Gulf Journal of Mathematics
Vol 23, Issue 1 (2026) 1-19

ISSN: 2309-4966
https://doi.org/10.56947/yyhq5w52

NORM INEQUALITIES IN DUNKL-TYPE TOTAL FOFANA-GULIYEV SPACES

POKOU NAGACY[1*], BÉRENGER AKON KPATA[2] and NOUFFOU DIARRA[3]

ABSTRACT. We introduce total Fofana-Guliyev spaces associated with the Dunkl operator on the real line called Dunkl-type total Fofana-Guliyev spaces and investigate their basic properties. We show that these spaces generalize Dunkl-Fofana spaces and are subspaces of Dunkl total Morrey spaces. As applications, we establish norm inequalities for the Dunkl-type Hardy-Littlewood maximal operator, Dunkl-type fractional integral operators and Dunkl-type fractional maximal operators in Dunkl-type total Fofana-Guliyev spaces.

Keywords. Dunkl-Fofana space, Dunkl-type total Fofana-Guliyev space, Dunkl-type maximal operator, Dunkl-type fractional integral operator, Dunkl-type fractional maximal operator.
2020 Mathematics Subject Classification. Primary: 42B20, 42B25; Secondary: 42B35.

1. INTRODUCTION

Fofana spaces, denoted by $(L^q, L^p)^\alpha(\mathbb{R}^d)$ $(1 \leq q \leq \alpha \leq p \leq \infty)$, are Banach spaces first introduced in [7]. It is proved in [6] that for $1 \leq q < \alpha < \infty$ fixed, $\{(L^q, L^p)^\alpha(\mathbb{R}^d) \ : \ \alpha \leq p \leq \infty\}$ is a non-decreasing family (in the sense of inclusion) of distinct Banach spaces whose minimal element and maximal element are the Lebesgue space $L^\alpha(\mathbb{R}^d)$ and the classical Morrey space $L^{q,\lambda}(\mathbb{R}^d) = (L^q, L^\infty)^\alpha(\mathbb{R}^d)$, where $\lambda = \frac{1}{q} - \frac{1}{\alpha}$, respectively. Other properties of Fofana spaces can be found in [2, 4]. Recall that the classical Morrey spaces were introduced in [18] by C.B. Morrey. They are related to regularity results for solutions of certain problems in partial differential equations and integral operators (see [22] and the references therein). Norm inequalities for classical operators, including Hardy-Littlewood maximal type operators, fractional maximal operators and fractional integral operators in Fofana spaces were extensively studied during these last decades (see, for example, [5, 7, 8, 9]. In a recent paper, Guliyev [10] introduced a new variant of Morrey spaces called total Morrey spaces and gave necessary and sufficient conditions for the boundedness of the maximal commutator and commutator of maximal operator in these spaces. Furthermore, he characterized

Date: Received: Feb 16, 2026; Revised: Mar 13, 2026; Accepted: Mar 29, 2026.
* Corresponding author

Lipschitz functions via the commutators of maximal function in total Morrey spaces (see [11]) and established necessary and sufficient conditions for the boundedness of the fractional maximal operator and its commutators on these spaces. Inspired by Guliyev's works, Kpata and Nagacy [15] studied total Fofana-Guliyev spaces and established the boundedness of some classical operators, namely the Hardy-Littlewood maximal operator, fractional integral operators and fractional maximal operators, in these spaces. In the last few years, various important results in classical harmonic analysis related to Morrey and Fofana spaces have been extended in the setting of rational Dunkl harmonic analysis on the real line (see, for example, [16, 20, 21] and the references therein). Recall that on the real line, the Dunkl operators are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$.

The aim of this paper is to extend, within the framework of the rational Dunkl harmonic analysis on the real line, the results obtained in [15]. To this end, we introduce the Dunkl-type total Fofana-Guliyev spaces and we investigate their basic properties. We show that these spaces generalize the Dunkl-Fofana spaces studied in [20] and are subspaces of total Morrey spaces associated with the Dunkl operator on $\mathbb{R}$ as defined in [19]. Furthermore, we establish norm inequalities for the Dunkl-type Hardy-Littlewood maximal operator, Dunkl-type fractional integral operators and Dunkl-type fractional maximal operators in Dunkl-type total Fofana-Guliyev spaces.

The remainder of this note is organized as follows. Section 2 contains preliminaries on rational Dunkl harmonic analysis on the real line. In Section 3, we define the Dunkl-type total Fofana-Guliyev spaces and examine some of their properties. Section 4 is devoted to norm inequalities for the Dunkl-type Hardy-Littlewood maximal operator. Norm inequalities for Dunkl-type fractional integral operators and Dunkl-type fractional maximal operators are established in Section 5.

The letter C will be used for positive constants not depending on the relevant variables, and these constants may change from one occurrence to another. The notation $A \lesssim$ B will mean that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$. We will adopt the convention $\frac{1}{\infty} = 0$.

2. Preliminaries on Dunkl analysis on the real line

Let $k > -\frac{1}{2}$ be a fixed number. We denote by μ, the weighted Lebesgue measure on $\mathbb{R}$ defined by

$$d\mu(x) = \left(2^{k+1}\Gamma(k+1)\right)^{-1} |x|^{2k+1}\, dx.$$

By $L^0(\mu)$, we denote the complex vector space of equivalence classes (modulo equality μ-almost everywhere) of complex-valued functions μ-measurable on $\mathbb{R}$. For $1 \leq p \leq \infty$, we denote by $L^p(\mu)$ the Lebesgue space associated with the measure μ, also called Dunkl-Lebesgue space. For $f \in L^p(\mu)$, $\|f\|_p$ stands for the classical norm of f. The class of locally integrable functions with respect to μ is denoted by $L^1_{loc}(\mu)$. For any subset A of $\mathbb{R}$, χ_A denotes its characteristic function. For any $x \in \mathbb{R}$ and for any $r > 0$, we set

$$B(x,\, r) = \{y \in \mathbb{R} \,:\, \max\{0,\, |x| - r\} < |y| < |x| + r\}$$

if $x \neq 0$ and

$$B_r = B(0,\, r) = (-r,\, r).$$

It is well-known that

$$\mu(B_r) = d_k r^{2k+2}, \quad r > 0,$$

where $d_k = \left(2^{k+1}(k+1)\Gamma(k+1)\right)^{-1}$.
The Dunkl operator associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ is given by

$$\Lambda_k f(x) = \frac{df}{dx}(x) + \frac{2k+1}{x}\left(\frac{f(x)-f(-x)}{2}\right).$$

Let $x, y, z \in \mathbb{R}$. We put

$$W_k(x,y,z) = [1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}]\Delta_k(|x|,|y|,|z|),$$

where

$$\sigma_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } x, y \in \mathbb{R}\setminus\{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

and Δ_k is a Bessel kernel. For the explicit definition of Δ_k, we refer the reader to [14]).
Let us consider the signed measure $\nu_{x,y}$ on $\mathbb{R}$ given by

$$d\nu_{x,y}(z) = \begin{cases} W_k(x,y,z)d\mu(z) & \text{if} \quad x, y \in \mathbb{R}\setminus\{0\}, \\ d\delta_x(z) & \text{if} \quad y = 0, \\ d\delta_y(z) & \text{if} \quad x = 0. \end{cases}$$

For $x,\, y \in \mathbb{R}$ and a continuous function f on $\mathbb{R}$, we put

$$\tau_x f(y) = \int_{\mathbb{R}} f(z) d\nu_{x,y}(z).$$

The operator τ_x is called the Dunkl translation operator on $\mathbb{R}$. It is known (see [14]) that for any $x \in \mathbb{R}$ and for any $r > 0$, the map $y \mapsto \tau_x \chi_{B_r}(y)$ is supported in $B(x,r)$ and

$$0 \le \tau_x \chi_{B_r}(y) \le \min\left\{1, \frac{2C_k}{2\kappa+1}\left(\frac{r}{|x|}\right)^{2\kappa+1}\right\}, \quad y \in B(x,r),$$

It is established in [23] that for all $x \in \mathbb{R}$, the operator τ_x extends to $L^p(\mu)$, $p \ge 1$, and

$$\|\tau_x f\|_p \le 4\,\|f\|_p$$

for all $f \in L^p(\mu)$. Furthermore, for any $f \in L^1_{loc}(\mu)$,

$$\lim_{r\to 0} \frac{1}{\mu(B_r)} \int_{B_r} |\tau_x f(y) - f(x)| d\mu(y) = 0 \quad \text{for a.e.} \quad x \in \mathbb{R}$$

and

$$\lim_{r\to 0} \frac{1}{\mu(B_r)} \int_{B_r} \tau_x f(y) d\mu(y) = f(x) \quad \text{for a.e.} \quad x \in \mathbb{R}.$$

3. Total Fofana-Guliyev spaces associated with the Dunkl operator on the real line

Let $1 \leq q,\, p \leq \infty$ and let $r > 0$. For $f \in L^0(\mu)$, we define

$$_r\left\|f\right\|_{q,p} = \begin{cases} \left\|\left(\int_{\mathbb{R}}(\tau_y\left|f\right|^q)\chi_{B_r}(x)d\mu(x)\right)^{\frac{1}{q}}\right\|_p & \text{if} \quad q < \infty, \\ \left\|\left\|f\chi_{B(y,r)}\right\|_\infty\right\|_p & \text{if} \quad q = \infty, \end{cases}$$

with the $L^p(\mu)$-norm taken with respect to the variable y and

$$(L^q, L^p)_r(\mu) = \left\{\ f \in L^0(\mu)\ :\ _r\|f\|_{q,p} < \infty\ \right\}.$$

Here and in the sequel, τ_y $(y \in \mathbb{R})$ stands for the Dunkl translation operator. Note that Proposition 4.13 in [21] and a slight adaptation of Proposition 3.5 in [20], as well as Proposition 4.1 in [21] yield the following result.

Proposition 3.1. *Let $1 \leq q, p \leq \infty$ and let $r > 0$. Then, the following assertions hold.*

(1) *$(L^q, L^p)_r(\mu)$ is a complex subspace of the space $L^0(\mu)$.*
(2) *The map $f \mapsto {}_r\|f\|_{q,p}$ define a norm on $(L^q, L^p)_r(\mu)$.*
(3) *$((L^q, L^p)_r(\mu),\ {}_r\|\cdot\|_{q,p})$ is a complex Banach space.*
(4) $_r\left\|f\right\|_{q,q} = (\mu(B_r))^{\frac{1}{q}}\left\|f\right\|_q, \qquad f \in L^0(\mu).$
(5) *If $1 \leq q_1 \leq q_2 \leq \infty$ then*

$$_r\left\|f\right\|_{q_1,p} \leq (\mu(B_r))^{\frac{1}{q_1}-\frac{1}{q_2}}\ _r\left\|f\right\|_{q_2,p}, \qquad f \in L^0(\mu).$$

(6) *If $1 \leq p_1 \leq p_2 \leq \infty$ then there exists a constant $C > 0$ not depending on r such that*

$$_r\left\|f\right\|_{q,p_2} \leq C(\mu(B_r))^{\frac{1}{p_2}-\frac{1}{p_1}}\ _r\left\|f\right\|_{q,p_1}, \qquad f \in L^0(\mu).$$

We shall use the following notation.

Notation 3.2. For any $r > 0$, we put $[r]_1 = \min\{1, r\}$.

Definition 3.3. Let $1 \leq q \leq \alpha, \lambda \leq p \leq \infty$. The Dunkl-Fofana space and the Dunkl-type total Fofana-Guliyev space are defined as

$$(L^q, L^p)^\alpha(\mu) = \left\{\ f \in L^0(\mu)\ :\ \|f\|_{q,p,\alpha} < \infty\ \right\}$$

and

$$(L^q, L^p)^{\alpha,\lambda}(\mu) = \left\{\ f \in L^0(\mu)\ :\ \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)} < \infty\ \right\},$$

respectively, where

$$\left\|f\right\|_{q,p,\alpha} = \sup_{r>0}(\mu(B_r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\ _r\left\|f\right\|_{q,p}$$

while

$$\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)} = \sup_{r>0}(\mu(B_{[r]_1}))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}(\mu(B_{[1/r]_1}))^{-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p}}\ _r\left\|f\right\|_{q,p}.$$

It is well-known (see [20]) that equipped with the norm $\|\cdot\|_{q,p,\alpha}$, $(L^q, L^p)^\alpha(\mu)$ is a complex Banach space, and for $1 \le s \le t \le \infty$ we have $(L^s, L^t)^s(\mu) = L^s(\mu)$. In addition, $L^\alpha(\mu)$ is continuously embedded in $(L^q, L^p)^\alpha(\mu)$ and $(L^q, L^\infty)^\alpha(\mu)$ is the Dunkl-Morrey space $L_{q,(2k+2)(1-\frac{q}{\alpha}),k}$ defined in [12].

Remark 3.4. For $1 \le q \le \alpha, \lambda < \infty$, the space $(L^q, L^\infty)^{\alpha,\lambda}(\mu)$ is the Dunkl-type total Morrey space $L_{q,(2k+2)\left(1-\frac{q}{\alpha}\right),(2k+2)\left(1-\frac{q}{\lambda}\right)}(\mu)$ as defined in [19].

As a simple application of Proposition 3.1 and Definition 3.3, we derive the following corollary.

Corollary 3.5. *The following assertions hold:*

1. *for* $1 \le q \le \alpha, \lambda \le p \le \infty$, $(L^q, L^p)^{\alpha,\lambda}(\mu)$ *is a complex vector subspace of* $L^0(\mu)$*;*
2. *the map* $f \mapsto \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)}$ *defines a norm on* $(L^q, L^p)^{\alpha,\lambda}(\mu)$.

Remark 3.6. Replacing $(\mu(B_r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}$ by

$$(\mu(B_{[r]_1}))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}(\mu(B_{[1/r]_1}))^{-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p}}$$

in the proof of Proposition 4.16 in [21], it is easy to see that endowed with the norm $\|\cdot\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)}$, $(L^q, L^p)^{\alpha,\lambda}(\mu)$ is a complex Banach space.

The following results state the relationship between the Dunkl-Fofana spaces and the Dunkl-type total Fofana-Guliyev spaces.

Proposition 3.7. *Let* $1 \le q \le \alpha, \lambda \le p \le \infty$. *Then*

$$(L^q, L^p)^\alpha(\mu) \cap (L^q, L^p)^\lambda(\mu) \hookrightarrow (L^q, L^p)^{\alpha,\lambda}(\mu)$$

and for $f \in (L^q, L^p)^\alpha(\mu) \cap (L^q, L^p)^\lambda(\mu)$,

$$\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)} \lesssim \max\{\|f\|_{q,p,\alpha}, \|f\|_{q,p,\lambda}\}.$$

Proof. Let $1 \le q \le \alpha, \lambda \le p \le \infty$ and let $f \in (L^q, L^p)^\alpha(\mu) \cap (L^q, L^p)^\lambda(\mu)$. We have

$$\begin{aligned}
&\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)}\\
=\;& \sup_{r>0}(\mu(B_{[r]_1}))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}(\mu(B_{[1/r]_1}))^{-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p}}\ {}_r\|f\|_{q,p}\\
\le\;& \max\Bigg\{\sup_{0<r\le 1}(\mu(B_r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}(\mu(B_1))^{-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p}}\ {}_r\|f\|_{q,p}\ ,\\
&\quad \sup_{r>1}(\mu(B_1))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}(\mu(B_{1/r}))^{-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p}}\ {}_r\|f\|_{q,p}\Bigg\}\\
\lesssim\;& \max\Bigg\{\sup_{0<r\le 1}(\mu(B_r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\ {}_r\|f\|_{q,p}\ ,\ \sup_{r>1}(\mu(B_r))^{\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p}}\ {}_r\|f\|_{q,p}\Bigg\}\\
\lesssim\;& \max\{\|f\|_{q,p,\alpha}, \|f\|_{q,p,\lambda}\}.
\end{aligned}$$

$\square$

Proposition 3.8. *Let $1 \leq q \leq \lambda \leq \alpha \leq p \leq \infty$. Then*

$$(L^q, L^p)^{\alpha,\lambda}(\mu) = (L^q, L^p)^{\alpha}(\mu) \cap (L^q, L^p)^{\lambda}(\mu)$$

and for $f \in (L^q, L^p)^{\alpha,\lambda}(\mu)$,

$$\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)} \approx \max\{\|f\|_{q,p,\alpha}, \|f\|_{q,p,\lambda}\}.$$

From Proposition 3.8, we have the following observation.

Remark 3.9. For $1 \leq q \leq \alpha \leq p \leq \infty$, the Dunkl-type total Fofana-Guliyev space $(L^q, L^p)^{\alpha,\alpha}(\mu)$ coincides with the Dunkl-Fofana space $(L^q, L^p)^{\alpha}(\mu)$.

Now, we investigate inclusion properties between Dunkl-type total Fofana-Guliyev spaces.

Proposition 3.10. *Let $1 \leq q \leq \alpha_1 \leq \alpha_2 \leq p \leq \infty$ and let $1 \leq q \leq \lambda_2 \leq \lambda_1 \leq p \leq \infty$. We have*

$$\|f\|_{(L^q,L^p)^{\alpha_1,\lambda_1}(\mu)} \lesssim \|f\|_{(L^q,L^p)^{\alpha_2,\lambda_2}(\mu)}, \qquad f \in L^0(\mu)$$

and consequently, $(L^q, L^p)^{\alpha_2,\lambda_2}(\mu) \subset (L^q, L^p)^{\alpha_1,\lambda_1}(\mu)$.

Proposition 3.11. *Let $1 \leq q_1 \leq q_2 \leq \alpha, \lambda \leq p \leq \infty$ and let $1 \leq q \leq \alpha, \lambda \leq p_1 \leq p_2 \leq \infty$. Then*

(1) $\|f\|_{(L^{q_1},L^p)^{\alpha,\lambda}(\mu)} \lesssim \|f\|_{(L^{q_2},L^p)^{\alpha,\lambda}(\mu)}, \qquad f \in L^0(\mu)$
and consequently, $(L^{q_2}, L^p)^{\alpha,\lambda}(\mu) \subset (L^{q_1}, L^p)^{\alpha,\lambda}(\mu)$;
(2) $\|f\|_{(L^q,L^{p_2})^{\alpha,\lambda}(\mu)} \lesssim \|f\|_{(L^q,L^{p_1})^{\alpha,\lambda}(\mu)}, \qquad f \in L^0(\mu)$
and consequently, $(L^q, L^{p_1})^{\alpha,\lambda}(\mu) \subset (L^q, L^{p_2})^{\alpha,\lambda}(\mu)$.

The proofs of Propositions 3.8, 3.10 and 3.11 follow a similar sequence of bounding the supremums over $0 < r \leq 1$ and $r > 1$ as in the proof of Proposition 3.7, and applying Proposition 3.1. They are omitted for the sake of brevity.

4. Norm inequalities for the Dunkl-type Hardy-Littlewood maximal operator

Let $f \in L^0(\mu)$ and let $0 \leq \beta < 2k+2$. The Dunkl-type fractional maximal operator M_β is defined by

$$M_\beta f(x) = \sup_{r>0} (\mu(B_r))^{\frac{\beta}{2k+2}-1} \int_{B_r} \tau_x |f|(y) d\mu(y), \qquad x \in \mathbb{R}.$$

For $\beta = 0$, the Dunkl-type fractional maximal operator reduces to the Dunkl-type Hardy-Littlewood maximal operator M. It is proved in [24] that M is bounded on $L^p(\mu)$ for $1 < p \leq \infty$ and it is of weak-type (1, 1). The boundedness of M on $(L^q, L^p)^{\alpha}(\mu)$ was established in [20].

4.1. **Key Lemmata.** In this subsection, we focus on results that will be used in the proof of norm inequalities for the Dunkl-type Hardy-Littlewood maximal operator in Dunkl-type total Fofana-Guliyev spaces.

Definition 4.1. Let X be a non-empty set. A map $\delta : X \times X \to [0, \infty)$ is called a quasi-distance on X if the following conditions hold:

(1) for every x and y in X, $\delta(x, y) = 0$ if and only if $x = y$;
(2) for every x and y in X, $\delta(x, y) = \delta(y, x)$;
(3) there exists a constant $\mathfrak{c}$ such that $\delta(x, y) \leq \mathfrak{c}(\delta(x, z) + \delta(z, y))$ for every x, y and z in X.

If δ is a quasi-distance on X, then (X, δ) is called a quasi-metric space.

Let (X, δ) be a quasi-metric space. For $x \in X$ and $r > 0$, the ball of center x and radius r is denoted by $I(x, r)$ and is defined as

$$I(x, r) = \{y \in X : \delta(x, y) < r\}.$$

For every $x \in X$, the family $\{I(x, r)\}_{r>0}$ forms a basis of neighborhoods of x for the topology induced by δ.

Definition 4.2. A space of homogeneous type (X, δ, ν) is a quasi-metric space endowed with a non-negative Borel measure ν satisfying the following doubling condition:

$$0 < \nu(I(x, 2r)) \leq C\nu(I(x, r)) < \infty, \qquad x \in X \text{ and } r > 0, \tag{4.1}$$

where C is a constant not depending on x and r.

For a ν-measurable function f in the space of homogeneous type (X, δ, ν), we define the Hardy-Littlewood maximal function $M_\nu f$ by

$$M_\nu f(x) = \sup_{r>0} (\nu(I(x, r)))^{-1} \int_{I(x,r)} |f(y)|\, d\nu(y), \quad x \in X.$$

Lemma 4.3. *Let (X, δ, ν) be a space of homogeneous type. There exists a constant $C > 0$ such that for all $r > 0$,*

$$M_\nu \chi_{I(y,r)}(x) \leq C \frac{\nu(I(y, r))}{\nu(I(y, \delta(x, y) + r))}, \quad x, y \in X.$$

Proof. Let $r > 0$ and let $(x, y) \in X^2$. We have

$$M_\nu(\chi_{I(y,r)})(x) = \sup_{\rho>0} \frac{\nu(I(y, r) \cap I(x, \rho))}{\nu(I(x, \rho))} \leq 1.$$

Let us fix $\rho > 0$. If $\rho \geq \delta(x, y) + r$ then the result is obvious. Assume $\rho < \delta(x, y) + r$. We also suppose that $I(x, r) \cap I(y, \rho) \neq \emptyset$, since otherwise there is nothing to prove. Let $u \in u \in I(x, r) \cap I(y, \rho)$. We have

$$\delta(x, y) \leq \mathfrak{c}(\delta(x, u) + \delta(u, y)) < \mathfrak{c}(\rho + r) \leq 2\mathfrak{c} \max\{\rho, r\}.$$

It follows from (4.1) that

$$\begin{aligned} \nu(I(x, \delta(x, y) + r)) &\approx \nu(I(y, \delta(x, y) + r)) \\ &\leq \begin{cases} \nu(I(y, 2\mathfrak{c}r)) \leq C\nu(I(y, r)) & \text{if} \quad \rho \leq r \\ \nu(I(x, 2\mathfrak{c}\rho)) \leq C\nu(I(x, \rho)) & \text{if} \quad \rho > r. \end{cases} \end{aligned}$$

This ends the proof. □

Remark 4.4. Notice that $(\mathbb{R}, |\cdot|, \mu)$ is a space of homogeneous type.

Lemma 4.5. *Let $x \in \mathbb{R}$ and let $r > 0$. For all $y \in \mathbb{R}$,*

$$M_\mu(\tau_x \chi_{I(0,r)})(y) \approx M_\mu(\chi_{I(-x,r)})(y).$$

Proof. Let $x \in \mathbb{R}$ and let $r > 0$. For all $y \in \mathbb{R}$,

$$\begin{aligned}
\int_{I(y,\rho)} \tau_x \chi_{I(0,r)}(z)d\mu(z) &= \int_{I(0,r)} \tau_{-x}\chi_{I(y,\rho)}(z)d\mu(z) \\
&\approx \int_{I(-x,r)} \chi_{I(y,\rho)}(z)d\mu(z) \quad \text{(see [17])} \\
&= \int_{I(y,\rho)} \chi_{I(-x,r)}(z)d\mu(z).
\end{aligned}$$

□

Lemma 4.6. *Let $r > 0$ and let $y \in \mathbb{R}$. Then*

$$\chi_{B(y,r)} \leq \chi_{I(-y,r)} + \chi_{I(y,r)}.$$

Proof. The claim follows from the fact that for all $r > 0$ and for all $y \in \mathbb{R}$, we have $B(y,r) \subset I(y,r) \cup I(-y,r)$. □

For all locally μ-integrable functions f, we put

$$\widetilde{M}f(x) = \sup_{\rho>0} \frac{1}{\mu(B(x,\rho))} \int_{B(x,\rho)} |f|(z)d\mu(z), \qquad x \in \mathbb{R}.$$

We recall the following result.

Lemma 4.7 ([20])**.** *Let $f \in L^1_{loc}(\mu)$. For all $y \in \mathbb{R}$,*

$$Mf(y) \approx \widetilde{M}f(y) \approx M_\mu f(y).$$

4.2. **Statements and proofs of norm inequalities.** The boundedness result of the Dunkl-type Hardy-Littlewood maximal operator on Dunkl-type total Fofana-Guliyev spaces reads as follows.

Theorem 4.8. *Let $1 < q \leq \alpha, \lambda < p < \infty$ satisfying $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. Then, for all $f \in (L^q, L^p)^{\alpha,\lambda}(\mu)$, we have*

$$\|Mf\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)} .$$

Proof. Let $1 < q \leq \alpha, \lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. Let f be an element of $(L^q, L^p)^{\alpha,\lambda}(\mu)$. Fix $y \in \mathbb{R}$ and $r > 0$. Applying successively Lemma 4.7, Lemma 4.6 and the triangular inequality for the $L^q(\mu)$-norm, we have

$$\begin{aligned}
\left[\int_{\mathbb{R}} (\tau_y (Mf)^q)\chi_{B_r}(x)d\mu(x)\right]^{\frac{1}{q}} &\lesssim \left(\int_{\mathbb{R}} [\widetilde{M}f(x)\chi_{I(-y,r)}(x)]^q d\mu(x)\right)^{\frac{1}{q}} \\
&\quad + \left(\int_{\mathbb{R}} [\widetilde{M}f(x)\chi_{I(y,r)}(x)]^q d\mu(x)\right)^{\frac{1}{q}}.
\end{aligned}$$

Thanks to [3, Theorem 2.35], we get

$$\left(\int_{\mathbb{R}}[\widetilde{M}f(x)\chi_{I(-y,r)}(x)]^q d\mu(x)\right)^{\frac{1}{q}} \lesssim \left(\int_{\mathbb{R}}|f(x)|^q\widetilde{M}\chi_{I(-y,r)}(x)d\mu(x)\right)^{\frac{1}{q}}$$

and

$$\left(\int_{\mathbb{R}}[\widetilde{M}f(x)\chi_{I(y,r)}(x)]^q d\mu(x)\right)^{\frac{1}{q}} \lesssim \left(\int_{\mathbb{R}}|f(x)|^q\widetilde{M}\chi_{I(y,r)}(x)d\mu(x)\right)^{\frac{1}{q}}.$$

It follows that

$$\begin{aligned}\left[\int_{\mathbb{R}}(\tau_y\,(Mf)^q)\chi_{B_r}(x)d\mu(x)\right]^{\frac{1}{q}} \;&\lesssim\; \left(\int_{\mathbb{R}}|f(x)|^q\widetilde{M}\chi_{I(-y,r)}(x)d\mu(x)\right)^{\frac{1}{q}} \\ &\quad+\left(\int_{\mathbb{R}}|f(x)|^q\widetilde{M}\chi_{I(y,r)}(x)d\mu(x)\right)^{\frac{1}{q}}.\end{aligned} \tag{4.2}$$

Applying Lemma 4.5, Lemma 4.7 and the fact that Dunkl translation commutes with Dunkl-type Hardy-Littlewood maximal operator (see [12]), we have

$$\int_{\mathbb{R}}|f|^q(x)\widetilde{M}\chi_{I(-y,r)}(x)d\mu(x) \lesssim \int_{\mathbb{R}}\tau_{-y}|f|^q(x)(M_\mu\chi_{I(0,r)})(x)d\mu(x).$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}}|f|^q(x)\widetilde{M}\chi_{I(-y,r)}(x)d\mu(x) \;&\lesssim\; \int_{I(0,2r)}\tau_{-y}\,|f(x)|^q\,M_\mu\chi_{I(0,r)}(x)d\mu(x) \\ &\quad+\int_{\mathbb{R}\setminus I(0,2r)}\tau_{-y}\,|f(x)|^q\,M_\mu\chi_{I(0,r)}(x)d\mu(x).\end{aligned}$$

We have

$$\int_{I(0,2r)}\tau_{-y}\,|f(x)|^q\,M_\mu\chi_{I(0,r)}(x)d\mu(x) \le \int_{I(0,2r)}\tau_{-y}\,|f(x)|^q\,d\mu(x) = K(-y)$$

and, by Lemma 4.3,

$$\begin{aligned}&\int_{\mathbb{R}\setminus I(0,2r)}\tau_{-y}\,|f(x)|^q\,M_\mu\chi_{I(0,r)}(x)d\mu(x) \\ &\lesssim \sum_{i=1}^{\infty}\int_{2^i r\le|x|<2^{i+1}r}\tau_{-y}\,|f(x)|^q\,\frac{\mu(I(0,r))}{\mu(I(0,|x|+r))}d\mu(x) \\ &\lesssim \sum_{i=1}^{\infty}\frac{\mu(I(0,r))}{\mu(I(0,(2^i+1)r))}\int_{I(0,2^{i+1}r)}\tau_{-y}\,|f(x)|^q\,d\mu(x) = L(-y).\end{aligned}$$

It follows that

$$\left(\int_{\mathbb{R}}|f|^q(x)\widetilde{M}\chi_{I(-y,r)}(x)d\mu(x)\right)^{\frac{1}{q}} \lesssim (K(-y)+L(-y))^{\frac{1}{q}}. \tag{4.3}$$

Using the same arguments as above, we obtain

$$\left(\int_{\mathbb{R}}|f|^q(x)\widetilde{M}\chi_{I(y,r)}(x)d\mu(x)\right)^{\frac{1}{q}} \lesssim (K(y)+L(y))^{\frac{1}{q}}. \tag{4.4}$$

Taking into account (4.3) and (4.4), (4.2) becomes

$$\left[\int_{\mathbb{R}}(\tau_y\left(Mf\right)^q)\chi_{B_r}(x)d\mu(x)\right]^{\frac{1}{q}} \lesssim (K(-y)+L(-y))^{\frac{1}{q}}+(K(y)+L(y))^{\frac{1}{q}}.$$

Since μ is a symmetric invariant measure, by taking the $L^p(\mu)$-norm of both sides in the above relation and applying Minkowski's inequality for integral, we get

$$\left\|\left[\int_{\mathbb{R}}(\tau_{(\cdot)}\left(Mf\right)^q)\chi_{B_r}(x)d\mu(x)\right]^{\frac{1}{q}}\right\|_p \lesssim I+II,$$

where

$$I=\left(\int_{\mathbb{R}}\left(\int_{I(0,2r)}\tau_y\left|f(x)\right|^q d\mu(x)\right)^{\frac{p}{q}}d\mu(y)\right)^{\frac{1}{p}}$$

and

$$II=\sum_{i=1}^{\infty}\frac{\mu(I(0,r))^{\frac{1}{q}}}{\mu(I(0,(2^i+1)r))^{\frac{1}{q}}}\left(\int_{\mathbb{R}}\left(\int_{I(0,2^{i+1}r)}\tau_y\left|f(x)\right|^q d\mu(x)\right)^{\frac{p}{q}}d\mu(y)\right)^{\frac{1}{p}}.$$

On the one hand,

$$\begin{aligned}
I &= \frac{(\mu(I(0,[2r]_1)))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}(\mu(I(0,[1/2r]_1)))^{-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p}}}{(\mu(I(0,[2r]_1)))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}(\mu(I(0,[1/2r]_1)))^{-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p}}}\\
&\quad\times\left(\int_{\mathbb{R}}\left(\int_{I(0,2r)}\tau_y\left|f(x)\right|^q d\mu(x)\right)^{\frac{p}{q}}d\mu(y)\right)^{\frac{1}{p}}\\
&\le (\mu(I(0,[2r]_1)))^{-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p}}(\mu(I(0,[1/2r]_1)))^{\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p}}\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)}.
\end{aligned}$$

Hence,

$$I \lesssim (\mu(I(0,[r]_1)))^{-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p}}(\mu(I(0,[1/r]_1)))^{\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p}}\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)} \tag{4.5}$$

since $-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p}>0$, $\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p}<0$ and μ is doubling.
On the other hand,

$$\begin{aligned}
II &= \sum_{i=1}^{\infty}\frac{1}{(2^i+1)^{\frac{2k+2}{q}}}\left(\int_{\mathbb{R}}\left(\int_{I(0,2^{i+1}r)}\tau_y\left|f(x)\right|^q d\mu(x)\right)^{\frac{p}{q}}d\mu(y)\right)^{\frac{1}{p}}\\
&\le \left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)}\\
&\quad\times\sum_{i=1}^{\infty}\frac{1}{(2^i)^{\frac{2k+2}{q}}}(\mu(I(0,[2^{i+1}r]_1)))^{-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p}}(\mu(I(0,[1/2^{i+1}r]_1)))^{\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p}}.
\end{aligned}$$

Keeping in mind that μ is doubling, and $\sum_{i=1}^{\infty}(2^i)^{(2k+2)(-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{1}{q}+\frac{2}{p})}<\infty$, we obtain through simple calculations that

$$II \lesssim (\mu(I(0,[r]_1)))^{-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p}}(\mu(I(0,[1/r]_1)))^{\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p}}\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)}. \tag{4.6}$$

From (4.5) and (4.6), we deduce that

$$\begin{aligned}
&\left\| \left[\int_{\mathbb{R}} (\tau_{(\cdot)} (Mf)^q) \chi_{B_r}(x) d\mu(x) \right]^{\frac{1}{q}} \right\|_p \\
\lesssim\ & (\mu(I(0,[r]_1)))^{-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p}} (\mu(I(0,[1/r]_1)))^{\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p}} \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)} .
\end{aligned}$$

This implies that

$$\begin{aligned}
& (\mu(I(0,[r]_1)))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} (\mu(I(0,[1/r]_1)))^{-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p}} \\
& \times \left\| \left[\int_{\mathbb{R}} (\tau_{(\cdot)} (Mf)^q) \chi_{B_r}(x) d\mu(x) \right]^{\frac{1}{q}} \right\|_p \\
\lesssim\ & \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)} .
\end{aligned}$$

We end the proof by taking the supremum over all $r > 0$ in the left-hand side of the previous relation. □

Let $1 < \alpha,\ \lambda \le p < \infty$. For a locally μ-measurable function f, we put

$$\begin{aligned}
&\|f\|_{(L^{1,\infty},L^p)^{\alpha,\lambda}(\mu)} \\
=\ & \sup_{r>0} (\mu(B_{[r]_1}))^{\frac{1}{\alpha}-1-\frac{1}{p}} (\mu(B_{[1/r]_1}))^{-\frac{1}{\lambda}+1+\frac{1}{p}} \| \|f\tau_{-y}\chi_{B_r}\|_{L^{1,\infty}(\mu)} \|_p,
\end{aligned}$$

where $L^{1,\infty}(\mu)$ denotes the weighted weak-Lebesgue space whose elements are the functions $f \in L^0(\mu)$ satisfying the condition

$$\|f\|_{L^{1,\infty}(\mu)} = \sup_{\lambda>0} \lambda \mu(\{\ x \in \mathbb{R} : |f(x)| > \lambda\ \}) < \infty.$$

We have the following weak-type inequality.

Theorem 4.9. *Let $1 < \alpha, \lambda < p < \infty$ such that $1 + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. Then, for all $f \in (L^1, L^p)^{\alpha,\lambda}(\mu)$, we have*

$$\|Mf\|_{(L^{1,\infty},L^p)^{\alpha,\lambda}(\mu)} \lesssim \|f\|_{(L^1,L^p)^{\alpha,\lambda}(\mu)}, \quad f \in L^1_{loc}(\mu).$$

Proof. Let $1 < \alpha,\ \lambda < p < \infty$ such that $1 + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$ and let $f \in (L^1, L^p)^{\alpha,\lambda}(\mu)$. It follows from Lemma 4.7 that

$$\|Mf\|_{(L^{1,\infty},L^p)^{\alpha,\lambda}(\mu)} \lesssim \|\widetilde{M}f\|_{(L^{1,\infty},L^p)^{\alpha,\lambda}(\mu)}. \tag{4.7}$$

Let $y \in \mathbb{R}$ and $r > 0$. A slight adaptation of the proof of Theorem 5.2 in [20], in which Lemma 6.2 in [20] is replaced by Lemma 4.6, leads to the following estimate:

$$\begin{aligned}
\|(\widetilde{M}f)\tau_{-y}\chi_{B_r}\|_{L^{1,\infty}(\mu)} \ \lesssim\ & \int_{\mathbb{R}} |f|(x) \widetilde{M}\chi_{I(-y,r)}(x) d\mu(x) \\
& + \int_{\mathbb{R}} |f|(x) \widetilde{M}\chi_{I(y,r)}(x) d\mu(x).
\end{aligned} \tag{4.8}$$

Using the same argument as in the proof of Theorem 4.8, we get

$$\begin{aligned}\int_{\mathbb{R}} |f|(x)\widetilde{M}\chi_{I(-y,r)}(x)d\mu(x) &\lesssim \int_{I(0,2r)} \tau_{-y}\,|f(x)|\,d\mu(x)\\ &\quad + \sum_{i=1}^{\infty} \frac{\mu(I(0,r))}{\mu(I(0,(2^i+1)r))}\int_{I(0,2^{i+1}r)} \tau_{-y}\,|f(x)|\,d\mu(x)\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}} |f|(x)\widetilde{M}\chi_{I(y,r)}(x)d\mu(x) &\lesssim \int_{I(0,2r)} \tau_{y}\,|f(x)|\,d\mu(x)\\ &\quad + \sum_{i=1}^{\infty} \frac{\mu(I(0,r))}{\mu(I(0,(2^i+1)r))}\int_{I(0,2^{i+1}r)} \tau_{y}\,|f(x)|\,d\mu(x).\end{aligned}$$

Therefore, we deduce from (4.8) that

$$\begin{aligned}&\|(\widetilde{M}f)\tau_{-y}\chi_{B_r}\|_{L^{1,\infty}(\mu)}\\ &\lesssim \int_{I(0,2r)} \tau_{y}\,|f(x)|\,d\mu(x) + \sum_{i=1}^{\infty} \frac{\mu(I(0,r))}{\mu(I(0,(2^i+1)r))}\int_{I(0,2^{i+1}r)} \tau_{y}\,|f(x)|\,d\mu(x)\\ &\quad + \int_{I(0,2r)} \tau_{-y}\,|f(x)|\,d\mu(x) + \sum_{i=1}^{\infty} \frac{\mu(I(0,r))}{\mu(I(0,(2^i+1)r))}\int_{I(0,2^{i+1}r)} \tau_{-y}\,|f(x)|\,d\mu(x).\end{aligned}$$

We end the proof as we did in Theorem 4.8. □

Combining Theorem 4.8 with Theorem 4.9, in the case $\alpha = \lambda$, we get the following result.

Corollary 4.10. *Let $1 \leq q \leq \alpha < p < \infty$ satisfying $\frac{1}{q} + \frac{2}{p} < \frac{2}{\alpha}$.*

(1) *If $q > 1$, then for all $f \in (L^q, L^p)^\alpha(\mu)$ we have*
$$\|Mf\|_{(L^q,L^p)^\alpha(\mu)} \lesssim \|f\|_{(L^q,L^p)^\alpha(\mu)}\,.$$
(2) *If $q = 1 < \alpha$, then for all $f \in L^1_{loc}(\mu)$, we have*
$$\|Mf\|_{(L^{1,\infty},L^p)^\alpha(\mu)} \lesssim \|f\|_{(L^1,L^p)^\alpha(\mu)}.$$

Remark 4.11. Note that:

(1) Corollary 4.10 was established in [20] without the assumption $\frac{1}{q} + \frac{2}{p} < \frac{2}{\alpha}$;
(2) part (1) of Corollary 4.10, in the case $\alpha = q$, was proved in [1].

5. Norm inequalities for Dunkl-type fractional integral and maximal operators

The Dunkl-type fractional integral operator I_β, $0 < \beta < 2k + 2$, is defined as

$$I_\beta f(x) = \int_{\mathbb{R}} \tau_x f(z)|z|^{\beta-(2k+2)}d\mu(z), \qquad x \in \mathbb{R}.$$

Its boundedness properties have been investigated in Dunkl-Lebesgue spaces (see [13]) and in Dunkl-Fofana spaces (see [21]). We have the following norm inequalities for Dunkl-type fractional integral operators in Dunkl-type total Fofana-Guliyev spaces.

Theorem 5.1. *Let $1 < q \leq \alpha, \lambda < p < \infty$ satisfying $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. Let $0 < \beta < (2k+2)(\frac{1}{\alpha} + \frac{1}{\lambda} - \frac{1}{q})$. Put*

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{q\lambda\beta}{(2k+2)(q\lambda + q\alpha - \alpha\lambda)}, \quad \frac{1}{\lambda^*} = \frac{1}{\lambda} - \frac{q\alpha\beta}{(2k+2)(q\lambda + q\alpha - \alpha\lambda)},$$

$$\frac{1}{\bar{p}} = \frac{1}{p}\left(1 - \frac{q\alpha\lambda\beta}{(2k+2)(q\lambda + q\alpha - \alpha\lambda)}\right) \text{ and } \frac{1}{\bar{q}} = \frac{1}{q} - \frac{\lambda\alpha\beta}{(2k+2)(q\lambda + q\alpha - \alpha\lambda)}.$$

Then, for all $f \in (L^q, L^p)^{\alpha,\lambda}(\mu)$, we have

$$\|I_\beta f\|_{(L^{\bar{q}}, L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)}^{1-\frac{q\alpha\lambda\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}} \|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)}^{\frac{q\alpha\lambda\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}}$$

and

$$\|I_\beta f\|_{(L^{\bar{q}}, L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)}.$$

Proof. Let $1 < q \leq \alpha, \lambda < p < \infty$ satisfying $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. Let $0 < \beta < (2k+2)(\frac{1}{\alpha} + \frac{1}{\lambda} - \frac{1}{q})$ and let $f \in (L^q, L^p)^{\alpha,\lambda}(\mu)$. Assume that $f \neq 0$ μ-almost everywhere, otherwise the result is obvious. Let $r > 0$ and let $x \in \mathbb{R}$. We have

$$\begin{aligned} I_\beta f(x) &= \int_{B_r} \tau_x f(z)|z|^{\beta-2k-2} d\mu(z) + \int_{\mathbb{R}\setminus B_r} \tau_x f(z)|z|^{\beta-2k-2} d\mu(z) \\ &= A(x) + B(x). \end{aligned}$$

Following the proof of Theorem 1.1 in [21], we may write that

$$|A(x)| \leq Cr^\beta Mf(x) \tag{5.1}$$

and

$$|B(x)| \leq \sum_{i=0}^{\infty} (2^i r)^{\beta-2k-2} \mu(B(0, 2^{i+1}r))^{1-\frac{1}{q}} \left\| \left(\int_{B(0,2^{i+1}r)} \tau_{(\cdot)} |f(z)|^q d\mu(z) \right)^{\frac{1}{q}} \right\|_\infty.$$

It follows that

$$\begin{aligned} &|B(x)| \\ \leq\ & \sum_{i=0}^{\infty} (2^i r)^{\beta-2k-2} \mu(B(0, 2^{i+1}r))^{1-\frac{1}{q}} \mu(B(0, [2^{i+1}r]_1))^{-\frac{1}{\alpha}+\frac{1}{q}} \\ & \times \mu(B(0, [1/2^{i+1}r]_1))^{\frac{1}{\lambda}-\frac{1}{q}} \mu(B(0, [2^{i+1}r]_1))^{\frac{1}{\alpha}-\frac{1}{q}} \\ & \times \mu(B(0, [1/2^{i+1}r]_1))^{-\frac{1}{\lambda}+\frac{1}{q}} \left\| \left(\int_{B(0,2^{i+1}r)} \tau_{(\cdot)} |f(z)|^q d\mu(z) \right)^{\frac{1}{q}} \right\|_\infty \\ \leq\ & \sum_{i=0}^{\infty} (2^i r)^{\beta-2k-2} \mu(B(0, 2^{i+1}r))^{1-\frac{1}{q}} \mu(B(0, 2^{i+1}r))^{-\frac{1}{\alpha}+\frac{1}{q}} \mu(B(0, \frac{1}{2^{i+1}}[1/r]_1))^{\frac{1}{\lambda}-\frac{1}{q}} \\ & \times \|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)} \\ \leq\ & \|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)} \sum_{i=0}^{\infty} (2^i r)^{\beta-2k-2} \mu(B(0, 2^{i+1}r))^{1-\frac{1}{\alpha}} \mu(B(0, \frac{1}{2^{i+1}}[1/r]_1))^{\frac{1}{\lambda}-\frac{1}{q}}. \end{aligned}$$

Since μ is doubling and $\sum_{i=0}^{\infty}(2^i)^{\beta-(2k+2)(\frac{1}{\alpha}+\frac{1}{\lambda}-\frac{1}{q})} < \infty$, we get

$$|B(x)| \leq Cr^{\beta-\frac{2k+2}{\alpha}}[1/r]_1^{(2k+2)(\frac{1}{\lambda}-\frac{1}{q})}\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)}. \tag{5.2}$$

It follows from (5.1) and (5.2) that

$$\begin{aligned}
|I_\beta f(x)| &\leq C\left(r^\beta Mf(x) + r^{\beta-\frac{2k+2}{\alpha}}[1/r]_1^{(2k+2)(\frac{1}{\lambda}-\frac{1}{q})}\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)}\right)\\
&\leq C\min\Big\{ r^\beta Mf(x) + r^{\beta-\frac{2k+2}{\alpha}}(1/r)^{(2k+2)(\frac{1}{\lambda}-\frac{1}{q})}\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)} \quad ,\\
&\qquad r^\beta Mf(x) + r^{\beta-\frac{2k+2}{\alpha}}\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)} \Big\}\\
&\leq C\min\Big\{ r^\beta Mf(x) + r^{\beta-(2k+2)(\frac{1}{\alpha}+\frac{1}{\lambda}-\frac{1}{q})}\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)} \quad ,\\
&\qquad r^\beta Mf(x) + r^{\beta-\frac{2k+2}{\alpha}}\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)} \Big\}.
\end{aligned}$$

Minimizing with respect to

$$r = \left(\frac{\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)}}{Mf(x)}\right)^{\frac{q\alpha\lambda}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}} \quad \text{or} \quad r = \left(\frac{\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)}}{Mf(x)}\right)^{\frac{\alpha}{2k+2}}$$

we obtain

$$\begin{aligned}
|I_\beta f(x)| &\leq C\min\Big\{ \|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)}^{\frac{q\alpha\lambda\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}}(Mf(x))^{1-\frac{q\lambda\alpha\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}} \quad ,\\
&\qquad (Mf(x))^{1-\frac{\alpha\beta}{2k+2}}\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)}^{\frac{\alpha\beta}{2k+2}} \Big\}.
\end{aligned}$$

This implies that

$$|I_\beta f(x)| \leq C\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)}^{\frac{q\alpha\lambda\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}}(Mf(x))^{1-\frac{q\lambda\alpha\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}}. \tag{5.3}$$

From (5.3), we have

$$\begin{aligned}
{}_r\|I_\beta f\|_{\bar{q},\bar{p}} &\leq C\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)}^{\frac{q\alpha\lambda\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}}\ {}_r\|(Mf)^{1-\frac{q\lambda\alpha\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}}\|_{\bar{q},\bar{p}}\\
&\leq C\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)}^{\frac{q\alpha\lambda\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}}\ {}_r\|Mf\|_{q,p}^{1-\frac{q\lambda\alpha\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}}.
\end{aligned}$$

Then, multiplying both sides of the previous inequality by

$$\mu(B(0,[r]_1))^{\frac{1}{\alpha^*}-\frac{1}{\bar{q}}-\frac{1}{\bar{p}}}\mu(B(0,[1/r]_1))^{-\frac{1}{\lambda^*}+\frac{1}{\bar{q}}+\frac{1}{\bar{p}}},$$

we get

$$\begin{aligned}
&\mu(B(0,[r]_1))^{\frac{1}{\alpha^*}-\frac{1}{\bar{q}}-\frac{1}{\bar{p}}}\mu(B(0,[1/r]_1))^{-\frac{1}{\lambda^*}+\frac{1}{\bar{q}}+\frac{1}{\bar{p}}}\ {}_r\|I_\beta f\|_{\bar{q},\bar{p}}\\
\leq\ & C\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)}^{\frac{q\alpha\lambda\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}}[\mu(B(0,[r]_1))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}]^{1-\frac{q\lambda\alpha\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}}\\
&\times[\mu(B(0,[1/r]_1))^{-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p}}]^{1-\frac{q\lambda\alpha\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}}\ {}_r\|Mf\|_{q,p}^{1-\frac{q\lambda\alpha\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}}.
\end{aligned}$$

Taking the supremum over all $r > 0$ in both sides of the above inequality, we obtain

$$\|I_\beta f\|_{(L^{\bar{q}},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \leq C\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)}^{\frac{q\alpha\lambda\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}} \|Mf\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)}^{1-\frac{q\alpha\lambda\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}}.$$

From Theorem 4.8, we deduce that

$$\|I_\beta f\|_{(L^{\bar{q}},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \leq C\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)}^{1-\frac{q\alpha\lambda\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}} \|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)}^{\frac{q\alpha\lambda\beta}{(2k+2)(q\lambda+q\alpha-\alpha\lambda)}}. \tag{5.4}$$

Since Proposition 3.11 asserts that there exists a constant $C > 0$ not depending on f such that

$$\|f\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mu)} \leq C\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)}, \tag{5.5}$$

we end the proof by combining (5.4) with (5.5). $\square$

For $q = 1$, we have the following statement.

Theorem 5.2. *Let $1 < \alpha, \lambda < p < \infty$ satisfying $1 + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$ and let $0 < \beta < (2k+2)(\frac{1}{\alpha} + \frac{1}{\lambda} - 1)$. Put*

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\lambda\beta}{(2k+2)(\lambda+\alpha-\alpha\lambda)}, \quad \frac{1}{\lambda^*} = \frac{1}{\lambda} - \frac{\alpha\beta}{(2k+2)(\lambda+\alpha-\alpha\lambda)},$$

$$\frac{1}{\bar{p}} = \frac{1}{p}\left(1 - \frac{\alpha\lambda\beta}{(2k+2)(\lambda+\alpha-\alpha\lambda)}\right) \text{ and } \frac{1}{\bar{q}} = 1 - \frac{\lambda\alpha\beta}{(2k+2)(\lambda+\alpha-\alpha\lambda)}.$$

Then, for all $f \in (L^1, L^p)^{\alpha,\lambda}(\mu)$, we have

$$\|I_\beta f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \lesssim \|f\|_{(L^1,L^p)^{\alpha,\lambda}(\mu)}^{1-\frac{\alpha\lambda\beta}{(2k+2)(\lambda+\alpha-\alpha\lambda)}} \|f\|_{(L^1,L^\infty)^{\alpha,\lambda}(\mu)}^{\frac{\alpha\lambda\beta}{(2k+2)(\lambda+\alpha-\alpha\lambda)}},$$

where

$$\begin{aligned} &\|I_\beta f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \\ = \;& \sup_{r>0} (\mu(B_{[r]_1}))^{\frac{1}{\alpha^*}-\frac{1}{\bar{q}}-\frac{1}{\bar{p}}} (\mu(B_{[1/r]_1}))^{-\frac{1}{\lambda^*}+\frac{1}{\bar{q}}+\frac{1}{\bar{p}}} \| \| I_\beta f \tau_{-y}^{\frac{1}{\bar{q}}} \chi_{B_r} \|_{L^{\bar{q},\infty}(\mu)} \|_{\bar{p}} \end{aligned}$$

and

$$\|I_\beta f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \lesssim \|f\|_{(L^1,L^p)^{\alpha,\lambda}(\mu)}.$$

Proof. Let $1 < \alpha, \lambda < p < \infty$ satisfying $1 + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$ and let $0 < \beta < (2k+2)(\frac{1}{\alpha} + \frac{1}{\lambda} - 1)$. Let $f \in (L^1, L^p)^{\alpha,\lambda}(\mu)$. We assume that $f \neq 0$ μ-almost everywhere, otherwise the result is obvious. From (5.3), we have

$$\|I_\beta f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \leq C\|f\|_{(L^1,L^\infty)^{\alpha,\lambda}(\mu)}^{\frac{\alpha\lambda\beta}{(2k+2)(\lambda+\alpha-\alpha\lambda)}} \|(Mf)^{1-\frac{\lambda\alpha\beta}{(2k+2)(\lambda+\alpha-\alpha\lambda)}}\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)}.$$

Let $r > 0$. Thanks to the proof of Theorem 1.2 in [21], we may write

$$\|\|(Mf)^{1-\frac{\lambda\alpha\beta}{(2k+2)(\lambda+\alpha-\alpha\lambda)}} \tau_{-y}^{\frac{1}{\bar{q}}} \chi_{B_r}\|_{L^{\bar{q},\infty}(\mu)}\|_{\bar{p}} = \|\|(Mf)\tau_{-y}\chi_{B_r}\|_{L^{1,\infty}(\mu)}\|_p^{1-\frac{\lambda\alpha\beta}{(2k+2)(\lambda+\alpha-\alpha\lambda)}}.$$

Following the same steps as in the proof of Theorem 5.1, but replacing the strong-type maximal bounds with the weak-type bounds from Theorem 4.9, we obtain

$$\|I_\beta f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \leq C\|f\|_{(L^1,L^p)^{\alpha,\lambda}(\mu)}^{1-\frac{\alpha\lambda\beta}{(2k+2)(\lambda+\alpha-\alpha\lambda)}} \|f\|_{(L^1,L^\infty)^{\alpha,\lambda}(\mu)}^{\frac{\alpha\lambda\beta}{(2k+2)(\lambda+\alpha-\alpha\lambda)}}. \tag{5.6}$$

Since Proposition 3.11 asserts that there exists a constant $C > 0$ such that

$$\|f\|_{(L^1,L^\infty)^{\alpha,\lambda}(\mu)} \le C\|f\|_{(L^1,L^p)^{\alpha,\lambda}(\mu)}, \tag{5.7}$$

we end the proof by combining (5.6) with (5.7). □

Theorem 5.1 and Theorem 5.2, in the case $\alpha = \lambda$, yield the following result.

Corollary 5.3. *Let $1 \le q \le \alpha < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{2}{\alpha}$ and let $0 < \beta < (2k+2)(\frac{2}{\alpha} - \frac{1}{q})$. Put*

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\beta}{2k+2}\frac{1}{\alpha(\frac{2}{\alpha} - \frac{1}{q})}, \quad \frac{1}{\bar{p}} = \frac{1}{p} - \frac{\beta}{2k+2}\frac{1}{p(\frac{2}{\alpha} - \frac{1}{q})}$$

$$\text{and } \frac{1}{\bar{q}} = \frac{1}{q} - \frac{\beta}{2k+2}\frac{1}{q(\frac{2}{\alpha} - \frac{1}{q})}.$$

(1) If $1 < q$, then for all $f \in (L^q, L^p)^\alpha(\mu)$ we have

$$\|I_\beta f\|_{(L^{\bar{q}},L^{\bar{p}})^{\alpha^*}(\mu)} \lesssim \|f\|_{(L^q,L^p)^\alpha(\mu)}^{1-\frac{\beta}{2k+2}\frac{1}{\frac{2}{\alpha}-\frac{1}{q}}} \|f\|_{(L^q,L^\infty)^\alpha(\mu)}^{\frac{\beta}{2k+2}\frac{1}{\frac{2}{\alpha}-\frac{1}{q}}}$$

and

$$\|I_\beta f\|_{(L^{\bar{q}},L^{\bar{p}})^{\alpha^*}(\mu)} \lesssim \|f\|_{(L^q,L^p)^\alpha(\mu)}.$$

(2) If $q = 1 < \alpha$, then for all $f \in (L^1, L^p)^\alpha(\mu)$ we have

$$\|I_\beta f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \lesssim \|f\|_{(L^1,L^p)^\alpha(\mu)}^{1-\frac{\beta}{2k+2}\frac{1}{\frac{2}{\alpha}-1}} \|f\|_{(L^1,L^\infty)^\alpha(\mu)}^{\frac{\beta}{2k+2}\frac{1}{\frac{2}{\alpha}-1}}$$

and

$$\|I_\beta f\|_{(L^{\bar{q},\infty},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \lesssim \|f\|_{(L^1,L^p)^\alpha(\mu)}.$$

As a consequence of Theorem 5.1, we obtain the corresponding boundedness result for M_β.

Corollary 5.4. *Let $1 < q \le \alpha, \lambda < p < \infty$ satisfying $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. Let $0 < \beta < (2k+2)(\frac{1}{\alpha} + \frac{1}{\lambda} - \frac{1}{q})$. Put*

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{q\lambda\beta}{(2k+2)(q\lambda + q\alpha - \alpha\lambda)}, \quad \frac{1}{\lambda^*} = \frac{1}{\lambda} - \frac{q\alpha\beta}{(2k+2)(q\lambda + q\alpha - \alpha\lambda)},$$

$$\frac{1}{\bar{p}} = \frac{1}{p}\left(1 - \frac{q\alpha\lambda\beta}{(2k+2)(q\lambda + q\alpha - \alpha\lambda)}\right) \text{ and } \frac{1}{\bar{q}} = \frac{1}{q} - \frac{\lambda\alpha\beta}{(2k+2)(q\lambda + q\alpha - \alpha\lambda)}.$$

Then, for all $f \in (L^q, L^p)^{\alpha,\lambda}(\mu)$, we have

$$\|M_\beta f\|_{(L^{\bar{q}},L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mu)}.$$

Proof. Let $1 < q \le \alpha, \lambda < p < \infty$ satisfying $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$ and let $0 < \beta < (2k+2)(\frac{1}{\alpha} + \frac{1}{\lambda} - \frac{1}{q})$. Let $f \in (L^q, L^p)^{\alpha,\lambda}(\mu)$. It is well known that

$$M_\beta f(x) \le d_k^{\frac{\beta}{2k+2}-1} I_\beta |f|(x), \qquad x \in \mathbb{R} \tag{5.8}$$

(see [21]). So, we end the proof by taking the norm $\|\cdot\|_{(L^{\bar{q}}, L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)}$ on the both sides of (5.8) and by applying Theorem 5.1. □

An immediate consequence of Theorem 5.2 and inequality (5.8) is the following weak-type inequality for M_β.

Corollary 5.5. *Let $1 < \alpha, \lambda < p < \infty$ satisfying $1 + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. Let $0 < \beta < (2k+2)(\frac{1}{\alpha} + \frac{1}{\lambda} - 1)$. Put*

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\lambda\beta}{(2k+2)(\lambda + \alpha - \alpha\lambda)}, \quad \frac{1}{\lambda^*} = \frac{1}{\lambda} - \frac{\alpha\beta}{(2k+2)(\lambda + \alpha - \alpha\lambda)},$$

$$\frac{1}{\bar{p}} = \frac{1}{p}\left(1 - \frac{\alpha\lambda\beta}{(2k+2)(\lambda + \alpha - \alpha\lambda)}\right) \text{ and } \frac{1}{\bar{q}} = 1 - \frac{\lambda\alpha\beta}{(2k+2)(\lambda + \alpha - \alpha\lambda)}.$$

Then there exists a constant $C > 0$ such that

$$\|M_\beta f\|_{(L^{\bar{q},\infty}, L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \le C\|f\|_{(L^1, L^p)^{\alpha,\lambda}(\mu)}, \qquad f \in (L^1, L^p)^{\alpha,\lambda}(\mu).$$

By Corollary 5.3 and inequality (5.8) we have the following result.

Corollary 5.6. *Let $1 \le q \le \alpha < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{2}{\alpha}$ and let $0 < \beta < (2k+2)(\frac{2}{\alpha} - \frac{1}{q})$. Put*

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\beta}{2k+2}\frac{1}{\alpha(\frac{2}{\alpha} - \frac{1}{q})}, \quad \frac{1}{\bar{p}} = \frac{1}{p} - \frac{\beta}{2k+2}\frac{1}{p(\frac{2}{\alpha} - \frac{1}{q})}$$

$$\text{and } \frac{1}{\bar{q}} = \frac{1}{q} - \frac{\beta}{2k+2}\frac{1}{q(\frac{2}{\alpha} - \frac{1}{q})}.$$

(1) If $1 < q$, then for all $f \in (L^q, L^p)^\alpha(\mu)$ we have

$$\|M_\beta f\|_{(L^{\bar{q}}, L^{\bar{p}})^{\alpha^*}(\mu)} \lesssim \|f\|_{(L^q, L^p)^\alpha(\mu)}^{1 - \frac{\beta}{2k+2}\frac{1}{\frac{2}{\alpha} - \frac{1}{q}}} \|f\|_{(L^q, L^\infty)^\alpha(\mu)}^{\frac{\beta}{2k+2}\frac{1}{\frac{2}{\alpha} - \frac{1}{q}}}$$

and

$$\|M_\beta f\|_{(L^{\bar{q}}, L^{\bar{p}})^{\alpha^*}(\mu)} \lesssim \|f\|_{(L^q, L^p)^\alpha(\mu)}.$$

(2) If $q = 1 < \alpha$, then for all $f \in (L^1, L^p)^\alpha(\mu)$ we have

$$\|M_\beta f\|_{(L^{\bar{q},\infty}, L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \lesssim \|f\|_{(L^1, L^p)^\alpha(\mu)}^{1 - \frac{\beta}{2k+2}\frac{1}{\frac{2}{\alpha} - 1}} \|f\|_{(L^1, L^\infty)^\alpha(\mu)}^{\frac{\beta}{2k+2}\frac{1}{\frac{2}{\alpha} - 1}}$$

and

$$\|M_\beta f\|_{(L^{\bar{q},\infty}, L^{\bar{p}})^{\alpha^*,\lambda^*}(\mu)} \lesssim \|f\|_{(L^1, L^p)^\alpha(\mu)}.$$

Remark 5.7. Note that part (1) of Corollary 5.3 and Corollary 5.4, in the case $\alpha = q$, were proved in [13].

6. Conclusion

In summary, we have introduced a new class of function spaces, called Dunkl-type total Fofana-Guliyev spaces, which merge the frameworks of Dunkl harmonic analysis on the real line and total Morrey/Fofana-Guliyev spaces. By closely following existing literature on total Morrey spaces and total Fofana-Guliyev spaces, we have investigated the structural properties of these spaces, showing that they generalize classical Dunkl-Fofana spaces and can be characterized as intersections of specific Dunkl-Fofana spaces. Furthermore, we have established norm inequalities for the Dunkl-type Hardy-Littlewood maximal operator, fractional maximal operators, and fractional integral operators within these new Banach spaces. We plan to study the role that Dunkl-type total Fofana-Guliyev spaces could play in the qualitative analysis of partial differential equations.

Acknowledgement. The authors sincerely thank the anonymous reviewers for their careful reading and constructive comments, which have significantly contributed to improving the clarity and quality of this paper.

References

1. Abdelkefi, C., & Sifi, M. (2007). Dunkl translation and uncentered maximal operator on the real line. International Journal of Mathematics and Mathematical Sciences, Article ID 87808, 9 p. https://doi.org/10.1155/2007/87808
2. Coulibaly, S., Douyon, D., & Sanogo, M. (2025). Some properties of Besov-Fofana type spaces. Gulf Journal of Mathematics, 20(1), 96-119. https://doi.org/10.56947/gjom.v20i.2842
3. Deleaval, L. (2010). Analyse harmonique associée a des systemes de racines et aux opérateurs de Dunkl rationnels. Mathématiques. Université Pierre et Marie Curie-Paris VI. Français. tel-00558751. https://theses.hal.science/tel-00558751v1
4. Diarra, N., & Fofana, I. (2021). On preduals and Köthe duals of some subspaces of Morrey spaces. Journal of Mathematical Analysis and Applications, 496(2), 124842. https://doi.org/10.1016/j.jmaa.2020.124842
5. Feuto, J. (2014). Norm inequalities in generalized Morrey spaces. Journal of Fourier Analysis and Applications, 20, 896-909. https://doi.org/10.1007/s00041-014-9337-2
6. Fofana, I. (1988). Étude d'une classe d'espaces de fonctions contenant les espaces de Lorentz. Afrika Matematika, 2(1), 29-50.
7. Fofana, I. (1989). Continuité de l'intégrale fractionnaire et espace $(L^q, l^p)^\alpha$. Comptes Rendus. Mathématique. Académie des Sciences, Paris, 308, 525-527.
8. Fofana, I. (2001). Espace $(L^q, l^p)^\alpha$ et continuité de l'opérateur maximal fractionnaire de Hardy-Littlewood. Afrika mathematica, 3(12), 23-37.
9. Fofana, I., Faléa, F. R., & Kpata, B. A. (2015). A class of subspaces of Morrey spaces and norm inequalities on Riesz potential operators, Afrika mathematica, 26, 717-739. doi:10.1007/s13370-014-0241-3
10. Guliyev, V. S. (2022). Maximal commutator and commutator of maximal function on total Morrey spaces. Journal of Mathematical Inequalities, 16(4), 1509-1524. https://doi.org/10.7153/jmi-2022-16-98
11. Guliyev, V. S. (2024). Characterizations of Lipschitz functions via the commutators of maximal function in total Morrey spaces. Mathematical Methods in the Applied Sciences, 47(11), 8669-8682. https://doi.org/10.1002/mma.10038
12. Guliyev, V. S., Eroglu, A., & Mammadov, Y. Y. (2010). Necessary and Sufficient conditions for the boundedness of Dunkl-type fractional maximal operator in the Dunkl-Type Morrey

spaces. Abstract and Applied Analysis, 2010, Article ID 976493, 10 pp.. https://doi.org/10.1155/2010/976493

13. Guliyev, V. S., & Mammadov, Y. Y. (2009). On fractional maximal function and fractional integrals associated with the Dunkl operator on the real line. Journal of Mathematical Analysis and Applications. 353(1), 449-459. https://doi.org/10.1016/j.jmaa.2008.11.083
14. Guliyev, V. S., & Mammadov, Y. Y. (2010). $(Lp,\ Lq)$ boundedness of the fractional maximal operator associated with the Dunkl operator on the real line. Integral Transforms and Special Functions, 21(8), 629-639. https://doi.org/10.1080/10652460903507248
15. Kpata, B. A., & Nagacy, P. (2025). Total Fofana-Guliyev spaces with applications to norm inequalities for classical operators. Khayyam Journal of Mathematics, 11(2), 285-301. doi:10.22034/KJM.2025.528689.3573
16. Mammadov, Y. Y., Guliyev, V. S., & Muslumova, F. A. (2025). Commutator of the maximal function in total Morrey spaces for the Dunkl operator on the real line. Azerbaijan Journal of Mathematics, 15(2), 85-104. https://doi.org/10.59849/2218-6816.2025.2.85
17. Mammadov, Y. Y., & Hasanli, S. A. (2018). On the Boundedness of Commutators of Dunkl-type Maximal operator in the Dunkl-type Morrey spaces. Caspian Journal of Applied Mathematics, Ecology and Economics, 6(1), 37-52.
18. Morrey, C. B. (1938). On the solutions of quasi-linear elliptic partial differential equations. Transactions of the American Mathematical Society, 43, 126-166. https://doi.org/10.1090/s0002-9947-1938-1501936-8
19. Muslumova, F. A. (2023). Some embeddings into the total Morrey spaces associated with the Dunkl operator on $\mathbb{R}^d$. Transactions of National Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences, 43(1), 94-102. https://doi.org/10.30546/2617-7900.43.1.2023.94
20. Nagacy, P., & Feuto, J. (2021). Maximal operator in Dunkl-Fofana spaces. Advances in Pure and Applied Mathematics, 12(2), 30-52. https://doi.org/10.21494/iste.op.2021.0647
21. Nagacy, P., Feuto, J., & Kpata, B. A. (2022). Norm inequalities for Dunkl-type fractional integral and fractional maximal operators in the Dunkl-Fofana spaces. Poincare Journal of Analysis and Applications, 9(2), 295-319. https://doi.org/10.46753/pjaa.2022.v09i02.011
22. Sawano, Y., Fazio, G. D., & Hakim, D. I.(2020). Morrey Spaces-Introduction and Applications to Integral Operators and PDE's. Vol. II, Monographs and Research Notes in Mathematics, Chapman and Hall/CRC Press, 428 pp. 10.1201/9781003029076
23. Soltani, F. (2004). L^p-Fourier multipliers for the Dunkl operator on the real line. Journal of Functional Analysis, 209(1), 16-35. https://doi.org/10.1016/j.jfa.2003.11.009
24. Thangavelu, S., & Xu, Y. (2005). Convolution operator and maximal function for the Dunkl transform. Journal d'Analyse Mathématique, 97, 25-55. https://doi.org/10.1007/bf02807401

[1] Laboratoire des Sciences et Technologies de l'Environnement, UFR Environnement, Université Jean Lorougnon GUEDE, BP 150 Daloa, Côte d'Ivoire.
Email address: pokounagacy@yahoo.com

[2] Laboratoire de Mathématiques et Informatique, UFR Sciences Fondamentales et Appliquées, Université Nangui Abrogoua, 02 BP 801 Abidjan 02, Côte d'Ivoire.
Email address: kpata_akon@yahoo.fr

[3] Laboratoire de Mathématiques et Applications, UFR Mathématiques et Informatique, Université Félix Houphouët Boigny, Abidjan-Cocody, 22 BP 582 Abidjan 22, Côte d'Ivoire.
Email address: nouffoud@yahoo.fr