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EXISTENCE OF SOLUTIONS FOR MIXED TYPE FRACTIONAL DIFFERENTIAL EQUATIONS DEPENDING ON MIXED BOUNDARY CONDITIONS

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Abstract. This paper investigates a class of mixed non-linear fractional boundary value problems where the derivative is taken in the conformable sense. The primary contribution of our work is the application of the fixed point theorem of Benmezai et al. to this conformable fractional framework. By doing so, we establish novel existence and uniqueness results for solutions to this broader class of problems, thereby extending the applicability of recent theoretical tools to modern fractional derivative models.

Keywords. Conformable fractional derivative,Riemann-Liouville fractional derivative,fixed point theorems,boundary value problem.
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1. Introduction and preliminaries

2. Introduction

This paper investigates a fractional differential equation with nonlinear terms (NFDE) of the type

$$\mathfrak{D}_{1^-}^{(\beta)}\big({}^C\mathfrak{D}_{0+}^{\alpha}y\big)(s) = g(s, y(s)), \qquad 0 < s < 1, \tag{2.1}$$

accompanied by the boundary conditions

$$\begin{cases} y(0) = \gamma \displaystyle\int_0^1 y(s)\, ds, \\ {}^C\mathfrak{D}_{0+}^{\alpha}y(1) = 0, \end{cases} \tag{2.2}$$

where the parameters satisfy $\alpha, \beta \in (0, 1]$ and $\gamma \in (0, 1)$. Here $\mathfrak{D}_{1^-}^{(\beta)}$ represents the conformable fractional derivative of the right type, while ${}^C\mathfrak{D}_{0+}^{\alpha}$ denotes the left Caputo fractional derivative of order α. The function $g : [0, 1] \times [0, \infty) \to [0, \infty)$ is assumed continuous.

Fractional differential equations have attracted considerable research attention due to their ability to capture memory effects in a wide range of scientific and

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engineering applications, including control theory, anomalous transport, diffusion phenomena, and the modelling of biological and physical processes [5, 10, 16, 18, 20, 22].

Over recent years, boundary value problems involving both left and right fractional derivatives have become an active field of study. A substantial number of works have examined mixed fractional derivatives of various types; for representative contributions, see [12, 24]. The analysis of nonlinear fractional boundary value problems continues to generate significant interest, with many studies focusing on the existence of solutions or mild solutions [3, 4, 7, 9, 25]. As for the question of nonexistence of solutions for fractional boundary value problems though not specifically in the context of nonlinear mixed fractional differential boundary value problems the reader may refer to [11, 13, 21, 23].

A notable contribution by Khalil et al. [14] introduced a fractional derivative known as the conformable derivative, which closely mirrors the classical derivative and satisfies standard properties such as the chain rule. A natural and significant question arises: can the nonexistence of solutions for problem (2.1)-(2.2) be established under weaker assumptions? The present work answers this question in the affirmative, thereby addressing an open issue and extending the existing body of literature.

As far as we are aware, no prior study has examined fractional differential equations involving both left and right fractional derivatives through the lens of strongly positive-like operators, subsequently applying the fixed point theorem of Benmezai et al. [2] to the conformable and left Caputo derivatives appearing in our mixed boundary value problem. The only related works are those of [2], which treat Riemann–Liouville derivatives under local and nonlocal conditions. By filling this void, our investigation not only advances the field but also supplements earlier research.

Inspired by the aforementioned studies, the aim of this paper is to establish both existence and nonexistence results for solutions of the nonlinear mixed fractional nonlocal boundary value problem (2.1)-(2.2).

The paper is organized in the following manner. Section 2 is dedicated to recalling essential notation, definitions, and preliminary facts. Section 3 provides the proofs of our main existence and nonexistence results for problem (2.1)-(2.2).

3. Preliminaries

In the first section, we now review some essential definitions and preliminary concepts that will be utilized in the subsequent analysis.

Definition 3.1 ([8])**.** Let $\mathfrak{E}$ be a real Banach space. A nonempty closed subset $\mathcal{G} \subset \mathfrak{E}$ is called a cone if it satisfies the following properties:

(i) For any nonnegative scalars $c_1, c_2 \geq 0$ and any elements $y, \chi \in \mathcal{G}$, the combination $c_1 y + c_2 \chi$ belongs to $\mathcal{G}$;

(ii) Whenever both y and $-y$ are in $\mathcal{G}$, then necessarily $y = 0$.

The cone $\mathcal{G}$ induces a partial order on $\mathfrak{E}$ by defining $y \leq \chi$ if and only if $\chi - y \in \mathcal{G}$.

Definition 3.2 ([16, 20]). Let $\alpha \in (0,1]$. The left and right Riemann-Liouville fractional integrals of a function y are respectively defined by

$$I_{a^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)\, ds, \qquad I_{b^-}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} y(s)\, ds,$$

where $\Gamma(\cdot)$ is called the Euler gamma function, given by the integral representation

$$\Gamma(\alpha) = \int_0^{\infty} \tau^{\alpha-1} e^{-\tau}\, d\tau,$$

and it fulfills $\frac{1}{\Gamma(\alpha)} = 0$ for α belonging to the set $\{0,-1,-2,\ldots\}$ (i.e., at nonpositive integers).

Definition 3.3 ([16]). Let $y \in AC([0,\infty),\mathbb{R})$. The Caputo fractional derivative of order $0 < \varpi \le 1$ on the left side for a function $y : [a,b] \to \mathbb{R}$ is defined by

$${}^{C}\mathfrak{D}_{a^+}^{\varpi} y(t) = \frac{1}{\Gamma(1-\varpi)} \int_a^t (t-s)^{-\varpi} y'(s)\, ds.$$

Similarly, the Caputo fractional derivative of order $0 < \varpi \le 1$on the right side is terminating at b is defined as

$${}^{C}\mathfrak{D}_{b^-}^{\varpi} y(t) = \frac{1}{\Gamma(1-\varpi)} \int_t^b (s-t)^{-\varpi} y'(s)\, ds.$$

Definition 3.4 ([1, 14]). Let $y : [a,\infty) \to \mathbb{R}$ be a given function and let $0 < \beta \le 1$. The left conformable fractional derivative of order β starting from a is defined by

$$\big(\mathfrak{D}_{a^+}^{(\beta)} y\big)(t) = \lim_{\varepsilon \to 0} \frac{y\big(t + \varepsilon (t-a)^{1-\beta}\big) - y(t)}{\varepsilon}, \qquad t > a.$$

If this derivative exists on (a,b), then we set $\big(\mathfrak{D}_{a^+}^{(\beta)} y\big)(a) = \lim_{t \to a^+} \big(\mathfrak{D}_{a^+}^{(\beta)} y\big)(t)$.

The right-hand version of the conformable fractional derivative of order β terminating at b is defined by

$$\big(\mathfrak{D}_{b^-}^{(\beta)} y\big)(t) = -\lim_{\varepsilon \to 0} \frac{y\big(t + \varepsilon (b-t)^{1-\beta}\big) - y(t)}{\varepsilon}, \qquad t < b.$$

When this derivative exists on (a,b), we define $\big(\mathfrak{D}_{b^-}^{(\beta)} y\big)(b) = \lim_{t \to b^-} \big(\mathfrak{D}_{b^-}^{(\beta)} y\big)(t)$.

The corresponding left and right conformable fractional integrals of order β are given by

$$J_{a^+}^{\beta} y(t) = \int_a^t (s-a)^{\beta-1} y(s)\, ds, \qquad J_{b^-}^{\beta} y(t) = \int_t^b (b-s)^{\beta-1} y(s)\, ds.$$

Lemma 3.5 ([1, 16, 14]). *We have the following properties for the conformable and Caputo fractional operators:*

(i) *For a continuous function y on (a,b), we have*

$$\mathfrak{D}_{b^-}^{(\beta)}\big(J_{b^-}^{\beta} y(t)\big) = \mathfrak{D}_{a^+}^{(\beta)}\big(J_{a^+}^{\beta} y(t)\big) = {}^{C}\mathfrak{D}_{b^-}^{\alpha}\big(I_{b^-}^{\alpha} y(t)\big) = {}^{C}\mathfrak{D}_{a^+}^{\alpha}\big(I_{a^+}^{\alpha} y(t)\big) = y(t).$$

(ii) *If* $\mathfrak{D}_{b^-}^{(\beta)}y$, $\mathfrak{D}_{a^+}^{(\beta)}y$, ${}^C\mathfrak{D}_{b^-}^{\alpha}y$, *and* ${}^C\mathfrak{D}_{a^+}^{\alpha}y$ *are continuous on* (a,b), *then*

$$I_{b^-}^{\beta}\big(\mathfrak{D}_{b^-}^{(\beta)}y(t)\big) = J_{b^-}^{\alpha}\big({}^C\mathfrak{D}_{b^-}^{\alpha}y(t)\big) = y(t) - y(b),$$

$$I_{a^+}^{\beta}\big(\mathfrak{D}_{a^+}^{(\beta)}y(t)\big) = J_{a^+}^{\alpha}\big({}^C\mathfrak{D}_{a^+}^{\alpha}y(t)\big) = y(t) - y(a).$$

(iii) *Provided* y *is differentiable on* (a,b), *we obtain*

$$\mathfrak{D}_{b^-}^{(\beta)}y(t) = -(b-t)^{1-\beta}y'(t), \qquad \mathfrak{D}_{a^+}^{(\beta)}y(t) = (t-a)^{1-\beta}y'(t).$$

Definition 3.6 ([20])**.** Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$. Define C_α^n as the class of functions $h(t)$ for $t > 0$ satisfying $h(t) = t^p h_1(t)$, where $p > \alpha$ and $h_1 \in C^n([0,\infty))$.

Definition 3.7 ([2])**.** Let $L_C(\mathfrak{E})$ denote the collection of all linear compact operators mapping a Banach space $\mathfrak{E}$ into itself.The operator $L \in L_C(\mathfrak{E})$ is called positive if $L(\mathcal{G}) \subset \mathcal{G}$, where $\mathcal{G} \subset \mathfrak{E}$ is a cone. It is said to be strongly positive whenever the interior of $\mathcal{G}$ is nonempty and $L(\mathcal{G} \setminus \{0\}) \subset \operatorname{int}(\mathcal{G})$.

Definition 3.8 ([2])**.** Let $\mathfrak{L} \in L_C(\mathfrak{E})$ be a positive operator. Then $\mathfrak{L}$ is termed lower bounded whenever

$$\inf\big\{\|\mathfrak{L}y\| : y \in \mathcal{G} \cap \partial B(0,1)\big\} > 0,$$

where $\partial B(0,1)$ denotes the boundary of the open unit ball in $\mathfrak{E}$.

For any positive operator $\mathfrak{L} \in L_C(\mathfrak{E})$, we introduce the following sets:

$$\Lambda_{\mathfrak{L}} = \big\{\lambda \geq 0 : \exists y > 0_{\mathfrak{E}} \text{ such that } \mathfrak{L}y \geq \lambda y\big\},$$

and

$$\Gamma_{\mathfrak{L}} = \big\{\lambda \geq 0 : \exists y > 0_{\mathfrak{E}} \text{ such that } \mathfrak{L}y \leq \lambda y\big\}.$$

Lemma 3.9 ([2])**.** *If* $\mathfrak{L} \in L_C(\mathfrak{E})$ *is strongly positive, then its spectral radius satisfies*

$$r(\mathfrak{L}) = \sup \Lambda_{\mathfrak{L}} = \inf \Gamma_{L\mathfrak{L}}.$$

Definition 3.10. A positive operator $\mathfrak{L} \in L_C(\mathfrak{E})$ is called a strongly positive-like operator whenever

$$r(\mathfrak{L}) = \sup \Lambda_{\mathfrak{L}} = \inf \Gamma_{\mathfrak{L}} > 0.$$

The results presented below, due to Benmezai, Chentout, and Henderson [2], form the cornerstone upon which our main findings are built. The first theorem concerns the existence of positive fixed points, while the second addresses the question of nonexistence.

Theorem 3.11. *Let* $T : \mathcal{G} \to \mathcal{G}$ *is a completely continuous mapping. Suppose there exist two operators that are strongly positive in a similar sense* $L_1, L_2 \in L_C(\mathfrak{E})$ *accompanied by two functions* $F_1, F_2 : \mathcal{G} \to \mathcal{G}$ *satisfying the following conditions:*

- L_1 *is the lower bounded on* $\mathcal{G}$*;*
- $r(L_1) < 1 < r(L_2)$*;*
- *For every* $y \in \mathcal{G}$, *we have*

$$L_1y - F_1y \leq Ty \leq L_2y - F_2y,$$

and moreover $Ty \geq Ly$ *for all* $y \in \mathcal{G}$.

If either

$$F_1 y = o(\|y\|) \text{ as } \|y\| \to \infty \quad \text{and} \quad F_2 y = o(\|y\|) \text{ as } \|y\| \to 0,$$

or

$$F_1 y = o(\|y\|) \text{ as } \|y\| \to 0 \quad \text{and} \quad F_2 y = o(\|y\|) \text{ as } \|y\| \to \infty,$$

then T possesses at least one fixed point in $\mathcal{G}$.

Theorem 3.12. *Let $T : \mathcal{G} \to \mathcal{G}$ be a continuous mapping and suppose $L \in L_C(\mathfrak{E})$ is an operator of strongly positive-like type. If either*

$$r(L) > 1 \quad \text{and} \quad Ty \geq Ly \text{ for all } y \in \mathcal{G},$$

or

$$r(L) < 1 \quad \text{and} \quad Ty \leq Ly \text{ for all } y \in \mathcal{G},$$

then T admits no fixed points in $\mathcal{G}$.

4. Existence and nonexistence results

Let $\mathfrak{E} = C([0,1])$ denote the Banach space equipped with the usual supremum norm $\|y\| = \sup_{t\in[0,1]} |y(t)|$. We further introduce the Banach space X defined by

$$X = \left\{ y \in \mathfrak{E} : \lim_{t\to 0} \frac{y(t)}{t^\alpha} \text{ exists} \right\},$$

endowed with the norm

$$\|y\|_X = \sup_{t\in[0,1]} \left| \frac{y(t)}{t^\alpha} \right|.$$

Fix a constant η with $0 < \eta < 1$. Consider the following cones:

$$\mathfrak{E}_+ = \{y \in \mathfrak{E} : y(t) \geq 0,\ t \in [0,1]\},$$
$$\mathcal{G} = \{y \in \mathfrak{E} : y(t) \geq \eta^\alpha \|y\|_0,\ t \in (\eta, 1)\},$$

and

$$X_+ = \{y \in X : y(t) \geq 0,\ t \in [0,1]\}.$$

We also define the sets

$$\mathcal{L}^1_+ = \left\{ m \in L^1(0,1) : m(t) \geq 0 \text{ a.e. on } [0,1] \right\},$$
$$\mathcal{L}^1_{++} = \left\{ m \in L^1(0,1) : m > 0 \text{ on a subset of positive measure} \right\}.$$

Additionally, we introduce the subset $S \subset X$ given by

$$S = \left\{ y \in X : y(t) > 0 \ \forall t \in [0,1] \text{ and } \lim_{t\to 0} \frac{y(t)}{t^\alpha} > 0 \right\}.$$

To establish our main results, we require the following preliminary lemmas.

Lemma 4.1 ([6])**.** *Let $\alpha, \beta \in (0,1]$, $\gamma \in (0,1)$ and suppose $h \in C[0,1]$. Then the boundary value problem*

$$\begin{cases} y(0) = \gamma \displaystyle\int_0^1 y(s)\, ds, \\ {}^C\mathfrak{D}^\alpha_{0+} y(1) = 0, \\ \mathfrak{D}^{(\beta)}_{1-}\big({}^C\mathfrak{D}^\alpha_{0+} y\big)(s) = h(s), \quad 0 < s < 1, \end{cases} \tag{4.1}$$

has a unique solution $y \in \mathfrak{E}$ given by

$$y(s) = \int_0^1 G(s,r)h(r)\,dr, \tag{4.2}$$

where $G(s,r)$ is the Green's function defined by

$$G(s,r) = \begin{cases} \dfrac{\gamma(1-r)^{\beta-1}}{(1-\gamma)(\alpha+1)\Gamma(\alpha+1)}\Big[1-(1-r)^{\alpha+1}+s^{\alpha}-(s-r)^{\alpha}\Big], & 0 \le r \le s \le 1, \\ \dfrac{\gamma(1-r)^{\beta-1}}{(1-\gamma)(\alpha+1)\Gamma(\alpha+1)}\Big[1-(1-r)^{\alpha+1}+s^{\alpha}\Big], & 0 \le s \le r \le 1. \end{cases} \tag{4.3}$$

Remark 4.2. Observe that from (4.2), the Green's function can be expressed as $G(s,r) = s^{\alpha}\chi(s,r)$, where

$$\chi(s,r) = \begin{cases} \dfrac{\gamma(1-r)^{\beta-1}}{(1-\gamma)(\alpha+1)\Gamma(\alpha+1)}\left[\dfrac{1-(1-r)^{\alpha+1}}{s^{\alpha}}+1-\left(1-\dfrac{r}{s}\right)^{\alpha}\right], & 0 \le r \le s \le 1, \\ \dfrac{\gamma(1-r)^{\beta-1}}{(1-\gamma)(\alpha+1)\Gamma(\alpha+1)}\left[\dfrac{1-(1-r)^{\alpha+1}}{s^{\alpha}}+1\right], & 0 \le s \le r \le 1. \end{cases}$$

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$$\chi(s,r) = \begin{cases} \dfrac{\gamma(1-r)^{\beta-1}}{(1-\gamma)(\alpha+1)\Gamma(\alpha+1)}\left[\dfrac{1-(1-r)^{\alpha+1}}{s^{\alpha}}+1-\left(1-\dfrac{r}{s}\right)^{\alpha}\right], & 0 \le r \le s \le 1, \\ \dfrac{\gamma(1-r)^{\beta-1}}{(1-\gamma)(\alpha+1)\Gamma(\alpha+1)}\left[\dfrac{1-(1-r)^{\alpha+1}}{s^{\alpha}}+1\right], & 0 \le s \le r \le 1. \end{cases}$$

The next result establishes fundamental sign properties of the functions G and χ. A similar argument can be found in [6].

Lemma 4.4. *The Green's function $G(s,r)$ possesses the following properties:*

(i) *$G(s,r)$ is continuous on $[0,\infty) \times [0,\infty)$.*
(ii) *$G(s,r) > 0$ for all $(s,r) \in (0,1] \times (0,1)$, and $\chi(0,r) > 0$ for every $r \in [0,1)$.*
(iii) *For $(s,r) \in (0,1] \times [0,1)$, the inequalities $s^{\alpha}G(1,r) \le G(s,r) \le G(1,r)$ hold.*
(iv) *For any $\eta > 0$ and all $s \in [\eta,1]$, $r \in [0,1)$, we have $G(s,r) \ge \eta^{\alpha}G(1,r)$.*

The following result is taken from [2].

Lemma 4.5. *The set S is open in the space X.*

Let $m \in \mathcal{L}^1_{++}$. We define an operator $L_m : \mathfrak{E} \to \mathfrak{E}$ by

$$(L_m y)(s) = \int_0^1 G(s,r)m(r)y(r)\,dr, \quad s \in [0,1], \tag{4.4}$$

and denote by $L_m^X : X \to \mathfrak{E}$ its restriction given by $L_m^X y = L_m y$.

Lemma 4.6. *For any $m \in \mathcal{L}^1_{++}$, the operator L_m is compact and positive. Moreover, L_m maps $\mathfrak{E}_+$ into $\mathcal{G}$.*

Proof. Compactness of L_m follows from standard arguments. Take $y \in \mathfrak{E}_+$, so that $y(s) \geq 0$ for all $s \in [0,1]$. Since $m > 0$ almost everywhere on $[0,1]$, Lemma 4.4(ii) yields

$$(L_m y)(s) = \int_0^1 G(s,r)m(r)y(r)\,dr \geq 0,$$

hence $L_m y \in \mathfrak{E}_+$ and consequently $L_m : \mathfrak{E}_+ \to \mathfrak{E}_+$. Furthermore, employing Lemma 4.4(iv), we obtain

$$\|L_m y\| = |(L_m y)(1)|,$$

and

$$(L_m y)(s) = \int_0^1 G(s,r)m(r)y(r)\,dr \geq \eta^\alpha \int_0^1 G(1,r)m(r)y(r)\,dr = \eta^\alpha \|L_m y\|.$$

Thus $L_m y \in \mathcal{G}$, establishing $L_m : \mathfrak{E}_+ \to \mathcal{G}$. $\square$

Lemma 4.7. *Let $m \in \mathcal{L}^1_{++}$. Then L_m is a strongly positive-like operator and is lower bounded on the cone $\mathcal{G}$.*

Proof. Consider first $m \in \mathcal{L}^1_+ \cap C[0,1]$. By the Arzelà–Ascoli theorem and an argument similar to that in [2, 3], the operator L^X_m is compact. Now take $y \in X_+ \setminus \{0\}$. For any $s \in [0,1]$, Lemma 4.4 gives

$$(L^X_m y)(s) = \int_0^1 G(s,r)m(r)y(r)\,dr > 0,$$

and moreover

$$\lim_{s \to 0} \frac{(L^X_m y)(s)}{s^\alpha} = \int_0^\infty \chi(0,r)m(r)y(r)\,dr > 0.$$

Hence L^X_m maps $X \setminus \{0\}$ into $S \subset X_+$, showing that L^X_m is strongly positive. Lemma 2.6 then implies

$$r(L^X_m) = \sup \Lambda_{L^X_m} = \inf \Gamma_{L^X_m}.$$

Since L^X_m is essentially the restriction of L_m to X, we have the inclusions $\Lambda_{L^X_m} \subset \Lambda_{L_m}$ and $\Gamma_{L^X_m} \subset \Gamma_{L_m}$.

Now let $y \in \mathfrak{E}_+ \setminus \{0\}$ satisfy $L_m y \geq \lambda y$. A similar reasoning as above shows that $z = L_m y$ belongs to $X_+ \setminus \{0\}$. Then

$$L^X_m z = L^X_m(L_m y) = L_m(L_m y) \geq \lambda L_m y = \lambda z,$$

so $\lambda \in \Lambda_{L^X_m}$. Consequently $\Lambda_{L^X_m} = \Lambda_{L_m}$, and analogously $\Gamma_{L^X_m} = \Gamma_{L_m}$. Therefore

$$r(L_m) = \sup \Lambda_{L_m} = \inf \Gamma_{L_m},$$

which means L_m is a strongly positive-like operator.

Finally, for any $y \in \mathcal{G}$,

$$\|L_m y\| = (L_m y)(1) = \int_0^1 G(1,r)m(r)y(r)\,dr \geq \eta^\alpha \int_0^1 G(1,r)m(r)\eta^\alpha\,dr\,\|y\|.$$

Thus L_m is lower bounded on $\mathcal{G}$. $\square$

Define the operator $T : \mathfrak{E}_+ \to \mathfrak{E}$ by

$$(Ty)(s) = \int_0^1 G(s,r)g(r,y(r))\,dr. \tag{4.5}$$

Lemma 4.8. *The operator $T : \mathfrak{E}_+ \to \mathfrak{E}$ is compact and satisfies $T : \mathfrak{E}_+ \to \mathcal{G}$.*

Proof. Compactness of T follows from the Arzelà–Ascoli theorem. Take $y \in \mathfrak{E}_+$. By Lemma 4.3(ii)–(iii),

$$(Ty)(s) = \int_0^1 G(s,r)g(r,y(r))\,dr \geq 0.$$

Since $\|Ty\| = (Ty)(1)$, we obtain

$$\|Ty\| = (Ty)(1) = \int_0^1 G(1,r)g(r,y(r))\,dr \geq \eta^\alpha \int_0^1 G(1,r)g(r,y(r))\,dr = \eta^\alpha \|y\|.$$

Hence $Ty \in \mathcal{G}$, completing the proof. □

Consider the linear boundary value problem, for $m \in \mathcal{L}^1_{++}$,

$$\mathfrak{D}^\alpha_{0+} y(s) + \mu\, m(s)y(s) = 0, \quad \text{a.e. } s \in (0,1), \tag{4.6}$$

together with the boundary conditions (2.2), where μ is a real parameter.

Lemma 4.9. *For every $m \in \mathcal{L}^1_{++}$, problem (4.6)–(2.2) possesses a unique positive eigenvalue $\mu(m)$.*

Proof. Observe that (μ, y) solves (4.6)–(2.2) if and only if $L_m y = \mu^{-1} y$. Lemma 4.6 ensures that $\mu^{-1} = r(L_m)$ is the unique positive eigenvalue of L_m. Consequently, $\mu_\alpha(m) = \frac{1}{r(L_m)}$ is the unique positive eigenvalue of (4.6)–(2.2). □

Theorem 4.10. *Assume there exists $m \in \mathcal{L}^1_+$ such that one of the following conditions holds:*

$$\mu_\alpha(m) < 1 \quad \textit{and} \quad g(s,y) \geq m(s)y, \quad \textit{for all } y \geq 0 \textit{ and a.e. } s \in [0,1], \tag{4.7}$$

$$\mu_\alpha(m) > 1 \quad \textit{and} \quad g(s,y) \leq m(s)y, \quad \textit{for all } y \geq 0 \textit{ and a.e. } s \in [0,1]. \tag{4.8}$$

Then problem (2.1)-(2.2) admits no positive solutions.

Proof. Let $y \in \mathcal{G}$ and suppose that condition (4.7) is satisfied. Then we have $g(s,y) \geq m(s)y$, which immediately yields $Ty \geq L_m y$. Since L_m is a strongly positive-like operator with spectral radius $r(L_m) = 1/\mu_\alpha(m) > 1$, Theorem 2.9 is thus applicable, leading to the conclusion that T has no positive fixed points. The case where (4.8) holds can be treated analogously. □

Theorem 4.11. *Assume there exist functions $m_1, m_2 \in \mathcal{L}^1_{++}$, $q_1, q_2 \in \mathcal{L}^1_+$, and $\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ satisfying $\mu_\alpha(m_1) < 1 < \mu_\alpha(m_2)$ and such that for all $y \geq 0$ and almost every $s \in [0,1]$,*

$$m_1(s)y - q_1(s)\psi_1(y) \leq g(s,y) \leq m_2(s)y + q_2(s)\psi_2(y). \tag{4.9}$$

Furthermore, suppose that either

(H1) *$\psi_1(y) = o(\|y\|)$ as $\|y\| \to \infty$, $\psi_2(y) = o(\|y\|)$ as $\|y\| \to 0$, with ψ_1 nondecreasing and ψ_2 nondecreasing near 0;*

(H2) *$\psi_1(y) = o(\|y\|)$ as $\|y\| \to 0$, $\psi_2(y) = o(\|y\|)$ as $\|y\| \to \infty$, with ψ_1 nondecreasing near 0 and ψ_2 nondecreasing.*

Then problem (2.1)–(2.2) admits at least one positive solution.

Proof. For $i \in \{1, 2\}$, define the operators $F_i : \mathcal{G} \to \mathcal{G}$ by

$$(F_i y)(s) = \int_0^1 G(s, r)\psi_i(y(r))\, dr.$$

From (4.9) we obtain the inequality

$$L_{m_1} y - F_1 y \leq Ty \leq L_{m_2} y + F_2 y,$$

where the spectral radii satisfy

$$r(L_{m_2}) = \frac{1}{\mu_\alpha(m_2)} < 1 < r(L_{m_1}) = \frac{1}{\mu_\alpha(m_1)}.$$

Now consider hypothesis (H1). We have

$$\frac{\|F_i y\|_\infty}{\|y\|_\infty} \leq \int_0^1 G(1, r) q_i(r) \frac{\psi_i(y(r))}{\|y\|_\infty}\, dr \leq \int_0^1 G(1, r) q_i(r)\, dr,$$

which yields

$$F_1 y = o(\|y\|) \quad \text{as } \|y\| \to \infty, \qquad F_2 y = o(\|y\|) \quad \text{as } \|y\| \to 0.$$

Consequently, all conditions of Theorem 2.8 are fulfilled, guaranteeing that T possesses a fixed point. This fixed point is precisely a positive solution of (2.1)–(2.2). The argument under hypothesis (H2) proceeds in a completely analogous manner. □

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