^*, E_n^*) to the two-dimensional normal law is equivalent to the convergence of one-dimensional random variables

$$Z_n = t_1 Y_n^* + t_2 E_n^*$$

to the normal law for any t_1 and t_2. The value Z_n can be represented in the following form [3]:

$$Z_n = \sum_{j=1}^{n} \gamma_{n,j}\left(t_1, t_2\right)\varepsilon_j \tag{14}$$

where $\gamma_{n,j}\left(t_1, t_2\right) = \frac{t_1}{\sqrt{DY_n}}(n - j + 1) + \frac{t_2}{\sqrt{n}}$.

From here it is easy to show, using (14), that

$$E\left(Z_n^2\right) \sim \sigma^2 = t_1^2 + 2\rho t_1 t_2 + t_2^2$$

where $\rho = \frac{\sqrt{3}}{2}$.

Really

$$DZ_n = \sum_{j=1}^{n} \gamma_{n,j}^2\left(t_1, t_2\right) = \frac{t_1^2}{DY_n}\sum$$

$$\sum_{j=1}^{n}(n - j + 1)^2 + \frac{2t_1 t_2}{\sqrt{nDY_n}}\sum_{j=1}^{n}(n - j + 1) + \sum_{j=1}^{n}\frac{t_2^2}{n} \sim \sigma \sim t_1^2 + 2\rho t_1 t_2 + t_2^2$$

where $\rho = \frac{1}{\sqrt{nDY_n}}\sum_{j=1}^{n}(n - j + 1) = \frac{1}{\sqrt{nDY_n}}\left(\frac{n^2+n}{2}\right) \sim \frac{\sqrt{3}}{2}$.

We have from (9) and (12)

$$\sigma_c^2 = n^{-3}DY_n \sim \frac{1}{2|c|^3}\left[2|c| - 3 + 4e^{-|c|} - e^{-2|c|}\right], \quad \delta_c = \frac{1}{c^2\sigma_c}\left(e^{-|c|} + |c| - 1\right).$$

Let us calculate the following limits

$$\lim_{c\to 0}\sigma_c^2=\frac{1}{3},\quad \lim_{c\to 0}\delta_c=\frac{\sqrt{3}}{2}.$$

$$L_n=\frac{1}{\sigma^2}\sum_{j=1}^{n}\int_{|x|\geq\frac{\varepsilon\sigma}{\left|\gamma_{n,j}(t_1,t_2)\right|}}\gamma_{n,j}^2\left(t_1,t_2\right)x^2dF(x)\leq$$

$$\leq\frac{1}{\sigma^2}\sum_{j=1}^{n}\int_{|x|\geq\frac{\varepsilon\sigma}{\left|\gamma_{n,j}(t_1,t_2)\right|}}x^2dF_{\varepsilon_j}(x)\leq\int_{|x|\geq\frac{\varepsilon\sigma}{|\gamma_n(t_1,t_2)|}}x^2dF(x).$$

Now let's check the Lindeberg condition,
where $|\gamma_{n,k}\left(t_1,t_2\right)|\leq\gamma_n\left(t_1,t_2\right)=\frac{t_1n}{\sqrt{DY_n}}+\frac{t_2}{\sqrt{n}}\sim 0.$
From here

$$L_n\leq\int_{|x|\geq\frac{\varepsilon\sigma}{\left|\gamma_{n,k}(t_1,t_2)\right|}}x^2dF(x)\sim 0$$

Thus, for each, $|c|<\infty$ the statement of the theorem of the first case is fulfilled. The proof of the second and third cases is carried out similarly to the proof of the first case.

□

First-order spatial autoregressive model with one parameter [3].
Autoregressive process $\{X_{k,l}:k,l\in Z_+\}$under consideration is then defined as follows:

$$X_{k,l}=\begin{cases}\alpha\left(X_{k-1,l}+X_{k,l-1}\right)+\varepsilon_{k,l}, & \text{if } k,l\geq 1\\ 0, & \text{in other cases.}\end{cases}\tag{15}$$

This model is stable (asymptotically stationary) if $|\alpha|<1/2$ and unstable if $|\alpha|=1/2$.

Let's consider a rectangle

$$R_{m,n}:=\left\{(k,l)\in N^2:1\leq k\leq m\text{ and }1\leq l\leq n\right\}$$

squares $\hat{\alpha}_{m,n}$ estimator, based on observations $\{X_{k,l}:(k,l)\in R_{m,n}\}$, minimizes the sum of squares

$$\sum_{(k,l)\in R_{m,n}}\left(X_{k,l}-\alpha\left(X_{k-1,l}+X_{k,l-1}\right)\right)^2$$

and has the form

$$\hat{\alpha}_{m,n}=\frac{\sum_{(k,l)\in R_{m,n}}\left(X_{k-1,l}+X_{k,l-1}\right)X_{k,l}}{\sum_{(k,l)\in R_{m,n}}\left(X_{k-1,l}+X_{k,l-1}\right)^2}$$

Regarding the asymptotic behavior of this estimate, the following statement holds.

Theorem 0.3 ([14]). *Let $\{\varepsilon_{k,l} : k,l \in N\}$ be i.i.d. random variables such that*

$$\sup\left\{E\varepsilon_{k,l}^4 : k,l \in N\right\} < \infty.$$

If $|\alpha| < \frac{1}{2}$, then as $m,n \to \infty$ with $m/n \to const > 0$, we have

$$(mn)^{1/2}\left(\hat{\alpha}_{m,n} - \alpha\right) \xrightarrow{D} N\left(0, \sigma_\alpha^2\right),$$

where

$$\sigma_\alpha^2 := \begin{cases} \dfrac{\alpha^2}{\left(1-4\alpha^2\right)^{-1/2} - 1}, & \text{if } \alpha \neq 0, \\ \dfrac{1}{2}, & \text{if } \alpha = 0. \end{cases}$$

If $|\alpha| = \frac{1}{2}$, then as $m,n \to \infty$ with $m/n \to const > 0$, we have

$$(mn)^{5/8}\left(\hat{\alpha}_{m,n} - \alpha\right) \xrightarrow{D} N\left(0, \sigma^2\right),$$

where

$$\sigma^2 = \frac{15\sqrt{\pi}}{32\left(2^{5/2}+3\right)}.$$

Note that $\sigma_0^2 = \lim_{\alpha\to 0}\sigma_\alpha^2$, $\quad \sigma^2 \neq \lim_{\alpha \to 1/2}\sigma_\alpha^2 = 0$.

Further, an estimate of a simpler structure is proposed, using only a portion of the observations located along the diagonal in the rectangle. $R_{m,n}$. This approach with a significant reduction in the number of observations is relevant in geology problems and some other areas of application of spatial autoregressive models. A simpler estimate structure also allows for a reduction in momentary noise restrictions.

For simplicity, $m = n$. let us construct the estimate based on observations located along the diagonal of the square: $R_{n,n} = \{X_{k,l} : 1 \leq k \leq n, 1 \leq l \leq n\}$. In accordance with (15), these observations satisfy the following recurrence relation:

$$X_{k,k} = \begin{cases} \alpha\left(X_{k-1,k} + X_{k,k-1}\right) + \varepsilon_{k,k}, & \text{if} \quad k \geq 1, \\ 0, \quad \text{in other cases.} \end{cases} \tag{16}$$

Summing (16) over k from 1 to n, we obtain the estimation equation

$$\sum_{k=1}^{n} X_{k,k} = \alpha\sum_{k=1}^{n}\left(X_{k-1,k} + X_{k,k-1}\right) + \sum_{k=1}^{n}\varepsilon_{k,k}$$

dividing both parts of the resulting equation by the coefficient, α we obtain

$$\alpha_{n,n}^* - \alpha = Z_{n,n}/Y_{n,n} \tag{17}$$

where $Z_{n,n} = \sum_{k=1}^{n}\varepsilon_{k,k}$, $\quad Y_{n,n} = \sum_{k=1}^{n}\left(X_{k-1,k} + X_{k,k-1}\right)$,
$\alpha_{n,n}^* = \mathrm{X}_{n,n}/Y_{n,n}$ - proposed estimate of parameter α, a $\mathrm{X}_{n,n} = \sum_{k=1}^{n} X_{k,k}$.

The following statement holds for the constructed estimate.

Theorem 0.4. *Let $\{\varepsilon_{i,j} : i,j \geq 1\}$ be i.i.d. random variables [5] with*

$$E\left(\varepsilon_{i,j}\right) = 0, \quad E\left(\varepsilon_{i,j}^2\right) = 1.$$

(1) *In the critical case* $\alpha = \pm\frac{1}{2}$, *as* $n \to \infty$, *we have*

$$2\sigma^{-1}n^{3/4}\left(\alpha^*_{n,n} - \alpha\right) \Rightarrow \frac{\xi_1}{\xi_2}\, sign(\alpha),$$

where $\sigma^2 = \dfrac{16}{15\sqrt{\pi}}(7 + 8\sqrt{2})$, *and* ξ_1 *and* ξ_2 *are independent random variables with the standard normal distribution.*

(2) *If* $\alpha(n) = \frac{1}{2} - \dfrac{c}{n}$ *with* $c > 0$, *then as* $n \to \infty$,

$$P\left(\sigma_c^{-1}n^{3/4}\left(\alpha^*_{n,n} - \alpha\right) < x\right) \Rightarrow \frac{1}{\pi}\int_{-\infty}^{x} \frac{du}{1+u^2},$$

where $\sigma_c^2 = \dfrac{2}{\sqrt{c}}$.

Thus, the proposed estimates have a limit distribution of the Cauchy type, but the moment restrictions are brought to the second moment, in comparison with theorem, where the existence of the 4th moment is required, but with a normal limit law.

Conclusion

For the first-order ordinary autoregressive model, $Y_k = \alpha Y_{k-1} + \varepsilon_k$ the limiting distribution of non-standard parameter estimates was found α in the cases $\alpha = 1 - c/n$ and $\alpha = -1 + c/n$ [13],[7],[5].

The constructed estimates have, as a rule, simpler limit distributions in comparison with traditional least squares estimates. At the same time, the simpler structure of the proposed estimates also allows us to reduce the moment restrictions on the "noise" that define the stochastic structure of the models.

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