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EFFICIENT TAYLOR-BASED SOLVER FOR PROPORTIONAL-DELAY VOLTERRA EQUATIONS

CHAHINAZ HENNOUS[1] and HAFIDA LAIB[2*]

Abstract. This study introduces a Taylor Collocation Method for Volterra integral equations with proportional delay. The proposed method employs Taylor polynomials to approximate the solution through collocation at selected nodes, achieving high-order accuracy for smooth problems. A rigorous convergence analysis establishes the theoretical foundation of the method. Numerical experiments evaluate performance against existing schemes, validating the Taylor Collocation Method as an efficient and accurate technique for solving proportional-delay integral equations.

Keywords. Delay integral equations, collocation method, Taylor polynomials, Volterra equations, proportional delay.
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1. Introduction

This paper investigates a numerical approach for solving Volterra integral equations with a proportional delay argument, expressed in the general form:

$$\Psi(\vartheta) = \Phi(\vartheta) + \int_0^{\vartheta} \mathcal{K}_1(\vartheta,\varsigma)\Psi(\varsigma)\,d\varsigma + \int_0^{\alpha\vartheta} \mathcal{K}_2(\vartheta,\varsigma)\Psi(\varsigma)\,d\varsigma, \quad \vartheta \in \mathcal{I} := [0,\Theta], \tag{1.1}$$

where $0 < \alpha < 1$ is the delay parameter. The functions $\Phi(\vartheta)$, $\mathcal{K}_1(\vartheta,\varsigma)$, and $\mathcal{K}_2(\vartheta,\varsigma)$ are assumed to be sufficiently smooth on their respective domains.

The theoretical questions concerning the existence and uniqueness of a solution for (1.1) are addressed in prior work, such as [9]. A notable special case of equation (1.1) is the delay integral equation, which is obtained by setting $\mathcal{K}_1 = -\mathcal{K}_2$.

Equations of this type provide a robust mathematical framework for modeling systems whose present state $\Psi(\vartheta)$ is influenced by its entire history and a specific past state $\Psi(\alpha\vartheta)$. Such models are indispensable across numerous scientific domains, including control theory with signal propagation delays [12], epidemiology accounting for incubation periods [5], population dynamics incorporating maturation times [10], and material science describing memory effects [15].

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* Corresponding author

The development of numerical methods for these equations remains a vibrant research area. Various techniques have been successfully applied, including Legendre spectral methods [1], collocation approaches [14], iterated collocation schemes [3, 4], least-squares collocation [6], multilevel correction methods [16], and Chebyshev spectral collocation [18]. For further examples of numerical methods applied to integral equations, the reader may refer to recent works published in this journal, such as [11] and [8].

This work is devoted to the development and analysis of a Taylor Collocation Method (TCM) for the class of equations represented by (1.1). Our approach approximates the unknown solution $\Psi(\vartheta)$ locally using Taylor polynomials, whose coefficients are determined by enforcing the equation at a set of collocation points. The proposed technique offers several distinct benefits, most notably its potential for high-order accuracy when the true solution is smooth, its conceptual simplicity that avoids complex mesh refinement near the proportional delay $\alpha\vartheta$, and its computational efficiency.

While Taylor series have been used in various numerical contexts, their application as a direct collocation technique for this specific class of proportional-delay Volterra equations remains relatively unexplored. This gap in the literature establishes the novelty of our contribution. We support the proposed method with a rigorous convergence analysis and demonstrate its practical performance through a series of numerical experiments.

2. Methodology

For a given positive integer $\mathcal{N}$, let $\mathcal{P}_{\mathcal{N}}$ be a uniform partition of the solution domain $\mathcal{I} = [0, \Theta]$ defined by the nodal points $\vartheta_\nu = \nu\Delta\vartheta$, for $\nu = 0, 1, \ldots, \mathcal{N}-1$, where the step size is $\Delta\vartheta = \Theta/\mathcal{N}$. We define the subintervals $\mathcal{I}_\nu = [\vartheta_\nu, \vartheta_{\nu+1})$ for $\nu = 0, 1, \ldots, \mathcal{N}-2$, and $\mathcal{I}_{\mathcal{N}-1} = [\vartheta_{\mathcal{N}-1}, \vartheta_{\mathcal{N}}]$.

Moreover, we denote by $\mathbb{P}_{\mathfrak{d}-1}$ the set of all real polynomials of degree not exceeding $\mathfrak{d}-1$. We define the piecewise polynomial space of degree $\mathfrak{d}-1$ with no continuity constraints at the partition points as follows:

$$\mathcal{S}_{\mathfrak{d}-1}^{(-1)}(\mathcal{P}_{\mathcal{N}}) = \{\mathfrak{w} : \mathfrak{w}_\nu = \mathfrak{w}|_{\mathcal{I}_\nu} \in \mathbb{P}_{\mathfrak{d}-1},\ \nu = 0, \ldots, \mathcal{N}-1\}.$$

It follows that $\mathcal{S}_{\mathfrak{d}-1}^{(-1)}(\mathcal{P}_{\mathcal{N}})$ is a real vector space of dimension $\mathcal{N}\mathfrak{d}$.

Our objective is to find an approximate solution $\mathfrak{w} \in \mathcal{S}_{\mathfrak{d}-1}^{(-1)}(\mathcal{P}_{\mathcal{N}})$ to the exact solution Ψ of (1.1). The coefficients of the local polynomials $\mathfrak{w}_\nu$, for $\nu = 0, \ldots, \mathcal{N}-1$, are determined using Taylor series expansions on each subinterval $\mathcal{I}_\nu$.

First, we approximate the exact solution Ψ on the initial interval $\mathcal{I}_0$ by the polynomial:

$$\mathfrak{w}_0(\vartheta) = \sum_{\jmath=0}^{\mathfrak{d}-1} \frac{\Psi^{(\jmath)}(0)}{\jmath!}\vartheta^\jmath, \quad \vartheta \in \mathcal{I}_0. \tag{2.1}$$

The derivatives $\Psi^{(\jmath)}(0)$ are found by differentiating the original equation (1.1) $\jmath$ times. Differentiating (1.1) yields:

$$\begin{aligned}\Psi^{(\jmath)}(\vartheta) = \Phi^{(\jmath)}(\vartheta) + \sum_{i=0}^{\jmath-1} \left[\partial_\vartheta^{(\jmath-1-i)} \mathcal{K}_1(\vartheta,\vartheta)\Psi(\vartheta)\right]^{(i)} + \int_0^\vartheta \partial_\vartheta^{(\jmath)} \mathcal{K}_1(\vartheta,\varsigma)\Psi(\varsigma)\,d\varsigma \\ + \alpha \sum_{i=0}^{\jmath-1} \left[\partial_\vartheta^{(\jmath-1-i)} \mathcal{K}_2(\vartheta,\alpha\vartheta)\Psi(\alpha\vartheta)\right]^{(i)} + \int_0^{\alpha\vartheta} \partial_\vartheta^{(\jmath)} \mathcal{K}_2(\vartheta,\varsigma)\Psi(\varsigma)\,d\varsigma.\end{aligned} \tag{2.2}$$

Applying the Leibniz rule to the terms involving products, we obtain:

$$\begin{aligned}\Psi^{(\jmath)}(\vartheta) = \Phi^{(\jmath)}(\vartheta) + \sum_{i=0}^{\jmath-1}\sum_{\ell=0}^{i} \binom{i}{\ell} \left[\partial_\vartheta^{(\jmath-1-i)} \mathcal{K}_1(\vartheta,\vartheta)\right]^{(i-\ell)} \Psi^{(\ell)}(\vartheta) \\ + \int_0^\vartheta \partial_\vartheta^{(\jmath)} \mathcal{K}_1(\vartheta,\varsigma)\Psi(\varsigma)\,d\varsigma \\ + \alpha \sum_{i=0}^{\jmath-1}\sum_{\ell=0}^{i} \binom{i}{\ell} \alpha^\ell \left[\partial_\vartheta^{(\jmath-1-i)} \mathcal{K}_2(\vartheta,\alpha\vartheta)\right]^{(i-\ell)} \Psi^{(\ell)}(\alpha\vartheta) \\ + \int_0^{\alpha\vartheta} \partial_\vartheta^{(\jmath)} \mathcal{K}_2(\vartheta,\varsigma)\Psi(\varsigma)\,d\varsigma.\end{aligned} \tag{2.3}$$

Evaluating at $\vartheta = 0$ gives:

$$\begin{aligned}\Psi^{(\jmath)}(0) = \Phi^{(\jmath)}(0) + \sum_{i=0}^{\jmath-1}\sum_{\ell=0}^{i} \binom{i}{\ell} \left[\partial_\vartheta^{(\jmath-1-i)} \mathcal{K}_1(\vartheta,\vartheta)\right]^{(i-\ell)}(0)\, \Psi^{(\ell)}(0) \\ + \alpha \sum_{i=0}^{\jmath-1}\sum_{\ell=0}^{i} \binom{i}{\ell} \alpha^\ell \left[\partial_\vartheta^{(\jmath-1-i)} \mathcal{K}_2(\vartheta,\alpha\vartheta)\right]^{(i-\ell)}(0)\, \Psi^{(\ell)}(0),\end{aligned} \tag{2.4}$$

for $\jmath = 0, 1, \ldots, \mathfrak{d}-1$. This recursive relation allows us to compute the Taylor coefficients sequentially.

Subsequently, for $\nu \in \{1, 2, \ldots, \mathcal{N}-1\}$, we approximate Ψ on the interval $\mathcal{I}_\nu$ by the polynomial:

$$\mathfrak{w}_\nu(\vartheta) = \sum_{\jmath=0}^{\mathfrak{d}-1} \frac{\widetilde{\mathfrak{w}}_\nu^{(\jmath)}(\vartheta_\nu)}{\jmath!}(\vartheta-\vartheta_\nu)^\jmath, \quad \vartheta \in \mathcal{I}_\nu, \tag{2.5}$$

where $\widetilde{\mathfrak{w}}_\nu$ is the exact solution of the local integral equation:

$$\begin{aligned}\widetilde{\mathfrak{w}}_\nu(\vartheta) = \Phi(\vartheta) + \sum_{\kappa=0}^{\nu-1} \int_{\vartheta_\kappa}^{\vartheta_{\kappa+1}} \mathcal{K}_1(\vartheta,\varsigma)\mathfrak{w}_\kappa(\varsigma)\,d\varsigma + \int_{\vartheta_\nu}^{\vartheta} \mathcal{K}_1(\vartheta,\varsigma)\widetilde{\mathfrak{w}}_\nu(\varsigma)\,d\varsigma \\ + \sum_{\kappa=0}^{\nu_0-1} \int_{\vartheta_\kappa}^{\vartheta_{\kappa+1}} \mathcal{K}_2(\vartheta,\varsigma)\mathfrak{w}_\kappa(\varsigma)\,d\varsigma + \int_{\vartheta_{\nu_0}}^{\alpha\vartheta} \mathcal{K}_2(\vartheta,\varsigma)\mathfrak{w}_{\nu_0}(\varsigma)\,d\varsigma,\end{aligned} \tag{2.6}$$

with $\alpha\vartheta \in \mathcal{I}_{\nu_0}$.

The derivatives $\widetilde{\mathfrak{w}}_\nu^{(\jmath)}(\vartheta_\nu)$ are obtained by differentiating (2.6) $\jmath$ times. Differentiating (2.6) yields:

$$\begin{aligned}\widetilde{\mathfrak{w}}_\nu^{(\jmath)}(\vartheta) &= \Phi^{(\jmath)}(\vartheta) + \sum_{\kappa=0}^{\nu-1}\int_{\vartheta_\kappa}^{\vartheta_{\kappa+1}} \partial_\vartheta^{(\jmath)}\mathcal{K}_1(\vartheta,\varsigma)\mathfrak{w}_\kappa(\varsigma)\,d\varsigma \\ &+ \sum_{i=0}^{\jmath-1}\left[\partial_\vartheta^{(\jmath-1-i)}\mathcal{K}_1(\vartheta,\vartheta)\widetilde{\mathfrak{w}}_\nu(\vartheta)\right]^{(i)} + \int_{\vartheta_\nu}^{\vartheta}\partial_\vartheta^{(\jmath)}\mathcal{K}_1(\vartheta,\varsigma)\widetilde{\mathfrak{w}}_\nu(\varsigma)\,d\varsigma \\ &+ \sum_{\kappa=0}^{\nu_0-1}\int_{\vartheta_\kappa}^{\vartheta_{\kappa+1}} \partial_\vartheta^{(\jmath)}\mathcal{K}_2(\vartheta,\varsigma)\mathfrak{w}_\kappa(\varsigma)\,d\varsigma \\ &+ \alpha\sum_{i=0}^{\jmath-1}\left[\partial_\vartheta^{(\jmath-1-i)}\mathcal{K}_2(\vartheta,\alpha\vartheta)\mathfrak{w}_{\nu_0}(\alpha\vartheta)\right]^{(i)} + \int_{\vartheta_{\nu_0}}^{\alpha\vartheta}\partial_\vartheta^{(\jmath)}\mathcal{K}_2(\vartheta,\varsigma)\mathfrak{w}_{\nu_0}(\varsigma)\,d\varsigma.\end{aligned} \tag{2.7}$$

Applying the Leibniz rule to the product terms, we obtain:

$$\begin{aligned}\widetilde{\mathfrak{w}}_\nu^{(\jmath)}(\vartheta) &= \Phi^{(\jmath)}(\vartheta) + \sum_{\kappa=0}^{\nu-1}\int_{\vartheta_\kappa}^{\vartheta_{\kappa+1}} \partial_\vartheta^{(\jmath)}\mathcal{K}_1(\vartheta,\varsigma)\mathfrak{w}_\kappa(\varsigma)\,d\varsigma \\ &+ \sum_{i=0}^{\jmath-1}\sum_{\ell=0}^{i}\binom{i}{\ell}\left[\partial_\vartheta^{(\jmath-1-i)}\mathcal{K}_1(\vartheta,\vartheta)\right]^{(i-\ell)}\widetilde{\mathfrak{w}}_\nu^{(\ell)}(\vartheta) \\ &+ \int_{\vartheta_\nu}^{\vartheta}\partial_\vartheta^{(\jmath)}\mathcal{K}_1(\vartheta,\varsigma)\widetilde{\mathfrak{w}}_\nu(\varsigma)\,d\varsigma + \sum_{\kappa=0}^{\nu_0-1}\int_{\vartheta_\kappa}^{\vartheta_{\kappa+1}}\partial_\vartheta^{(\jmath)}\mathcal{K}_2(\vartheta,\varsigma)\mathfrak{w}_\kappa(\varsigma)\,d\varsigma \\ &+ \alpha\sum_{i=0}^{\jmath-1}\sum_{\ell=0}^{i}\binom{i}{\ell}\alpha^\ell\left[\partial_\vartheta^{(\jmath-1-i)}\mathcal{K}_2(\vartheta,\alpha\vartheta)\right]^{(i-\ell)}\mathfrak{w}_{\nu_0}^{(\ell)}(\alpha\vartheta) \\ &+ \int_{\vartheta_{\nu_0}}^{\alpha\vartheta}\partial_\vartheta^{(\jmath)}\mathcal{K}_2(\vartheta,\varsigma)\mathfrak{w}_{\nu_0}(\varsigma)\,d\varsigma.\end{aligned} \tag{2.8}$$

Evaluating at $\vartheta = \vartheta_\nu$ gives:

$$\begin{aligned}\widetilde{\mathfrak{w}}_\nu^{(\jmath)}(\vartheta_\nu) = \Phi^{(\jmath)}(\vartheta_\nu) &+ \sum_{\kappa=0}^{\nu-1} \int_{\vartheta_\kappa}^{\vartheta_{\kappa+1}} \partial_\vartheta^{(\jmath)} \mathcal{K}_1(\vartheta_\nu, \varsigma) \mathfrak{w}_\kappa(\varsigma)\, d\varsigma \\ &+ \sum_{i=0}^{\jmath-1} \sum_{\ell=0}^{i} \binom{i}{\ell} \left[\partial_\vartheta^{(\jmath-1-i)} \mathcal{K}_1(\vartheta, \vartheta)\right]^{(i-\ell)} (\vartheta_\nu)\, \widetilde{\mathfrak{w}}_\nu^{(\ell)}(\vartheta_\nu) \\ &+ \sum_{\kappa=0}^{\nu_0-1} \int_{\vartheta_\kappa}^{\vartheta_{\kappa+1}} \partial_\vartheta^{(\jmath)} \mathcal{K}_2(\vartheta_\nu, \varsigma) \mathfrak{w}_\kappa(\varsigma)\, d\varsigma \\ &+ \alpha \sum_{i=0}^{\jmath-1} \sum_{\ell=0}^{i} \binom{i}{\ell} \alpha^\ell \left[\partial_\vartheta^{(\jmath-1-i)} \mathcal{K}_2(\vartheta, \alpha\vartheta)\right]^{(i-\ell)} (\vartheta_\nu)\, \mathfrak{w}_{\nu_0}^{(\ell)}(\alpha\vartheta_\nu) \\ &+ \int_{\vartheta_{\nu_0}}^{\alpha\vartheta_\nu} \partial_\vartheta^{(\jmath)} \mathcal{K}_2(\vartheta, \varsigma) \mathfrak{w}_{\nu_0}(\varsigma)\, d\varsigma,\end{aligned} \tag{2.9}$$

for $\jmath \in \{0, 1, \ldots, \mathfrak{d} - 1\}$. This recursive relation allows us to compute the Taylor coefficients on each subinterval sequentially.

Remark 2.1. The proposed method requires the computation of high-order derivatives of the kernel functions $\mathcal{K}_1$, $\mathcal{K}_2$ and the source term Φ. For problems involving smooth or analytic functions, these derivatives can be computed efficiently using symbolic or automatic differentiation techniques, making the method practical and highly accurate.

However, for problems with non-smooth or piecewise-defined data, the computation of high-order derivatives may become computationally expensive. In such situations, numerical differentiation techniques can be employed, although this may affect the overall accuracy. Alternatively, the method is particularly advantageous when high precision is required, as it achieves accurate results with relatively coarse discretizations, thereby reducing the need for fine meshes and partially compensating for the additional computational cost.

3. Convergence Analysis

This section is dedicated to a rigorous convergence analysis of the proposed Taylor Collocation Method. Our primary objective is to establish that the numerical solution $\mathfrak{w}$ converges to the exact solution Ψ of (1.1) at a rate of $O((\Delta\vartheta)^{\mathfrak{d}})$, where $\mathfrak{d}$ is the degree of the local Taylor polynomials, provided the data is sufficiently smooth.

The following lemmas will be used in this section.

Lemma 3.1 ([2]). *Let $\{\beta_\jmath\}_{\jmath=0}^n$ be a non-negative sequence and suppose the sequence $\{\zeta_n\}$ satisfies*

$$\zeta_n \le \gamma_0 + \sum_{i=0}^{n-1} \beta_i \zeta_i, \quad n \ge 0,$$

with $\gamma_0 \geq 0$. *Then,*

$$\zeta_n \leq \gamma_0 \exp\left(\sum_{\jmath=0}^{n-1} \beta_\jmath\right), \quad n \geq 0.$$

Lemma 3.2 ([7]). *Let* $\delta(\vartheta)$ *be a positive, monotonic nondecreasing function, and let* $\mathfrak{z}(\vartheta) \geq 0$, $\psi(\vartheta) \geq 0$ *be continuous. If*

$$\mathfrak{z}(\vartheta) \leq \delta(\vartheta) + \int_{\mathfrak{a}}^{\vartheta} \psi(\mathfrak{s})\mathfrak{z}(\mathfrak{s})\, d\mathfrak{s}, \quad \mathfrak{a} \leq \vartheta \leq \mathfrak{b},$$

then,

$$\mathfrak{z}(\vartheta) \leq \delta(\vartheta) \exp\left(\int_{\mathfrak{a}}^{\vartheta} \psi(\mathfrak{s})\, d\mathfrak{s}\right), \quad \mathfrak{a} \leq \vartheta \leq \mathfrak{b}.$$

The following lemma is pivotal for establishing the convergence of the numerical scheme.

Lemma 3.3. *Let* $\Phi, \mathcal{K}_1$, *and* $\mathcal{K}_2$ *be* $\mathfrak{d}$*-times continuously differentiable on their respective domains. Then, there exists a constant* $\beta(\mathfrak{d}) > 0$ *such that for all* $\nu = 0, 1, \ldots, \mathcal{N}-1$ *and* $\jmath = 0, 1, \ldots, \mathfrak{d}$,

$$\|\widetilde{\mathfrak{w}}_\nu^{(\jmath)}\|_{L^\infty(\mathcal{I}_\nu)} \leq \beta(\mathfrak{d}),$$

provided $\Delta\vartheta$ *is sufficiently small, where* $\widetilde{\mathfrak{w}}_0(\vartheta) = \Psi(\vartheta)$ *for* $\vartheta \in \mathcal{I}_0$.

Proof. The argument proceeds in two parts.

Part 1. There exists $\beta_1(\mathfrak{d}) > 0$ such that $\|\widetilde{\mathfrak{w}}_\nu^{(\jmath)}\|_{L^\infty(\mathcal{I}_\nu)} \leq \beta_1(\mathfrak{d})$.

Let $\mathfrak{b}_\nu^\jmath = \|\widetilde{\mathfrak{w}}_\nu^{(\jmath)}\|_{L^\infty(\mathcal{I}_\nu)}$. For all $\jmath = 0, 1, \ldots, \mathfrak{d}$, we have

$$\mathfrak{b}_0^\jmath \leq \max\left\{\|\Psi^{(\jmath)}\|_{L^\infty(\mathcal{I}_0)},\ \jmath = 0, 1, \ldots, \mathfrak{d}\right\} = \beta_1(\mathfrak{d}). \tag{3.1}$$

Part 2. For $\nu = 1, \ldots, \mathcal{N}-1$, since $0 < \alpha < 1$, it follows that $\nu_0 \leq \nu$.

Define $\Gamma_\nu = \max\{\mathfrak{b}_\nu^\jmath : \jmath = 0, \ldots, \mathfrak{d}\}$. From (2.8), for all $\jmath = 1, \ldots, \mathfrak{d}$, we obtain:

$$\mathfrak{b}_\nu^\jmath \leq \mathfrak{C}_1 + 2\mathfrak{C}_2\Delta\vartheta \sum_{\kappa=0}^{\nu-1} \Gamma_\kappa + (\mathfrak{C}_3 + \alpha\mathfrak{C}_3) \sum_{\ell=0}^{\jmath-1} \mathfrak{b}_\nu^\ell + 2\mathfrak{C}_2\Delta\vartheta\, \mathfrak{b}_\nu^0, \tag{3.2}$$

where the constants are defined as:

$$\begin{aligned}
\mathfrak{C}_1 &= \max\left\{\|\Phi^{(\jmath)}\|_{L^\infty(\mathcal{I})} : \jmath = 0, \ldots, \mathfrak{d}\right\},\\
\mathfrak{C}_2 &= \max\left\{\|\partial_\vartheta^{(\jmath)}\mathcal{K}_1\|_{L^\infty(\mathcal{D}_1)},\ \|\partial_\vartheta^{(\jmath)}\mathcal{K}_2\|_{L^\infty(\mathcal{D}_2)} : \jmath = 0, \ldots, \mathfrak{d}\right\},\\
\mathfrak{C}_3 &= \max\left\{\binom{\jmath-1-i}{\ell}\left\|\left[\partial_\vartheta^{(i)}\mathcal{K}_1(\vartheta,\vartheta)\right]^{(\jmath-1-i-\ell)}\right\|_{L^\infty(\mathcal{I})},\right.\\
&\qquad \left.\binom{\jmath-1-i}{\ell}\left\|\left[\partial_\vartheta^{(i)}\mathcal{K}_2(\vartheta,\alpha\vartheta)\right]^{(\jmath-1-i-\ell)}\right\|_{L^\infty(\mathcal{I})} : \jmath = 1, \ldots, \mathfrak{d};\right.\\
&\qquad \left. i = 0, \ldots, \jmath-1;\ \ell = 0, \ldots, \jmath-1-i\right\}.
\end{aligned}$$

Let $\mathfrak{C}_4 = \mathfrak{C}_3 + \alpha\mathfrak{C}_3 + 2\mathfrak{C}_2\Theta$. Then,

$$\mathfrak{b}_\nu^\jmath \le \mathfrak{C}_1 + 2\mathfrak{C}_2\Delta\vartheta \sum_{\kappa=0}^{\nu-1} \Gamma_\kappa + \mathfrak{C}_4 \sum_{\ell=0}^{\jmath-1} \mathfrak{b}_\nu^\ell. \tag{3.3}$$

Fixing $\nu \in \{1, \dots, \mathcal{N}-1\}$ and applying Lemma 3.1 to (3.3) yields:

$$\mathfrak{b}_\nu^\jmath \le \left(\mathfrak{C}_1 + 2\mathfrak{C}_2\Delta\vartheta \sum_{\kappa=0}^{\nu-1} \Gamma_\kappa\right) \exp\left(\sum_{\ell=0}^{\jmath-1} \mathfrak{C}_4\right) \le \underbrace{\mathfrak{C}_1 \exp(\mathfrak{d}\mathfrak{C}_4)}_{\mathfrak{C}_5} + \underbrace{2\mathfrak{C}_2 \exp(\mathfrak{d}\mathfrak{C}_4)}_{\mathfrak{C}_6} \Delta\vartheta \sum_{\kappa=0}^{\nu-1} \Gamma_\kappa$$

$$\le \mathfrak{C}_5 + \mathfrak{C}_6\Delta\vartheta \sum_{\kappa=0}^{\nu-1} \Gamma_\kappa.$$

Applying Lemma 3.1 once more gives, for all $\nu = 1, \dots, \mathcal{N}-1$:

$$\Gamma_\nu \le \mathfrak{C}_5 \exp\left(\mathfrak{C}_6 \sum_{\kappa=0}^{\nu-1} \Delta\vartheta\right) \le \mathfrak{C}_5 \exp(\mathfrak{C}_6\Theta).$$

The result follows by taking $\beta(\mathfrak{d}) = \max\{\beta_1(\mathfrak{d}),\ \mathfrak{C}_5 \exp(\mathfrak{C}_6\Theta)\}$. □

Theorem 3.4. *Let $\Phi, \mathcal{K}_1$, and $\mathcal{K}_2$ be $\mathfrak{d}$-times continuously differentiable on their respective domains. Then, equations (2.1) and (2.5) define a unique approximation $\mathfrak{w} \in \mathcal{S}_{\mathfrak{d}-1}^{(-1)}(\mathcal{P}_\mathcal{N})$, and the error $\mathfrak{E} := \Psi - \mathfrak{w}$ satisfies*

$$\|\mathfrak{E}\|_{L^\infty(\mathcal{I})} \le \mathfrak{C}(\Delta\vartheta)^\mathfrak{d}$$

for sufficiently small $\Delta\vartheta$, where $\mathfrak{C}$ is a constant independent of $\Delta\vartheta$.

Proof. We again proceed in two steps.
Part 1. There exists $\widetilde{\mathfrak{C}}_1 > 0$, independent of $\Delta\vartheta$, such that $\|\mathfrak{E}_0\|_{L^\infty(\mathcal{I}_0)} \le \widetilde{\mathfrak{C}}_1(\Delta\vartheta)^\mathfrak{d}$, where $\mathfrak{E}_0(\vartheta) = |\Psi(\vartheta) - \mathfrak{w}_0(\vartheta)|$ for $\vartheta \in \mathcal{I}_0$.

For $\vartheta \in \mathcal{I}_0$, Lemma 3.3 implies that for sufficiently small $\Delta\vartheta$,

$$|\mathfrak{E}_0(\vartheta)| = |\Psi(\vartheta) - \mathfrak{w}_0(\vartheta)| \le \frac{\|\Psi^{(\mathfrak{d})}\|_{L^\infty(\mathcal{I}_0)}}{\mathfrak{d}!}(\Delta\vartheta)^\mathfrak{d} \le \frac{\beta(\mathfrak{d})}{\mathfrak{d}!}(\Delta\vartheta)^\mathfrak{d}.$$

Part 2. There exists $\widetilde{\mathfrak{C}}_2 > 0$, independent of $\Delta\vartheta$, such that $\|\mathfrak{E}_\nu\|_{L^\infty(\mathcal{I}_\nu)} \le \widetilde{\mathfrak{C}}_2(\Delta\vartheta)^\mathfrak{d}$, where $\mathfrak{E}_\nu(\vartheta) = \Psi(\vartheta) - \mathfrak{w}_\nu(\vartheta)$ for $\nu \in \{1, \dots, \mathcal{N}-1\}$.

For $\vartheta \in \mathcal{I}_\nu$, it follows from (2.6) that

$$\Psi(\vartheta) - \widetilde{\mathfrak{w}}_\nu(\vartheta) = \sum_{\kappa=0}^{\nu-1} \int_{\vartheta_\kappa}^{\vartheta_{\kappa+1}} \mathcal{K}_1(\vartheta, \varsigma)\mathfrak{E}_\kappa(\varsigma)\, d\varsigma + \int_{\vartheta_\nu}^{\vartheta} \mathcal{K}_1(\vartheta, \varsigma)\left(\Psi(\varsigma) - \widetilde{\mathfrak{w}}_\nu(\varsigma)\right) d\varsigma$$

$$+ \sum_{\kappa=0}^{\nu_0-1} \int_{\vartheta_\kappa}^{\vartheta_{\kappa+1}} \mathcal{K}_2(\vartheta, \varsigma)\mathfrak{E}_\kappa(\varsigma)\, d\varsigma + \int_{\vartheta_{\nu_0}}^{\alpha\vartheta} \mathcal{K}_2(\vartheta, \varsigma)\mathfrak{E}_{\nu_0}(\varsigma)\, d\varsigma.$$

Since $\nu_0 \le \nu$ and $\alpha\vartheta \le \vartheta$, we obtain

$$\|\Psi - \widetilde{\mathfrak{w}}_\nu\| \le \sum_{\kappa=0}^{\nu-1} \Delta\vartheta\, \overline{\mathcal{K}}\, \|\mathfrak{E}_\kappa\| + \overline{\mathcal{K}} \int_{\vartheta_\nu}^{\vartheta} \|\Psi - \widetilde{\mathfrak{w}}_\nu\|\, d\mathfrak{s},$$

where $\overline{\mathcal{K}} = 2\max\left\{\|\mathcal{K}_1\|_{L^\infty(\mathcal{D}_1)},\ \|\mathcal{K}_2\|_{L^\infty(\mathcal{D}_2)}\right\}$.

Applying Lemma 3.2:

$$\|\Psi - \widetilde{\mathfrak{w}}_\nu\|_{L^\infty(\mathcal{I}_\nu)} \leq \left(\sum_{\kappa=0}^{\nu-1} \Delta\vartheta\, \overline{\mathcal{K}}\, \|\mathfrak{E}_\kappa\|\right) \exp\left(\int_{\vartheta_\nu}^{\vartheta} \overline{\mathcal{K}}\, d\mathfrak{s}\right) \leq \sum_{\kappa=0}^{\nu-1} \Delta\vartheta\, \underbrace{\overline{\mathcal{K}} \exp(\Theta\overline{\mathcal{K}})}_{\Lambda} \|\mathfrak{E}_\kappa\|$$
$$\leq \sum_{\kappa=0}^{\nu-1} \Delta\vartheta\, \Lambda\, \|\mathfrak{E}_\kappa\|.$$

Consequently,

$$\|\mathfrak{E}_\nu\| \leq \|\Psi - \widetilde{\mathfrak{w}}_\nu\| + \|\widetilde{\mathfrak{w}}_\nu - \mathfrak{w}_\nu\| \leq \sum_{\kappa=0}^{\nu-1} \Delta\vartheta\, \Lambda\, \|\mathfrak{E}_\kappa\| + \frac{\|\widetilde{\mathfrak{w}}_\nu^{(\mathfrak{d})}\|_{L^\infty(\mathcal{I}_\nu)}}{\mathfrak{d}!}(\Delta\vartheta)^{\mathfrak{d}}.$$

Invoking Lemma 3.3 and then Lemma 3.1, we find:

$$\|\mathfrak{E}_\nu\| \leq \frac{\beta(\mathfrak{d})}{\mathfrak{d}!}(\Delta\vartheta)^{\mathfrak{d}} \exp\left(\sum_{\kappa=0}^{\nu-1} \Delta\vartheta\, \Lambda\right) \leq \frac{\beta(\mathfrak{d})}{\mathfrak{d}!} \exp(\Theta\Lambda)(\Delta\vartheta)^{\mathfrak{d}}.$$

Taking $\widetilde{\mathfrak{C}}_2 = \frac{\beta(\mathfrak{d})}{\mathfrak{d}!} \exp(\Theta\Lambda)$ completes this part.

The theorem is proved by setting $\mathfrak{C} = \max\left\{\widetilde{\mathfrak{C}}_1, \widetilde{\mathfrak{C}}_2\right\}$. □

4. Numerical Experiments

This section is devoted to the numerical validation of the Taylor Collocation Method. The efficacy of the proposed scheme is assessed by applying it to a series of Volterra integral equations with proportional delays, for which exact solutions are known. All computations were performed in the Maple 17 software environment.

Example 4.1 ([17])**.** Consider the following pantograph-type Volterra integral equation:

$$\Psi(\vartheta) = (-\vartheta^2 + \vartheta + 1)e^{\vartheta} + \vartheta(\alpha\vartheta - 1)e^{\alpha\vartheta} + \int_{\alpha\vartheta}^{\vartheta} \vartheta\varsigma\, \Psi(\varsigma)\, d\varsigma,$$

where the kernels are $\mathcal{K}_1(\vartheta, \varsigma) = -\mathcal{K}_2(\vartheta, \varsigma) = \vartheta\varsigma$, and the exact solution is $\Psi(\vartheta) = e^{\vartheta}$.

Table 1 and display the absolute errors between the exact and approximate solutions for $\mathcal{N} = 10$ and various values of polynomial degree $\mathfrak{d}$ and α. Figure 1 display the absolute errors between the exact and approximate solutions for $\mathcal{N} = 10$ and $\alpha = 0.5$. Table 2 shows comparison between sinc-collocation method [17] and TCM for $\mathfrak{d} = 7$ and $\mathcal{N} = 100$.

Example 4.2. To provide a comprehensive evaluation of the proposed Taylor Collocation Method (TCM), we compare its performance with two classical quadrature methods: the repeated trapezoid quadrature method and the repeated Simpson quadrature method with variable step [13]. For this comparison, we consider the following Volterra integral equation:

TABLE 1. Absolute errors for Example 4.1

ϑ	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.99$	
	$\mathfrak{d} = 5$	$\mathfrak{d} = 7$	$\mathfrak{d} = 5$	$\mathfrak{d} = 7$	$\mathfrak{d} = 5$	$\mathfrak{d} = 7$
0.1	4.64×10^{-10}	1.87×10^{-11}	4.67×10^{-10}	1.87×10^{-11}	7.40×10^{-11}	2.06×10^{-11}
0.2	4.11×10^{-10}	6.40×10^{-11}	4.14×10^{-10}	6.40×10^{-11}	3.24×10^{-10}	1.10×10^{-9}
0.3	2.77×10^{-10}	4.67×10^{-10}	3.82×10^{-10}	4.37×10^{-10}	4.56×10^{-10}	8.33×10^{-10}
0.4	1.21×10^{-9}	4.89×10^{-10}	1.24×10^{-9}	4.83×10^{-10}	1.84×10^{-10}	1.96×10^{-10}
0.5	1.62×10^{-9}	5.75×10^{-10}	1.67×10^{-9}	7.85×10^{-10}	1.32×10^{-9}	2.08×10^{-10}
0.6	2.87×10^{-9}	3.22×10^{-10}	2.00×10^{-9}	8.20×10^{-10}	1.70×10^{-11}	8.31×10^{-10}
0.7	4.86×10^{-9}	1.10×10^{-9}	3.54×10^{-9}	4.71×10^{-10}	4.19×10^{-10}	6.39×10^{-10}
0.8	8.00×10^{-9}	1.98×10^{-10}	4.60×10^{-9}	4.58×10^{-10}	1.34×10^{-9}	6.85×10^{-10}
0.9	1.17×10^{-8}	1.29×10^{-10}	7.78×10^{-9}	1.29×10^{-9}	1.10×10^{-9}	1.24×10^{-9}
1.0	2.22×10^{-7}	2.63×10^{-10}	2.17×10^{-7}	1.43×10^{-9}	1.15×10^{-7}	1.52×10^{-9}

TABLE 2. Maximum error comparison between TCM and Sinc-Collocation for Example 4.1

α	TCM		Sinc-Collocation [17]	
	$\mathcal{N} = 10$	$\mathcal{N} = 30$	$\mathcal{N} = 10$	$\mathcal{N} = 30$
0.01	1.11×10^{-9}	1.84×10^{-9}	3.73×10^{-6}	5.50×10^{-10}
0.09	1.11×10^{-9}	1.92×10^{-9}	4.57×10^{-6}	3.61×10^{-9}
0.1	1.10×10^{-9}	1.91×10^{-9}	4.19×10^{-6}	5.25×10^{-9}
0.5	1.43×10^{-9}	1.87×10^{-9}	8.84×10^{-5}	4.09×10^{-7}
0.99	1.52×10^{-9}	1.52×10^{-9}	1.75×10^{-5}	7.55×10^{-8}

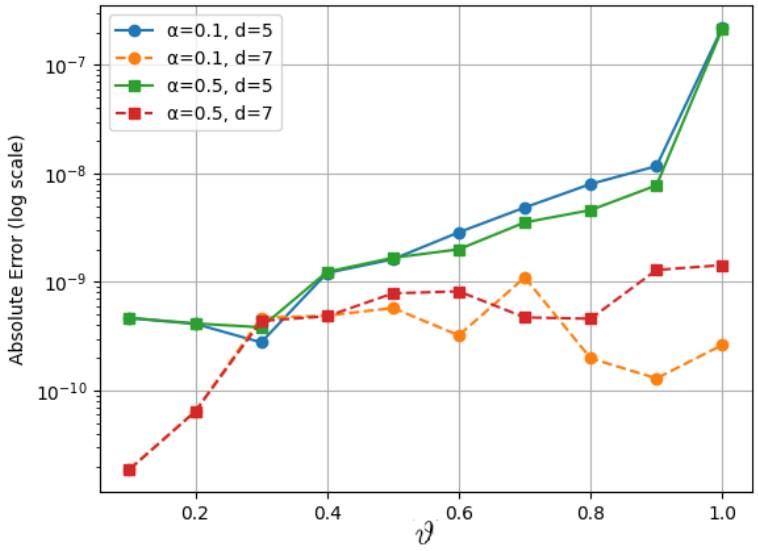

FIGURE 1. Absolute error of $\alpha = 0.1$ and $\alpha = 0.5$ for Example 4.1.

$$\Psi(\vartheta) = \vartheta + \int_0^{\vartheta} \mathcal{K}_1(\vartheta, \varsigma)\Psi(\varsigma)\, d\varsigma, \quad \vartheta \in \mathcal{I} := [0, \Theta] \tag{4.1}$$

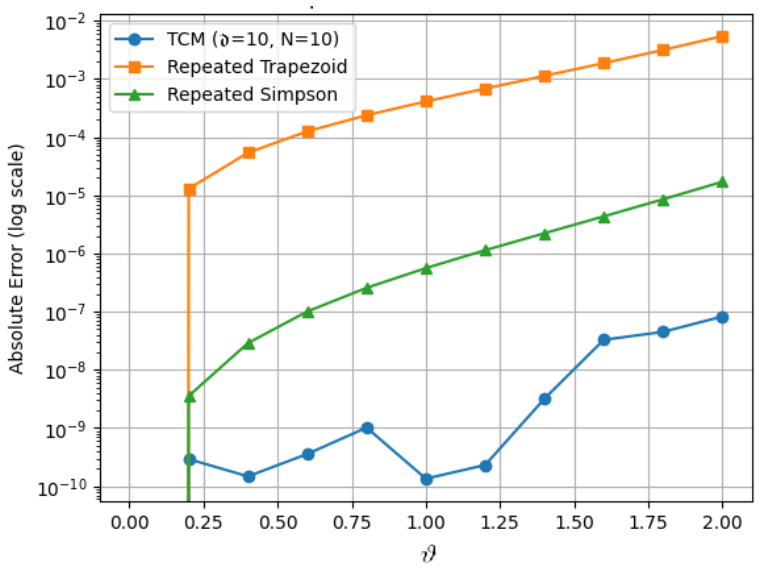

FIGURE 2. Comparison of absolute errors for Example 4.2.

with $\Theta = 2$ and the kernel $\mathcal{K}_1(\vartheta, \varsigma) = \frac{1}{5}\vartheta\varsigma$. The exact solution of this equation is $\Psi(\vartheta) = \vartheta e^{\vartheta^3/15}$. We compute the absolute errors at equally spaced points $\vartheta = 0, 0.2, \ldots, 2$ for each method. Table 3 and Figure 2 present the comparison of the TCM ($\mathfrak{d} = 10$) with the repeated trapezoid method and the repeated Simpson method.

TABLE 3. Comparison of absolute errors for Example 4.2.

ϑ	TCM ($\mathfrak{d} = 10, \mathcal{N} = 10$)	Repeated Trapezoid	Repeated Simpson
0.0	0	0	0
0.2	2.93×10^{-10}	1.30×10^{-5}	3.56×10^{-9}
0.4	1.48×10^{-10}	5.40×10^{-5}	2.89×10^{-8}
0.6	3.59×10^{-10}	1.26×10^{-4}	1.01×10^{-7}
0.8	1.02×10^{-9}	2.38×10^{-4}	2.57×10^{-7}
1.0	1.35×10^{-10}	4.11×10^{-4}	5.61×10^{-7}
1.2	2.32×10^{-10}	6.83×10^{-4}	1.14×10^{-6}
1.4	3.20×10^{-9}	1.13×10^{-3}	2.23×10^{-6}
1.6	3.29×10^{-8}	1.87×10^{-3}	4.35×10^{-6}
1.8	4.50×10^{-8}	3.15×10^{-3}	8.55×10^{-6}
2.0	8.27×10^{-8}	5.43×10^{-3}	1.71×10^{-5}

Example 4.3. Consider the following Volterra proportional delay integral equation:

$$\Psi(\vartheta) = \sin\vartheta + \frac{\vartheta}{2e^{\vartheta}}\left(\alpha\vartheta\cos(\alpha\vartheta) - \sin(\alpha\vartheta)\right) + \int_0^{\alpha\vartheta} \frac{1}{2}\varsigma\vartheta e^{-\vartheta}\Psi(\varsigma)\, d\varsigma,$$

for $\vartheta \in [0, 1]$, with $\mathcal{K}_1(\vartheta, \varsigma) = 0$ and $\mathcal{K}_2(\vartheta, \varsigma) = \frac{1}{2}\varsigma\vartheta e^{-\vartheta}$. The exact solution is $\Psi(\vartheta) = \sin\vartheta$.

Table 4 report the absolute errors for $\mathcal{N} = 100$ and $\alpha = 0.7$.

TABLE 4. Absolute errors for Example 4.3

ϑ	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.7$
0	0	0	0
0.1	8.43×10^{-14}	8.54×10^{-14}	2.14×10^{-13}
0.2	6.51×10^{-13}	1.49×10^{-12}	4.40×10^{-13}
0.3	6.41×10^{-12}	1.22×10^{-11}	2.05×10^{-11}
0.4	5.10×10^{-11}	5.70×10^{-11}	4.75×10^{-11}
0.5	3.00×10^{-11}	3.78×10^{-11}	3.71×10^{-11}
0.6	7.77×10^{-11}	5.32×10^{-11}	7.47×10^{-11}
0.7	1.14×10^{-10}	1.34×10^{-10}	8.14×10^{-11}
0.8	5.43×10^{-10}	3.71×10^{-11}	2.74×10^{-11}
0.9	1.75×10^{-10}	9.75×10^{-11}	5.17×10^{-11}

5. CONCLUSION

This paper has been devoted to the formulation, analysis, and numerical validation of a Taylor Collocation Method (TCM) for solving linear Volterra integral equations with a proportional delay. The core of our work was to present a high-order numerical scheme that is both conceptually straightforward and computationally efficient.

The significance of this method lies in its direct approach. By constructing a local Taylor polynomial approximation on a uniform mesh and determining its coefficients through collocation, we circumvent the need for complex mesh refinement near the delay point $\alpha\vartheta$ and avoid solving large, global algebraic systems. This results in a method that is not only simple to implement but also achieves high-order convergence for smooth solutions, as rigorously proven in our convergence analysis. The numerical experiments consistently demonstrated this high accuracy across various delay parameters, confirming the method's robustness.

A comparative analysis with Legendre spectral and Haar wavelet methods showed that the TCM achieves comparable or superior accuracy, validating the claim of efficiency made in the abstract. We have also discussed the computational cost associated with derivative evaluation and suggested strategies for handling non-analytic kernels.

The success of the TCM in this linear setting paves the way for several important extensions. The most immediate and promising direction is its application to the nonlinear case. While the fundamental framework remains applicable, this extension would involve solving systems of nonlinear equations for the Taylor coefficients, presenting an interesting challenge for future analysis. Furthermore, the method's potential can be explored for more complex functional equations, such as those with state-dependent delays, neutral integro-differential equations, and systems of coupled delay equations. The development of a variable step-size strategy to handle solutions with sharp gradients also represents a valuable avenue for enhancing the method's efficiency.

In summary, the Taylor Collocation Method presented herein provides a powerful and accessible numerical framework for proportional-delay integral equations.

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[1] Abdelhafid Boussouf University of Mila, Mila, Algeria.
Email address: c.hennous@centre-univ-mila.dz

[2] Higher Normal School of Kouba Cheikh Mohamed Elbachir Elibrahimi, Algiers, Algeria.
Email address: hafida.laib@gmail.com