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POLYNOMIAL DECAY RATE ESTIMATES IN BANACH SPACES FOR BILINEAR NEUTRAL SYSTEMS WITH DISTRIBUTED DELAY

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Abstract. The present paper focuses on the feedback stabilization of infinite dimensional bilinear neutral systems with distributed delay, evolving on a Banach state space. Under an appropriate state space decomposition, we consider a feedback control based only on the projected state onto a relevant subspace and analyze its ability to achieve strong and polynomial stabilization. An explicit decay estimate of the stabilized solution is derived. The effectiveness of the theoretical results is further demonstrated through examples and numerical simulations.

Keywords. Bilinear systems; Neutral systems; Distributed delay; Banach spaces; Duality mapping; Decomposition method; Strong stabilization; Decay estimate.
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1. Introduction

In the current work, we address the problem of feedback stabilization of the following bilinear neutral system with distributed delay

$$\begin{cases} \dfrac{d\mathcal{D}w_t}{dt} = \mathcal{A}\mathcal{D}w_t + v(t)\mathcal{F}(w_t), \quad t > 0, \\ w_0 = \phi \in \mathcal{C} := C\big([-r, 0], X\big), \end{cases} \tag{1.1}$$

where the linear operator $\mathcal{A} : \mathrm{D}(\mathcal{A}) \subset X \to X$ is the generator of a strongly continuous semigroup $S(t)$ on a Banach space X equipped with the norm $\|.\|$, while its dual space X^* is endowed with the norm $\|.\|_{X^*}$. We define the duality mapping $\mathcal{J}$ by

$$\mathcal{J}(x) = \left\{x^* \in X^* : \|x^*\|_{X^*} = \|x\|, \langle x, x^*\rangle = \|x\|^2\right\}, \; \forall x \in X,$$

where $\langle ., . \rangle$ refers to the duality product between X and its dual X^*. Let $w \in C\big([-r, +\infty[, X\big)$ with $r > 0$ and $t \geq 0$. The history function w_t is defined as the element of $\mathcal{C} = C\big([-r, 0], X\big)$ given by $w_t(\theta) = w(t + \theta)$, $\forall\theta \in [-r, 0]$. Here, $\mathcal{C}$ denotes the Banach space of continuous functions defined from $[-r, 0]$

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to X endowed with the supremum norm $\|\psi\|_{\mathcal{C}} = \sup_{\theta\in[-r,0]} \|\psi(\theta)\|, \forall\psi \in \mathcal{C}$. The positive real number r represents the delay. The notation w_t describes the past history (or trajectory segment) of the state up to time whereas $w(t)$ denotes the instantaneous state at time t. We define the bounded linear operator $\mathcal{D} : \mathcal{C} \to X$ by $\mathcal{D}\psi = \psi(0) - q\psi(-r)$, $\forall\psi \in \mathcal{C}$, where $|q| < 1$, ensuring that $\mathcal{D}$ is stable $\big($[4], Example 2.8$\big)$. Particularly, it holds that $\mathcal{D}w_t = w(t) - q\,w(t-r)$ for all $t \geq 0$. Moreover, the control operator $\mathcal{F} : \mathcal{C} \to X$ is linear and bounded, $\phi \in \mathcal{C}$ represents the initial function, and the mapping $t \mapsto v(t)$ is a scalar-valued feedback control. As a particular case of system (1.1), we consider the bilinear system without delay given by

$$\begin{cases} \dfrac{dw(t)}{dt} = \mathcal{A}w(t) + v(t)\mathcal{F}w(t), \quad t \geq 0, \\ w(0) = \phi(0) \in X, \end{cases} \tag{1.2}$$

where $\mathcal{F} : X \to X$ is linear and bounded. The stabilization of systems of the form (1.2) in Banach state spaces has been investigated in several previous works $\big($[5, 6, 7]$\big)$. In [5], the authors studied the weak and strong stabilization of bilinear systems without delay, under quadratic controls in a reflexive Banach space. Moreover, by employing a suitable decomposition of the state space, sufficient conditions ensuring exponential stabilization of bilinear systems were established in [6]. In the nonreflexive Banach setting, [7] provided sufficient conditions guaranteeing both strong and weak-star stabilization of bilinear systems. In Hilbert state spaces, the stabilization of systems similar to (1.2) has also been widely examined. For additional results and a comprehensive overview, we refer the reader to [2] and the references therein.

Recently, considerable interest has been directed toward the stabilization of infinite dimensional bilinear systems with time delay, with particular emphasis on retarded systems in which the delay influences only the state variable. In the setting of Hilbert spaces, several studies have investigated the stabilization of retarded systems, considering cases with variable delay $\big($[20]$\big)$, distributed delay $\big($[8, 10]$\big)$, as well as earlier works on systems with finite time delay $\big($[15, 25, 26]$\big)$, and related works. In the context of Banach state spaces, the authors in [9] considers the polynomial and weak stabilization for a class of bilinear systems with distributed delay, through a suitable decomposition of the state space. While, the study of exponential and weak stabilization for semilinear retarded systems with finite time delay was carried out in [23], under suitable assumptions, notably observability-type conditions. On the other hand, several works have been devoted to neutral systems, in which both the state and its derivatives are subject to time delays ([14, 16, 17, 19, 22]). In [19], the authors investigated the strong stabilization of a class of infinite dimensional bilinear neutral systems with finite time delay. They constructed a bounded feedback control defined as the solution of a minimization problem, which ensures a polynomial decay rate of the stabilized state. In contrast, [16] established sufficient conditions for the strong and exponential stabilization of the corresponding non-homogeneous system. In light of the above mentioned results, this work is devoted to the feedback stabilization of infinite dimensional neutral systems with distributed delays in a Banach space setting. We should mention here that the findings of [19] and [9] are not applicable to the illustrative examples considered in this work, as the first one deals only with finite delay systems in Hilbert spaces, while the second corresponds to a particular instance of the present study (i.e., $q = 0$). Therefore, we establish sufficient conditions for strong and polynomial stabilization of systems described by (1.1), using an adequate decomposition of the space X. Earlier works have applied this approach in Banach state spaces to undelayed systems $\big($[6, 24]$\big)$, and in [9] concerning bilinear retarded systems. Specifically, we assume that the space X can be expressed as the direct sum of two closed subspaces X_u and X_s which remain invariant under the action of $S(t)$, namely,

$$S(t)X_u \subset X_u \text{ and } S(t)X_s \subset X_s. \tag{1.3}$$

Let $S_u(t)$ and $S_s(t)$ denote the restrictions of $S(t)$ to X_u and X_s, respectively. The operators $S_u(t)$ and $S_s(t)$ form strongly continuous semigroups on X_u and X_s, with generators $\mathcal{A}_u$ and $\mathcal{A}_s$, respectively. Moreover, we assume that

$$\mathcal{F}(\mathcal{C}_u) \subset X_u \text{ and } \mathcal{F}(\mathcal{C}_s) \subset X_s, \tag{1.4}$$

such that $\mathcal{C}_u = C\big([-r, 0], X_u\big)$ and $\mathcal{C}_s = C\big([-r, 0], X_s\big)$. Let w^u and w^s denote the components of $w = w^u + w^s$ on X_u and X_s, respectively. In addition, we set $\phi_u \in \mathcal{C}_u$ and $\phi_s \in \mathcal{C}_s$ such that $\phi(t) = \phi_u(t) + \phi_s(t)$, $\forall t \in [-r, 0]$. The structure of the paper is outlined as follows: In Section 2, we study the existence and uniqueness of the solution for the corresponding controlled system. Section 3 is dedicated to the analysis of strong and polynomial stabilization of the system under consideration. Finally, we present several applications to illustrate the theoretical results obtained. Along this paper, we make the following assumption on the map $\mathcal{J}$:

$(\mathcal{H})$: The map $\mathcal{J}$ is assumed to be Lipschitz continuous, which implies that it is single-valued and the mapping $x \longmapsto \mathcal{J}(x)$ from X to X^* satisfies a Lipschitz condition [27]. That is, there exists a constant $L_{\mathcal{J}} > 0$ such that for all $x_1, x_2 \in X$, we have $\|\mathcal{J}(x_1) - \mathcal{J}(x_2)\|_{X^*} \leq L_{\mathcal{J}}\|x_1 - x_2\|$.

In the following, $L_{\mathcal{J}} > 0$ will denote a Lipschitz constant of $\mathcal{J}$.

Remark 1.1. Observe that, under assumption $(\mathcal{H})$, the squared norm $\|\cdot\|^2$ is Fréchet-differentiable and the space X is reflexive [1]. Moreover, this assumption is closely related to certain geometric properties of the Banach space X. Further results and detailed discussions concerning the properties of the duality mapping may be found, for instance, in [3] and [11].

2. A well-posedness result

In order to prove the main result of this section, we begin by recalling certain preliminary results that will be instrumental in the subsequent analysis.

Definition 2.1. ([12]). A function $w \in C([-r,T],X), T > 0$, is called a mild solution of the system (1.1), if it satisfies the abstract integral equation

$$\begin{cases} \mathcal{D}w_t = S(t)\mathcal{D}\phi + \int_0^t S(t-\sigma)v(\sigma)\mathcal{F}(w_\sigma)d\sigma, \quad t \in [0,T], \\ w_0 = \phi \in \mathcal{C}. \end{cases}$$

Lemma 2.2. $\big([21]\big)$. *Let $w : \mathbb{R} \to X$ be a function such that:*

(1) *The mapping $t \mapsto \|w(t)\|$ is differentiable almost everywhere on $\mathbb{R}$.*
(2) *The function w admits a weak derivative $\dot{w}(t)$ for almost every $t \in \mathbb{R}$.*

Then, for almost every $t \in \mathbb{R}$, it holds that

$$\frac{d}{dt}\|w(t)\|^2 = 2\left\langle \dot{w}(t), w^* \right\rangle, \forall w^* \in \mathcal{J}(w(t)).$$

The result of the well-posedness question is stated as follows:

Theorem 2.3. *Assume that:*

(1) *$S_u(t)$ is a contraction semigroup on X_u.*
(2) *$S_s(t)$ is exponentially stable on X_s.*

Then, the system (1.1) associated with the feedback law

$$v(t) = -\langle \mathcal{F}(w_t^u), \mathcal{J}(\mathcal{D}w_t^u)\rangle, \, \forall t \geq 0, \tag{2.1}$$

admits a unique global mild solution $w \in C([-r,+\infty[,X)$. In addition, for all $t \geq 0$, the following estimate holds:

$$\|\mathcal{D}w_t^u\|^2 - \|\mathcal{D}w_\tau^u\|^2 \leq -2\int_\tau^t |\langle \mathcal{F}(w_\sigma^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle|^2 d\sigma \leq 0, \ \forall \tau \in [0,t]. \tag{2.2}$$

Proof. Consider the system (1.1) under the feedback control (2.1). This leads to the closed loop system:

$$\begin{cases} \dfrac{d\mathcal{D}w_t}{dt} = \mathcal{A}\mathcal{D}w_t + f(w_t), \quad t > 0, \\ w_0 = \phi \in \mathcal{C}, \end{cases} \tag{2.3}$$

where the function $f : \mathcal{C} \longrightarrow X$ is given by

$$f(\psi) = -\langle \mathcal{F}(\psi^u), \mathcal{J}(\mathcal{D}\psi^u)\rangle \mathcal{F}(\psi). \tag{2.4}$$

Let $R > 0$ and $\psi, \varphi \in \mathcal{B}_R = \{\Phi \in \mathcal{C} : \|\Phi\|_{\mathcal{C}} \leq R\}$. Since $\mathcal{F} \in \mathcal{L}(\mathcal{C}, X)$ and using the estimate $\|\mathcal{D}\psi\| \leq 2\|\psi\|_{\mathcal{C}}$ for all $\psi \in \mathcal{C}$, it follows that

$$\|f(\psi) - f(\varphi)\| \leq 6L_{\mathcal{J}}R^2\|\mathcal{F}\|^2\|\psi - \varphi\|_{\mathcal{C}}.$$

As a consequence, the function f given by (2.4) satisfies a local Lipschitz condition. Using the approach developed in the proof of Theorem 2.1 in [16], and exploiting the local Lipschitz continuity of the duality mapping $\mathcal{J}$, we show that system (2.3) admits a unique mild solution $w \in C\big([-r, t_{max}), X\big)$, expressed by the following variation of constants formula:

$$\mathcal{D}w_t = S(t)\mathcal{D}\phi + \int_0^t S(t-\sigma)v(\sigma)\mathcal{F}(w_\sigma)d\sigma, \ t \in [0, t_{max}), \tag{2.5}$$

from which it follows that

$$\mathcal{D}w_t^u = S_u(t)\mathcal{D}\phi_u + \int_0^t S_u(t-\sigma)v(\sigma)\mathcal{F}(w_\sigma^u)d\sigma,\ t\in[0,t_{max}). \tag{2.6}$$

Let $T^*\in]0,t_{max}[$, and consider the map $g:[0,T^*]\longrightarrow X_u$ defined by

$$g: t\longmapsto -\langle \mathcal{F}(w_t^u),\mathcal{J}(\mathcal{D}w_t^u)\rangle \mathcal{F}(w_t^u),$$

so there exists a sequence $(g_n)_n\subset C^1([0,T^*],X_u)$ such that: $g_n\longrightarrow g$ uniformly in $C([0,T^*],X_u)$ as $n\longrightarrow +\infty$. Let $(h_n)_n\subset \mathrm{D}(\mathcal{A}_u)$ such that $(h_n)_n$ converges to $\mathcal{D}\phi_u$ in X_u and let us define the sequence $(w_n)_n\subset C([0,T^*],X_u)$ such that:

$$\begin{cases} w_n(t) = S_u(t)h_n + \displaystyle\int_0^t S_u(t-\sigma)g_n(\sigma)d\sigma, \quad t\in[0,T^*],\\ w_n(0)=h_n, \end{cases} \tag{2.7}$$

which represents the classical solution of the following system:

$$\begin{cases} \dfrac{dw_n(t)}{dt} = \mathcal{A}_u w_n(t) + g_n(t), \quad t\in[0,T^*],\\ w_n(0)=h_n. \end{cases}$$

Under assumption $(\mathcal{H})$, the function $\|\cdot\|^2$ is Fréchet differentiable and by Lemma 2.2, we obtain

$$\begin{aligned}\frac{d}{dt}\|w_n(t)\|^2 &= 2\big\langle \dot{w}_n(t),\mathcal{J}\left(w_n(t)\right)\big\rangle\\ &= 2\big\langle \mathcal{A}_u w_n(t),\mathcal{J}\left(w_n(t)\right)\big\rangle + 2\big\langle g_n(t),\mathcal{J}\left(w_n(t)\right)\big\rangle.\end{aligned}$$

The fact that $S_u(t)$ is a contraction semigroup implies that $\mathcal{A}_u$ is dissipative, which leads to

$$\frac{d}{dt}\|w_n(t)\|^2 \le 2\big\langle g_n(t),\mathcal{J}\left(w_n(t)\right)\big\rangle. \tag{2.8}$$

Integrating inequality (2.8), we obtain that

$$\|w_n(t)\|^2 - \|w_n(\tau)\|^2 \le 2\int_\tau^t \big\langle g_n(\sigma),\mathcal{J}\left(w_n(\sigma)\right)\big\rangle d\sigma,\ \forall\tau\in[0,t]. \tag{2.9}$$

Additionally, it comes from (2.6) and (2.7) for each $t\in[0,T^*]$ that

$$\begin{aligned}\|w_n(t)-\mathcal{D}w_t^u\| &\le \|h_n-\mathcal{D}\phi_u\| + \int_0^{T^*}\|g_n(\sigma)-g(\sigma)\|\,d\sigma\\ &\le \|h_n-\mathcal{D}\phi_u\| + T^*\sup_{\sigma\in[0,T^*]}\|g_n(\sigma)-g(\sigma)\|,\ \forall t\in[0,T^*].\end{aligned} \tag{2.10}$$

Hence, we infer that

$$w_n(t)\longrightarrow \mathcal{D}w_t^u,\ \text{as } n\longrightarrow+\infty. \tag{2.11}$$

Invoking the Lipschitz property $(\mathcal{H})$ of the duality mapping $\mathcal{J}$, we get that the sequence $\big(\langle g_n,\mathcal{J}(w_n)\rangle\big)_n$ converges uniformly to $\langle g,\mathcal{J}(\mathcal{D}w^u)\rangle$ on $[0,T^*]$. Taking the limit in (2.9) as $n\longrightarrow+\infty$, it yields by the dominated convergence theorem that

$$\|\mathcal{D}w_t^u\|^2 - \|\mathcal{D}w_\tau^u\|^2 \le -2\int_\tau^t |\langle\mathcal{F}(w_\sigma^u),\mathcal{J}(\mathcal{D}w_\sigma^u)\rangle|^2 d\sigma \le 0,\ \forall\tau\in[0,t]. \tag{2.12}$$

which implies that

$$\|\mathcal{D}w_t^u\| \leq \|\mathcal{D}\phi_u\|, \; \forall t \in [0, t_{max}). \tag{2.13}$$

The fact that $\mathcal{D}w_t^u = w^u(t) - qw^u(t-r)$ leads to

$$\|w^u(t)\| \leq \|\mathcal{D}\phi_u\| + |q| \, \|w^u(t-r)\| \leq \|\mathcal{D}\phi_u\| + |q| \, \|\mathcal{D}\phi_u\| + |q|^2 \|w^u(t-2r)\|.$$

Repeating this process, we obtain that $\|w^u(t)\| \leq \|\mathcal{D}\phi_u\| \sum_{n=0}^{+\infty} |q|^n + \|\phi_u\|_{\mathcal{C}_u}$. Using the fact that $|q| < 1$, we arrive at

$$\|w^u(t)\| \leq \frac{\|\mathcal{D}\phi_u\|}{1-|q|} + \|\phi_u\|_{\mathcal{C}_u}, \; \forall t \in [0, t_{max}). \tag{2.14}$$

Since $\|w^u(t)\| = \|\phi_u\|_{\mathcal{C}_u}$, $\forall t \in [-r, 0]$, we infer that

$$\|w^u(t)\| \leq M := \frac{\|\mathcal{D}\phi_u\|}{1-|q|} + \|\phi_u\|_{\mathcal{C}_u}, \; \forall t \in [-r, t_{max}). \tag{2.15}$$

From the variation of constants formula (2.5), we also obtain

$$\mathcal{D}w_t^s = S_s(t)\mathcal{D}\phi_s + \int_0^t S_s(t-\sigma)v(\sigma)\mathcal{F}(w_\sigma^s)d\sigma, \;\; t \in [0, t_{max}).$$

Exploiting the fact that $\mathcal{D}w_t^s = w^s(t) - qw^s(t-r)$, we get

$$w^s(t) = qw^s(t-r) + S_s(t)\mathcal{D}\phi_s + \int_0^t S_s(t-\sigma)v(\sigma)\mathcal{F}(w_\sigma^s)d\sigma, \;\; t \in [0, t_{max}). \tag{2.16}$$

The exponential stability of $S_s(t)$ ensures the existence of constants $N \geq 1$ and $\alpha > 0$ such that

$$\|S_s(t)\| \leq Ne^{-\alpha t}, \;\; \forall t \geq 0. \tag{2.17}$$

Thus, it comes from (2.16) that

$$\|w^s(t)\| \leq |q| \|w^s(t-r)\| + 2Ne^{-\alpha t}\|\phi_s\|_{\mathcal{C}_s} + N\|\mathcal{F}\| \int_0^t e^{-\alpha(t-\sigma)}|v(\sigma)| \|w_\sigma^s\|_{\mathcal{C}_s} d\sigma,$$

which gives

$$\|w^s(t)\|e^{\alpha t} \leq |q|e^{\alpha r}\|w^s(t-r)\|e^{\alpha(t-r)} + 2N\|\phi_s\|_{\mathcal{C}_s} + N\|\mathcal{F}\| \int_0^t |v(\sigma)|e^{\alpha\sigma}\|w_\sigma^s\|_{\mathcal{C}_s} d\sigma.$$

Now, let $\theta \in [0, t]$, we have

$$\|w^s(\theta)\|e^{\alpha \theta} \leq |q|e^{\alpha r}\|w^s(\theta-r)\|e^{\alpha(\theta-r)} + 2N\|\phi_s\|_{\mathcal{C}_s} + N\|\mathcal{F}\| \int_0^\theta |v(\sigma)|e^{\alpha\sigma}\|w_\sigma^s\|_{\mathcal{C}_s} d\sigma,$$

then

$$\|w^s(\theta)\|e^{\alpha \theta} \leq |q|e^{\alpha r} \sup_{-r \leq \xi \leq t} \|w^s(\xi)\|e^{\alpha\xi} + 2N\|\phi_s\|_{\mathcal{C}_s} + N\|\mathcal{F}\| \int_0^t |v(\sigma)|e^{\alpha\sigma}\|w_\sigma^s\|_{\mathcal{C}_s} d\sigma,$$

which implies that

$$\sup_{0 \leq \xi \leq t} \|w^s(\xi)\|e^{\alpha\xi} \leq |q|e^{\alpha r} \sup_{-r \leq \xi \leq t} \|w^s(\xi)\|e^{\alpha\xi} + 2N\|\phi_s\|_{\mathcal{C}_s} + N\|\mathcal{F}\| \int_0^t |v(\sigma)|e^{\alpha\sigma}\|w_\sigma^s\|_{\mathcal{C}_s} d\sigma. \tag{2.18}$$

Additionally, remarking that for all $\xi \in [-r, 0]$, we have

$$\|w^s(\xi)\| e^{\alpha \xi} \le \|\phi_s\|_{\mathcal{C}_s} \le 2N \|\phi_s\|_{\mathcal{C}_s}, \tag{2.19}$$

then, we obtain

$$\sup_{-r \le \xi \le 0} \|w^s(\xi)\| e^{\alpha \xi} \le |q| e^{\alpha r} \sup_{-r \le \xi \le t} \|w^s(\xi)\| e^{\alpha \xi} + 2N \|\phi_s\|_{\mathcal{C}_s} + N \|\mathcal{F}\| \int_0^t |v(\sigma)| e^{\alpha \sigma} \|w_\sigma^s\|_{\mathcal{C}_s} d\sigma. \tag{2.20}$$

Combining (2.18) and (2.20), we obtain that

$$\left(1 - |q| e^{\alpha r}\right) \sup_{-r \le \xi \le t} \|w^s(\xi)\| e^{\alpha \xi} \le 2N \|\phi_s\|_{\mathcal{C}_s} + N \|\mathcal{F}\| \int_0^t |v(\sigma)| e^{\alpha \sigma} \|w_\sigma^s\|_{\mathcal{C}_s} d\sigma. \tag{2.21}$$

Taking $\alpha > 0$ such that $1 - |q| e^{\alpha r} > 0$, it comes from (2.21) that

$$\|w^s(t)\| e^{\alpha t} \le N_1 + N_2 \int_0^t |v(\sigma)| e^{\alpha \sigma} \|w_\sigma^s\|_{\mathcal{C}_s} d\sigma, \tag{2.22}$$

where $N_1 = \dfrac{2N \|\phi_s\|_{\mathcal{C}_s}}{1 - |q| e^{\alpha r}}$ and $N_2 = \dfrac{N \|\mathcal{F}\|}{1 - |q| e^{\alpha r}}$. Let $t \in [0, t_{max})$ and $\theta \in [-r, 0]$ satisfy $t + \theta \ge 0$. It follows from (2.22) that

$$\|w^s(t+\theta)\| e^{\alpha(t+\theta)} \le N_1 + N_2 \int_0^{t+\theta} |v(\sigma)| e^{\alpha \sigma} \|w_\sigma^s\|_{\mathcal{C}_s} d\sigma,$$

then, we have

$$\|w^s(t+\theta)\| e^{\alpha t} \le N_1 e^{-\alpha \theta} + N_2 e^{-\alpha \theta} \int_0^t |v(\sigma)| e^{\alpha \sigma} \|w_\sigma^s\|_{\mathcal{C}_s} d\sigma,$$

which implies that

$$\|w^s(t+\theta)\| e^{\alpha t} \le N_1 e^{\alpha r} + N_2 e^{\alpha r} \int_0^t |v(\sigma)| e^{\alpha \sigma} \|w_\sigma^s\|_{\mathcal{C}_s} d\sigma. \tag{2.23}$$

Furthermore, for all $t \in [0, t_{max}[$ and $\theta \in [-r, 0]$ satisfy $t + \theta \le 0$, it holds that

$$\|w^s(t+\theta)\| = \|\phi_s(t+\theta)\|.$$

Therefore, it yields that

$$\|w^s(t+\theta)\| e^{\alpha t} \le e^{-\alpha \theta} \|\phi_s\|_{\mathcal{C}_s} \le e^{\alpha r} \|\phi_s\|_{\mathcal{C}_s}. \tag{2.24}$$

Using the fact that $N \ge 1$ and $1 - |q| e^{\alpha r} < 1$, it comes from (2.24) that

$$\|w^s(t+\theta)\| e^{\alpha t} \le \frac{2N e^{\alpha r} \|\phi_s\|_{\mathcal{C}_s}}{1 - |q| e^{\alpha r}} \le N_1 e^{\alpha r}. \tag{2.25}$$

Combining (2.23) and (2.25), we deduce that

$$\|w_t^s\|_{\mathcal{C}_s} e^{\alpha t} \le N_1 e^{\alpha r} + N_2 e^{\alpha r} \int_0^t |v(\sigma)| e^{\alpha \sigma} \|w_\sigma^s\|_{\mathcal{C}_s} d\sigma, \ \forall t \ge 0. \tag{2.26}$$

Applying Gronwall's inequality, we conclude from (2.26) that

$$\|w_t^s\|_{\mathcal{C}_s} e^{\alpha t} \le N_1 e^{\alpha r} \exp\left(N_2 e^{\alpha r} \int_0^t |v(\sigma)| d\sigma \right). \tag{2.27}$$

On the other hand, by applying Holder's inequality, we obtain that

$$\int_0^t |v(\sigma)|d\sigma \leq \sqrt{t}\Big(\int_0^t |v(\sigma)|^2 d\sigma\Big)^{\frac{1}{2}}. \tag{2.28}$$

Moreover, it yields from (2.12) that

$$\int_0^t |v(\sigma)|^2 d\sigma \leq \frac{\|\mathcal{D}\phi_u\|^2}{2}. \tag{2.29}$$

Employing (2.28) and (2.29), together with the fact that $\|\mathcal{D}\phi_u\| \leq 2\|\phi_u\|_{\mathcal{C}_u}$, we get from (2.27) that

$$\|w_t^s\|_{\mathcal{C}_s} e^{\alpha t} \leq N_1 e^{\alpha r} \exp\Big(\sqrt{2} N_2 e^{\alpha r}\|\phi_u\|_{\mathcal{C}_u}\sqrt{t}\Big),$$

which implies that

$$\|w_t^s\|_{\mathcal{C}_s} \leq N_1 e^{\alpha r} \exp\Big(\sqrt{2} N_2 e^{\alpha r}\|\phi_u\|_{\mathcal{C}_u}\sqrt{t} - \alpha t\Big).$$

Thus

$$\|w^s(t)\| \leq N_1 e^{\alpha r} \exp\Big(\sqrt{2} N_2 e^{\alpha r}\|\phi_u\|_{\mathcal{C}_u}\sqrt{t} - \frac{\alpha t}{2}\Big) \exp\Big(\frac{-\alpha t}{2}\Big), \ t \geq 0. \tag{2.30}$$

Remarking that

$$\frac{N_2^2 e^{2\alpha r}\|\phi_u\|_{\mathcal{C}_u}^2}{\alpha} - \Big(\sqrt{2} N_2 e^{\alpha r}\|\phi_u\|_{\mathcal{C}_u}\sqrt{t} - \frac{\alpha t}{2}\Big) = \frac{\alpha}{2}\Big(\sqrt{t} - \frac{\sqrt{2} N_2 e^{\alpha r}\|\phi_u\|_{\mathcal{C}_u}}{\alpha}\Big)^2 \geq 0,$$

it follows that

$$\exp\Big(\sqrt{2} N_2 e^{\alpha r}\|\phi_u\|_{\mathcal{C}_u}\sqrt{t} - \frac{\alpha t}{2}\Big) \leq \exp\Big(\frac{N_2^2 e^{2\alpha r}\|\phi_u\|_{\mathcal{C}_u}^2}{\alpha}\Big). \tag{2.31}$$

Hence, we obtain that

$$\|w^s(t)\| \leq C e^{-\beta t}, \tag{2.32}$$

where $C = N_1 e^{\alpha r} \exp\Big(\frac{N_2^2 e^{2\alpha r}\|\phi_u\|_{\mathcal{C}_u}^2}{\alpha}\Big)$ and $\beta = \frac{\alpha}{2}$. By combining (2.15) and (2.32), we conclude that $t_{\max} = +\infty$ as a consequence of the blow-up phenomenon. □

Remark 2.4.
(1) The control law (2.1) is determined exclusively by the unstable componentt $w^u(t)$. Therefore, it requires less informations about the state, as it is based on a partial projection of the state. This restriction leads to lower control energy, as only the critical dynamics are influenced.
(2) If X_u is a Hilbert space, the restriction of the duality mapping $\mathcal{J}$ to X_u coincides with the identity on X_u. In this case, the feedback control (2.1) reads as follows:

$$v(t) = -\langle \mathcal{F}(w_t^u), \mathcal{D}w_t^u\rangle, \ \forall t \geq 0. \tag{2.33}$$

where $\langle ., .\rangle$ denotes the inner product in X_u.
(3) Note that if condition (2.17) holds for some $\alpha > 0$, then it also holds for every $\alpha^* \in]0, \alpha[$. This fact justifies the choice of α in (2.22).

3. STABILIZATION RESULTS

We begin by recalling the following definition associated with the asymptotic behavior of system (1.1).

Definition 3.1. The system (1.1) is said to be strongly stabilizable if there exists a feedback control of the form $v(t) = \Gamma(w_t), \quad t \geq 0$, where $\Gamma : \mathcal{C} \to \mathbb{R}$ is appropriately defined, such that the following properties are satisfied:

(1) For every $\phi \in \mathcal{C}$, system (1.1) admits a unique mild solution $w \in C([-r, +\infty[, X)$.
(2) The origin is a Lyapunov stable equilibrium of system (1.1).
(3) The solution satisfies $w(t, \phi) \to 0$ as $t \to +\infty$.

To proceed with the stabilization analysis, we make use of the following lemma.

Lemma 3.2. *Assuming that $\mathcal{A}_u$ generates a contraction semigroup $S_u(t)$ on X_u and the operator $\mathcal{F} \in \mathcal{L}(\mathcal{C}, X)$. Thus, the solution of system (2.3) projected onto X_u fulfills the inequality*

$$\|w_t^u\|_{\mathcal{C}_u} \leq a e^{-bt} \|\phi_u\|_{\mathcal{C}_u} + \frac{4}{1-|q|} \|\mathcal{D} w_{\frac{t}{2}}^u\|, \; t \geq 0, \tag{3.1}$$

for some constants $a, b > 0$.

Proof. We apply the result of Lemma 3.1 in [19] to the system:

$$\begin{cases} \dfrac{d\mathcal{D}w_t^u}{dt} = \mathcal{A}_u \mathcal{D} w_t^u - \langle \mathcal{F}(w_t^u), \mathcal{J}(\mathcal{D}w_t^u)\rangle \mathcal{F}(w_t^u), \quad t > 0, \\ w_0^u = \phi_u \in \mathcal{C}_u. \end{cases}$$

Then, we obtain the desired estimate (3.1). □

Along this section, we assume that the operator $\mathcal{F}$ meets the following observability requirement: There exist $T > r$ and $\mu > 0$ such that

$$\int_r^T |\langle \mathcal{F}(S_u(\sigma + .)w), \mathcal{J}(S_u(\sigma)w)\rangle| d\sigma \geq \mu \|w\|^2, \; \forall w \in X_u. \tag{3.2}$$

The subsequent theorem furnishes sufficient conditions for achieving strong stabilization of system (1.1).

Theorem 3.3. *Given the hypotheses of Theorem 2.3, together with the observability assumption (3.2), the feedback control (2.1) yields strong stabilization of system (1.1).*

Proof. As stated in Theorem 2.3, the system (1.1) under the feedback (2.1) admits a unique global mild solution $w \in C([-r, +\infty[, X)$, which is given by the following variation of constants formula:

$$\begin{cases} \mathcal{D}w_t = S(t)\mathcal{D}\phi - \displaystyle\int_0^t S(t-\sigma)\langle \mathcal{F}(w_\sigma^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle \mathcal{F}(w_\sigma) d\sigma, \quad t \geq 0, \\ w_0 = \phi \in \mathcal{C}, \end{cases} \tag{3.3}$$

from where it comes that

$$\begin{cases} \mathcal{D}w_t^u = S_u(t)\mathcal{D}\phi_u - \displaystyle\int_0^t S_u(t-\sigma)\langle \mathcal{F}(w_\sigma^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle \mathcal{F}(w_\sigma^u) d\sigma, \quad t \geq 0, \\ w_0^u = \phi_u \in \mathcal{C}_u. \end{cases} \tag{3.4}$$

Now, let $t \geq 0$ and $T > r$. From (3.4), we get

$$\mathcal{D}w^u_\tau = S_u(\tau - t)\mathcal{D}w^u_t - \int_t^\tau S_u(\tau-\sigma)\langle \mathcal{F}(w^u_\sigma), \mathcal{J}(\mathcal{D}w^u_\sigma)\rangle \mathcal{F}(w^u_\sigma)d\sigma, \ \forall \tau \in [t, t+T]. \tag{3.5}$$

By combining (2.15) with the facts that the operator $\mathcal{F} \in \mathcal{L}(\mathcal{C}, X)$ and that $S_u(t)$ is a contraction semigroup, and by applying Schwarz's inequality, we obtain that

$$\begin{aligned} &\|\mathcal{D}w^u_\tau - S_u(\tau-t)\mathcal{D}w^u_t\| \\ &\leq \sqrt{T}M\|\mathcal{F}\|\left(\int_t^{t+T} |\langle \mathcal{F}(w^u_\sigma), \mathcal{J}(\mathcal{D}w^u_\sigma)\rangle|^2 d\sigma\right)^{\frac{1}{2}}, \ \forall \tau \in [t, t+T], \end{aligned} \tag{3.6}$$

Conversely, for all $\tau \in [t+r, t+T]$ and $\theta \in [-r, 0]$, it comes from (3.6) that

$$\|\mathcal{D}w^u_{\tau+\theta} - S_u(\tau+\theta-t)\mathcal{D}w^u_t\| \leq \sqrt{T}M\|\mathcal{F}\|\left(\int_t^{t+T} |\langle \mathcal{F}(w^u_\sigma), \mathcal{J}(\mathcal{D}w^u_\sigma)\rangle|^2 d\sigma\right)^{\frac{1}{2}}. \tag{3.7}$$

Remarking that

$$\|\mathcal{D}w^u_{\tau+.} - S_u(\tau-t+.)\mathcal{D}w^u_t\|_{\mathcal{C}_u} = \sup_{\theta\in[-r,0]}\|\mathcal{D}w^u_{\tau+\theta} - S_u(\tau+\theta-t)\mathcal{D}w^u_t\|, \tag{3.8}$$

where $\mathcal{D}w^u_{\tau+.} = w^u(\tau+.) - qw^u(\tau-r+.) = w^u_\tau - qw^u_{\tau-r}$, we deduce by employing (3.7) that

$$\begin{aligned} &\|\mathcal{D}w^u_{\tau+.} - S_u(\tau-t+.)\mathcal{D}w^u_t\|_{\mathcal{C}_u} \\ &\leq \sqrt{T}M\|\mathcal{F}\|\left(\int_t^{t+T} |\langle \mathcal{F}(w^u_\sigma), \mathcal{J}(\mathcal{D}w^u_\sigma)\rangle|^2 d\sigma\right)^{\frac{1}{2}}, \ \forall \tau \in [t+r, t+T]. \end{aligned} \tag{3.9}$$

Since $\|\mathcal{D}w^u_{\tau+.}\|_{\mathcal{C}_u} = \sup_{\theta\in[-r,0]}\|\mathcal{D}w^u_{\tau+\theta}\|$, we get by taking (2.13) into consideration

$$\|\mathcal{D}w^u_{\tau+.}\|_{\mathcal{C}_u} \leq \|\mathcal{D}\phi_u\|, \ \forall \tau \in [t+r, t+T]. \tag{3.10}$$

Next, in light of the observation that follows:

$$\begin{aligned} &\big\langle \mathcal{F}\big(S_u(\tau-t+.)\mathcal{D}w^u_t\big), \mathcal{J}\big(S_u(\tau-t)\mathcal{D}w^u_t\big)\big\rangle \\ &= -\big\langle \mathcal{F}\big(\mathcal{D}w^u_{\tau+.} - S_u(\tau-t+.)\mathcal{D}w^u_t\big), \mathcal{J}\big(S_u(\tau-t)\mathcal{D}w^u_t\big)\big\rangle \\ &\quad - \big\langle \mathcal{F}\big(\mathcal{D}w^u_{\tau+.}\big), \mathcal{J}\big(\mathcal{D}w^u_\tau\big) - \mathcal{J}\big(S_u(\tau-t)\mathcal{D}w^u_t\big)\big\rangle \\ &\quad + \big\langle \mathcal{F}\big(\mathcal{D}w^u_{\tau+.}\big), \mathcal{J}\big(\mathcal{D}w^u_\tau\big)\big\rangle, \end{aligned}$$

it yields that

$$\begin{aligned} &\big\langle \mathcal{F}\big(S_u(\tau-t+.)\mathcal{D}w^u_t\big), \mathcal{J}\big(S_u(\tau-t)\mathcal{D}w^u_t\big)\big\rangle \\ &= -\big\langle \mathcal{F}\big(\mathcal{D}w^u_{\tau+.} - S_u(\tau-t+.)\mathcal{D}w^u_t\big), \mathcal{J}\big(S_u(\tau-t)\mathcal{D}w^u_t\big)\big\rangle \\ &\quad - \big\langle \mathcal{F}\big(\mathcal{D}w^u_{\tau+.}\big), \mathcal{J}\big(\mathcal{D}w^u_\tau\big) - \mathcal{J}\big(S_u(\tau-t)\mathcal{D}w^u_t\big)\big\rangle \\ &\quad + \big\langle \mathcal{F}\big(w^u_\tau\big), \mathcal{J}\big(\mathcal{D}w^u_\tau\big)\big\rangle - q\big\langle \mathcal{F}\big(w^u_{\tau-r}\big), \mathcal{J}\big(\mathcal{D}w^u_\tau\big)\big\rangle. \end{aligned}$$

Since $\mathcal{J}$ is Lipschitz continuous with a Lipschitz constant $L_{\mathcal{J}}$, we obtain by using (3.10) that

$$\begin{aligned}
&\big|\big\langle \mathcal{F}\big(S_u(\tau-t+.)\mathcal{D}w_t^u\big), \mathcal{J}\big(S_u(\tau-t)\mathcal{D}w_t^u\big)\big\rangle\big| \\
&\le L_{\mathcal{J}}\|\mathcal{F}\|\,\|\mathcal{D}\phi_u\|\|\mathcal{D}w_{\tau+.}^u - S_u(\tau-t+.)\mathcal{D}w_t^u\|_{\mathcal{C}_u} \\
&\quad + L_{\mathcal{J}}\|\|\mathcal{F}\|\|\mathcal{D}\phi_u\|\|\mathcal{D}w_\tau^u - S_u(\tau-t)\mathcal{D}w_t^u\| \\
&\quad + \big|\big\langle \mathcal{F}\big(w_\tau^u\big), \mathcal{J}\big(\mathcal{D}w_\tau^u\big)\big\rangle\big| + |q|\big|\big\langle \mathcal{F}\big(w_{\tau-r}^u\big), \mathcal{J}\big(\mathcal{D}w_\tau^u\big)\big\rangle\big|.
\end{aligned}$$

Employing (3.9), it follows that

$$\begin{aligned}
&\big|\big\langle \mathcal{F}\big(S_u(\tau-t+.)\mathcal{D}w_t^u\big), \mathcal{J}\big(S_u(\tau-t)\mathcal{D}w_t^u\big)\big\rangle\big| \\
&\le 2L_{\mathcal{J}}\sqrt{T}M\|\mathcal{D}\phi_u\|\|\mathcal{F}\|^2\left(\int_t^{t+T}|\langle \mathcal{F}(w_\sigma^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle|^2 d\sigma\right)^{\frac{1}{2}} \\
&\quad + \big|\big\langle \mathcal{F}\big(w_\tau^u\big), \mathcal{J}\big(\mathcal{D}w_\tau^u\big)\big\rangle\big| + |q|\big|\big\langle \mathcal{F}\big(w_{\tau-r}^u\big), \mathcal{J}\big(\mathcal{D}w_\tau^u\big)\big\rangle\big|.
\end{aligned}$$

Integrating the above inequality with respect to τ over $[t+r, t+T]$, we deduce that

$$\begin{aligned}
&\int_{t+r}^{t+T}\big|\big\langle \mathcal{F}\big(S_u(\sigma-t+.)\mathcal{D}w_t^u\big), \mathcal{J}\big(S_u(\sigma-t)\mathcal{D}w_t^u\big)\big\rangle\big| d\sigma \\
&\le 2L_{\mathcal{J}}\sqrt{T}(T-r)M\|\mathcal{D}\phi_u\|\|\mathcal{F}\|^2\left(\int_t^{t+T}|\langle \mathcal{F}(w_\sigma^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle|^2 d\sigma\right)^{\frac{1}{2}} \\
&\quad + \int_{t+r}^{t+T}\big|\big\langle \mathcal{F}\big(w_\sigma^u\big), \mathcal{J}\big(\mathcal{D}w_\sigma^u\big)\big\rangle\big| d\sigma + |q|\int_{t+r}^{t+T}\big|\big\langle \mathcal{F}\big(w_{\sigma-r}^u\big), \mathcal{J}\big(\mathcal{D}w_\sigma^u\big)\big\rangle\big| d\sigma.
\end{aligned}$$

Applying Schwarz's inequality, it yields that

$$\begin{aligned}
&\int_r^T\big|\big\langle \mathcal{F}\big(S_u(\sigma+.)\mathcal{D}w_t^u\big), \mathcal{J}\big(S_u(\sigma)\mathcal{D}w_t^u\big)\big\rangle\big| d\sigma \\
&\le M_1\left(\int_t^{t+T}|\langle \mathcal{F}(w_\sigma^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle|^2 d\sigma\right)^{\frac{1}{2}} + |q|\int_{t+r}^{t+T}\big|\big\langle \mathcal{F}\big(w_{\sigma-r}^u\big), \mathcal{J}\big(\mathcal{D}w_\sigma^u\big)\big\rangle\big| d\sigma,
\end{aligned} \tag{3.11}$$

where $M_1 = 2L_{\mathcal{J}}\sqrt{T}(T-r)M\|\mathcal{D}\phi_u\|\|\mathcal{F}\|^2 + \sqrt{T-r}$. Utilizing (3.2), it follows from (3.11) that

$$\mu\|\mathcal{D}w_t^u\|^2 \le M_1\left(\int_t^{t+T}|\langle \mathcal{F}(w_\sigma^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle|^2 d\sigma\right)^{\frac{1}{2}} + |q|\int_{t+r}^{t+T}\big|\big\langle \mathcal{F}\big(w_{\sigma-r}^u\big), \mathcal{J}\big(\mathcal{D}w_\sigma^u\big)\big\rangle\big| d\sigma. \tag{3.12}$$

From (2.2), the function $\sigma \longmapsto \|\mathcal{D}w_\sigma^u\|$ is non-increasing, which gives

$$\big|\big\langle \mathcal{F}\big(w_{\sigma-r}^u\big), \mathcal{J}\big(\mathcal{D}w_\sigma^u\big)\big\rangle\big| \le L_{\mathcal{J}}\|\mathcal{F}\|\|\mathcal{D}w_t^u\|\|w_{\sigma-r}^u\|_{\mathcal{C}_u}, \ \forall\sigma \in [t+r, t+T].$$

Therefore, we infer that

$$|q|\int_{t+r}^{t+T}\big|\big\langle \mathcal{F}\big(w_{\sigma-r}^u\big), \mathcal{J}\big(\mathcal{D}w_\sigma^u\big)\big\rangle\big| d\sigma \le |q|(T-r)L_{\mathcal{J}}\|\mathcal{F}\|\|\mathcal{D}w_t^u\|\|w_{\sigma_t-r}^u\|_{\mathcal{C}_u}, \tag{3.13}$$

where $\left\|w^u_{\sigma_t-r}\right\|_{\mathcal{C}_u} = \sup_{t+r\leq\sigma\leq t+T}\|w^u_{\sigma-r}\|_{\mathcal{C}_u}$. By virtue of Lemma 3.2, we have

$$\left\|w^u_{\sigma_t-r}\right\|_{\mathcal{C}_u} \leq ae^{-b(\sigma_t-r)}\|\phi_u\|_{\mathcal{C}_u} + \frac{4}{1-|q|}\left\|\mathcal{D}w^u_{\frac{\sigma_t-r}{2}}\right\|, \tag{3.14}$$

for some constants $a, b > 0$. Combining (3.13) and (3.14), it comes from (3.12) that

$$\begin{aligned}\mu\|\mathcal{D}w^u_t\|^2 \leq & M_1\left(\int_t^{t+T}|\langle\mathcal{F}(w^u_\sigma),\mathcal{J}(\mathcal{D}w^u_\sigma)\rangle|^2 d\sigma\right)^{\frac{1}{2}} + |q|TL_{\mathcal{J}}\|\mathcal{F}\|\|\mathcal{D}w^u_t\| \\ & \times\left(ae^{-b(\sigma_t-r)}\|\phi_u\|_{\mathcal{C}_u} + \frac{4}{1-|q|}\left\|\mathcal{D}w^u_{\frac{\sigma_t-r}{2}}\right\|\right).\end{aligned} \tag{3.15}$$

Moreover, from (2.2), we have $\displaystyle\int_0^t|\langle\mathcal{F}(w^u_\sigma),\mathcal{J}(\mathcal{D}w^u_\sigma)\rangle|^2 d\sigma \leq \frac{\|\mathcal{D}\phi_u\|^2}{2}$, $\forall t \geq 0$, which implies that $\displaystyle\int_0^t|\langle\mathcal{F}(w^u_\sigma),\mathcal{J}(\mathcal{D}w^u_\sigma)\rangle|^2 d\sigma$ converges for all $t \geq 0$. Consequently, from the Cauchy criterion, we deduce that

$$\lim_{t\to+\infty}\int_t^{t+T}|\langle\mathcal{F}(w^u_\sigma),\mathcal{J}(\mathcal{D}w^u_\sigma)\rangle|^2 d\sigma = 0. \tag{3.16}$$

Using the fact that $t \longmapsto \|\mathcal{D}w^u_t\|$ is non-increasing, there exists a limit $\ell \in [0, +\infty[$ such that $\|\mathcal{D}w^u_t\| \longrightarrow \ell$, as $t \longrightarrow +\infty$. By taking the limit in (3.15) as $t \longrightarrow +\infty$, we obtain that

$$\mu\ell^2 \leq \frac{4|q|TL_{\mathcal{J}}\|\mathcal{F}\|}{1-|q|}\ell^2.$$

For $|q| < \dfrac{\mu}{\mu + 4TL_{\mathcal{J}}\|\mathcal{F}\|}$, it comes that $\ell = 0$. Finally, we assert that

$$\mathcal{D}w^u_t \longrightarrow 0, \text{ as } t \longrightarrow +\infty.$$

Reapplying Lemma 3.2, we deduce that

$$w^u(t) \longrightarrow 0, \text{ as } t \longrightarrow +\infty. \tag{3.17}$$

Combining (3.17) and (2.32), it yields that

$$w(t) \longrightarrow 0, \text{ as } t \longrightarrow +\infty. \tag{3.18}$$

□

Example 3.4. We consider the following uncoupled neutral systems given by:

$$\begin{cases} \dfrac{\partial}{\partial t}\big[y(x,t)-qy(x,t-r)\big] = -\dfrac{\partial}{\partial x}\big[y(x,t)-qy(x,t-r)\big] \\ +v(t)\Big(\alpha(x)y(x,t)+\displaystyle\int_{-r}^{0} g(\theta)y(x,t+\theta)d\theta\Big), & x\in(0,+\infty),\ t\geq 0, \\ \dfrac{\partial}{\partial t}\big[z(x,t)-qz(x,t-r)\big] = -\big[z(x,t)-qz(x,t-r)\big] \\ +v(t)\Big(\beta(x)z(x,t-r)+\displaystyle\int_{-r}^{0} z(x,t+\theta)d\theta\Big), & x\in(0,1),\ t\geq 0, \\ y(0,t)=0, & t\geq 0, \\ y(x,t)=\phi_1(x,t), & x\in(0,+\infty),\ t\in[-r,0], \\ z(x,t)=\phi_2(x,t), & x\in(0,1),\ t\in[-r,0], \end{cases} \tag{3.19}$$

where $-1<q<1$, $r>0$, $g\in C\big([-r,0],\mathbb{R}\big)$ and $(\alpha,\beta)\in L^\infty(0,+\infty)\times L^\infty(0,1)$ such that $\alpha(x)>c>rg_1$ a.e. $x\in(0,+\infty)$, where $g_1=\max\limits_{\theta\in[-r,0]}|g(\theta)|$. In addition, $\phi_1\in C\big([-r,0],\,L^p(0,+\infty)\big)$ and $\phi_2\in C\big([-r,0],\,L^p(0,1)\big)$ with $p>1$. Taking $X=L^p(0,+\infty)\times L^p(0,1)$ as a state space. System (3.19) can be written in the form of (1.1) by setting

$$\mathcal{A}=\begin{pmatrix} -\dfrac{\partial}{\partial x} & 0 \\ 0 & -\operatorname{Id}_{L^p(0,1)} \end{pmatrix} \text{ on the domain } \mathrm{D}(\mathcal{A})=\mathrm{D}_1\times L^p(\Omega),$$

where $\mathrm{D}_1=\big\{y\in W^{1,p}(0,+\infty): y(0)=0\big\}$. The control operator $\mathcal{F}: C\big([-r,0],X\big)\longrightarrow X$ is defined by

$$\mathcal{F}(\psi)=\begin{pmatrix} \alpha(.)\psi_1(0)+\displaystyle\int_{-r}^{0} g(\theta)\psi_1(\theta)d\theta \\ \beta(.)\psi_2(-r)+\displaystyle\int_{-r}^{0}\psi_2(\theta)d\theta \end{pmatrix}, \tag{3.20}$$

for every $\psi\in C\big([-r,0],X\big)$ $\big($i.e. $\psi(t)=\big(\psi_1(t),\psi_2(t)\big)\in L^p(0,+\infty)\times L^p(0,1)\big)$. The operator $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup given by $S(t)=\begin{pmatrix} U(t) & 0 \\ 0 & e^{-t}\end{pmatrix}$, where $U(t)$ denotes the semigroup of isometries defined by

$$(U(t)y)(x)=\begin{cases} y(x-t), & x\geq t, \\ 0, & x<t. \end{cases} \tag{3.21}$$

(see [18], p. 34). Consequently, the state space $X=L^p(0,+\infty)\times L^p(0,1)$ can be represented as a direct sum $X=X_u\oplus X_s$, where $X_u=L^p(0,+\infty)\times\{0\}$ and $X_s=\{0\}\times L^p(0,1)$. Moreover, the duality mapping $\mathcal{J}$ defined on $L^p(0,+\infty)\times L^p(0,1)$ is Lipschitz and is given by

$$\mathcal{J}(y,z)=\Big\{\frac{|y|^{p-2}}{\|y\|^{p-2}}y,\frac{|z|^{p-2}}{\|z\|^{p-2}}z\Big\}, \tag{3.22}$$

for all $(y,z) \in L^p(0,+\infty) \times L^p(0,1)$, (see [3], p. 82). Thus, for all $(y_2,z_2) \in L^p(0,+\infty) \times L^p(0,1)$ and $(y_1,z_1) \in L^q(0,+\infty) \times L^q(0,1)$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\big\langle (y_1,z_1), \mathcal{J}(y_2,z_2)\big\rangle_X = \int_0^{+\infty} y_1(x)\frac{|y_2|^{p-2}(x)}{\|y_2\|^{p-2}} y_2(x)dx + \int_0^1 z_1(x)\frac{|z_2|^{p-2}(x)}{\|z_2\|^{p-2}} z_2(x)dx.$$

Let $w = \begin{pmatrix} y \\ 0 \end{pmatrix} \in X_u$ where $y \in L^p(0,+\infty)$. It comes for all $t \geq r$ that

$$\begin{aligned} \big|\big\langle \mathcal{F}(S_u(t+.)w), \mathcal{J}\big(S_u(t)w\big)\big\rangle\big| &= \Big|\Big\langle \alpha(.)U(t)y + \int_{-r}^0 g(\theta)U(t+\theta)yd\theta, J(U(t)y)\Big\rangle\Big| \\ &\geq \big|\big\langle \alpha(.)U(t)y, J(U(t)y)\big\rangle\big| - \Big|\Big\langle \int_{-r}^0 g(\theta)U(t+\theta)yd\theta, J(U(t)y)\Big\rangle\Big|. \end{aligned} \tag{3.23}$$

where $J(y) = \frac{|y|^{p-2}}{\|y\|^{p-2}}y, \ \forall y \in L^p(0,+\infty)$. On the other hand, we have

$$\begin{aligned} \big|\big\langle \alpha(.)U(t)y, J(U(t)y)\big\rangle\big| &= \Big|\int_0^{+\infty} \alpha(x)U(t)y(x)\frac{|U(t)y(x)|^{p-2}}{\|U(t)y\|^{p-2}}U(t)y(x)dx\Big| \\ &\geq c\Big|\int_0^{+\infty} \frac{|U(t)y(x)|^p}{\|U(t)y\|^{p-2}}dx\Big| \\ &\geq c\|U(t)y\|^2 = c\|y\|^2. \end{aligned} \tag{3.24}$$

Furthermore,

$$\begin{aligned} \Big|\Big\langle \int_{-r}^0 g(\theta)U(t+\theta)yd\theta, J(U(t)y)\Big\rangle\Big| &\leq g_1\Big|\Big\langle \int_{-r}^0 U(t+\theta)yd\theta, J(U(t)y)\Big\rangle\Big| \\ &\leq g_1\Big\|\int_{-r}^0 U(t+\theta)yd\theta\Big\|\big\|J(U(t)y)\big\|_{L^q(0,+\infty)} \\ &\leq g_1\int_{-r}^0 \big\|U(t+\theta)y\big\|d\theta\big\|U(t)y\big\| \\ &\leq rg_1\|y\|^2. \end{aligned} \tag{3.25}$$

Combining (3.24) and (3.25), it yields from (3.23) for all $T > r$ that

$$\int_r^T \big|\big\langle \mathcal{F}(S_u(t+.)w), \mathcal{J}\big(S_u(t)w\big)\big\rangle\big|\, dt \geq \mu\|w\|^2,$$

where $\mu = (T-r)\,(c - rg_1)$. Thus, the condition (3.2) holds. Applying Theorem 3.3, one infers that, for $|q|$ sufficiently small, the feedback control

$$\begin{aligned} v(t) = -&\int_0^{+\infty}\Big(\alpha(x)y(x,t) + \int_{-r}^0 g(\theta)y(x,t+\theta)d\theta\Big)\frac{|y(x,t)-qy(x,t-r)|^{p-2}}{\|y(t)-qy(t-r)\|^{p-2}} \\ &\times\big(y(x,t)-qy(x,t-r)\big)dx, \end{aligned} \tag{3.26}$$

achieves strong stabilization of the system (3.19).

We now state a theorem which ensures the strong stabilization of system (1.1), while providing an explicit polynomial decay estimate.

Theorem 3.5. *Suppose that all the hypotheses of Theorem 3.3 hold, and assume further that there exists $\gamma > 0$ such that*

$$|\langle w, \mathcal{J}(S_u(r)z)\rangle| \leq \gamma |\langle w, \mathcal{J}(z)\rangle|, \text{ for all } w, z \in X_u. \tag{3.27}$$

Then, the feedback control (2.1) achieves strong stabilization of system (1.1), with the explicit decay estimate given by:

$$\|w(t)\| = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \text{ as } t \longrightarrow +\infty. \tag{3.28}$$

Proof. Consider $t \geq 0$ and $T > r$. It is evident that for all $\sigma \in [t+r, t+T]$, we have

$$\begin{aligned} &|\langle \mathcal{F}(w^u_{\sigma-r}), \mathcal{J}(\mathcal{D}w^u_\sigma)\rangle| \\ &\leq |\langle \mathcal{F}(w^u_{\sigma-r}), \mathcal{J}(\mathcal{D}w^u_\sigma) - \mathcal{J}(S_u(r)\mathcal{D}w^u_{\sigma-r})\rangle| + |\langle \mathcal{F}(w^u_{\sigma-r}), \mathcal{J}(S_u(r)\mathcal{D}w^u_{\sigma-r})\rangle| \\ &\leq L_{\mathcal{J}} M \|\mathcal{F}\| \|\mathcal{D}w^u_\sigma - S_u(r)\mathcal{D}w^u_{\sigma-r}\| + |\langle \mathcal{F}(w^u_{\sigma-r}), \mathcal{J}(S_u(r)\mathcal{D}w^u_{\sigma-r})\rangle|. \end{aligned} \tag{3.29}$$

By the variation of constants formula (3.4), it follows that

$$\mathcal{D}w^u_\sigma = S_u(r)\mathcal{D}w^u_{\sigma-r} - \int_{\sigma-r}^{\sigma} S_u(\sigma-\tau)\langle \mathcal{F}(w^u_\tau), \mathcal{J}(\mathcal{D}w^u_\tau)\rangle \mathcal{F}(w^u_\tau)d\tau, \ \forall \sigma \in [t+r, t+T],$$

which implies that

$$\|\mathcal{D}w^u_\sigma - S_u(r)\mathcal{D}w^u_{\sigma-r}\| \leq M\|\mathcal{F}\| \left(\int_t^{t+T} |\langle \mathcal{F}(w^u_\tau), \mathcal{J}(\mathcal{D}w^u_\tau)\rangle|^2 d\tau\right)^{\frac{1}{2}}, \ \forall \sigma \in [t+r, t+T]. \tag{3.30}$$

In addition, since $S_u(t)$ satisfies the assumption (3.27), we have

$$|\langle \mathcal{F}(w^u_{\sigma-r}), \mathcal{J}(S_u(r)\mathcal{D}w^u_{\sigma-r})\rangle| \leq \gamma |\langle \mathcal{F}(w^u_{\sigma-r}), \mathcal{J}(\mathcal{D}w^u_{\sigma-r})\rangle|. \tag{3.31}$$

Employing (3.30) and (3.31), it comes from (3.29) that

$$\begin{aligned} |\langle \mathcal{F}(w^u_{\sigma-r}), \mathcal{J}(\mathcal{D}w^u_\sigma)\rangle| \leq & L_{\mathcal{J}} M^2 \|\mathcal{F}\|^2 \left(\int_t^{t+T} |\langle \mathcal{F}(w^u_\tau), \mathcal{J}(\mathcal{D}w^u_\tau)\rangle|^2 d\tau\right)^{\frac{1}{2}} \\ &+ \gamma |\langle \mathcal{F}(w^u_{\sigma-r}), \mathcal{J}(\mathcal{D}w^u_{\sigma-r})\rangle|. \end{aligned}$$

Integrating the preceding inequality with respect to σ over $[t+r, t+T]$, we get

$$\begin{aligned} \int_{t+r}^{t+T} |\langle \mathcal{F}(w^u_{\sigma-r}), \mathcal{J}(\mathcal{D}w^u_\sigma)\rangle| d\sigma \leq & L_{\mathcal{J}}(T-r) M^2 \|\mathcal{F}\|^2 \left(\int_t^{t+T} |\langle \mathcal{F}(w^u_\tau), \mathcal{J}(\mathcal{D}w^u_\tau)\rangle|^2 d\tau\right)^{\frac{1}{2}} \\ &+ \gamma \int_{t+r}^{t+T} |\langle \mathcal{F}(w^u_{\sigma-r}), \mathcal{J}(\mathcal{D}w^u_{\sigma-r})\rangle| d\sigma, \end{aligned}$$

which gives

$$\int_{t+r}^{t+T} |\langle \mathcal{F}(w_{\sigma-r}^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle| d\sigma \leq T L_{\mathcal{J}} M^2 \|\mathcal{F}\|^2 \left(\int_t^{t+T} |\langle \mathcal{F}(w_\sigma^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle|^2 d\sigma \right)^{\frac{1}{2}} + \gamma \int_t^{t+T} |\langle \mathcal{F}(w_\sigma^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle| d\sigma.$$

Applying Schwarz's inequality, we obtain that

$$\int_{t+r}^{t+T} |\langle \mathcal{F}(w_{\sigma-r}^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle| d\sigma \leq M_2 \left(\int_t^{t+T} |\langle \mathcal{F}(w_\sigma^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle|^2 d\sigma \right)^{\frac{1}{2}}, \tag{3.32}$$

where $M_2 = T L_{\mathcal{J}} M^2 \|\mathcal{F}\|^2 + \gamma\sqrt{T}$. Combining (3.12) and (3.32), it follows that

$$\mu \|\mathcal{D}w_t^u\|^2 \leq (M_1 + |q| M_2) \left(\int_t^{t+T} |\langle \mathcal{F}(w_\sigma^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle|^2 d\sigma \right)^{\frac{1}{2}},$$

which leads to

$$\|\mathcal{D}w_t^u\|^4 \leq \eta \int_t^{t+T} |\langle \mathcal{F}(w_\sigma^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle|^2 d\sigma, \quad \eta = \left(\frac{M_1 + |q| M_2}{\mu} \right)^2. \tag{3.33}$$

Now, from (2.2), it comes that

$$\|\mathcal{D}w_{(k+1)T}^u\|^2 - \|\mathcal{D}w_{kT}^u\|^2 \leq -2 \int_{kT}^{(k+1)T} |\langle \mathcal{F}(w_\sigma^u), \mathcal{J}(\mathcal{D}w_\sigma^u)\rangle|^2 d\sigma, \ \forall k \in \mathbb{N}. \tag{3.34}$$

Employing (3.33), we deduce from (3.34) that

$$\|\mathcal{D}w_{(k+1)T}^u\|^2 - \|\mathcal{D}w_{kT}^u\|^2 \leq \frac{-2}{\eta} \|\mathcal{D}w_{kT}^u\|^4, \ \forall k \in \mathbb{N}.$$

If we set $\Psi_k = \|\mathcal{D}w_{kT}^u\|^2$, $\forall k \in \mathbb{N}$, we derive that

$$\Psi_k - \Psi_{k+1} \geq \frac{2}{\eta} \Psi_k^2, \tag{3.35}$$

which can be expressed as

$$\frac{\Psi_k - \Psi_{k+1}}{\Psi_k^2} \geq \frac{2}{\eta}, \ \forall k \in \mathbb{N}.$$

Remarking that $(\Psi_k)_{k\geq 0}$ is non-increasing, we infer that

$$\frac{\Psi_k - \Psi_{k+1}}{\Psi_k \Psi_{k+1}} \geq \frac{2}{\eta}, \ \forall k \in \mathbb{N},$$

which implies that

$$\frac{1}{\Psi_{k+1}} \geq \frac{2}{\eta} + \frac{1}{\Psi_k}, \ \forall k \in \mathbb{N},$$

then

$$\frac{1}{\Psi_k} \geq \frac{2k}{\eta} + \frac{1}{\Psi_0}, \ \forall k \in \mathbb{N},$$

thus, we conclude that

$$\Psi_k \leq \frac{\|\mathcal{D}\phi_u\|^2}{\frac{2\|\mathcal{D}\phi_u\|^2}{\eta}k+1}, \ \forall k \in \mathbb{N}. \tag{3.36}$$

Introducing the integer part $k = [\frac{t}{T}]$, it yields that

$$\|\mathcal{D}w_t^u\| = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \ \text{as } t \longrightarrow +\infty. \tag{3.37}$$

As a result, we deduce from Lemma 3.2 that

$$\|w^u(t)\| = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \ \text{as } t \longrightarrow +\infty. \tag{3.38}$$

Using both (3.38) and (2.32), we arrive at

$$\|w(t)\| = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \ \text{as } t \longrightarrow +\infty. \tag{3.39}$$

□

Remark 3.6. (1) In Theorem 3.3 and Theorem 3.5, $\dim X_u$ can be finite or infinite.

(2) Assumption (3.27) holds, for examples, when $S_u(t)$ is taken as the identity operator or as a periodic operator with period r.

(3) In the Hilbert space setting for X_u, the assumptions (3.2) and (3.27) are expressed, respectively, in the following forms:

$$\int_r^T |\langle \mathcal{F}(S_u(\sigma + .)w), S_u(\sigma)w\rangle| d\sigma \geq \mu\|w\|^2, \ \forall w \in X_u, \tag{3.40}$$

and

$$|\langle w, S_u(r)z\rangle| \leq \gamma|\langle w, z\rangle|, \ \forall w, z \in X_u, \tag{3.41}$$

where the notation $\langle ., .\rangle$ refers to the inner product in the space X_u.

(4) If $\dim X_u < +\infty$, the inequality (3.40) reduces to the following equivalent form:

$$\forall w \in X_u, \ \big\langle \mathcal{F}(S_u(t + .)w), S_u(t)w \big\rangle = 0, \ \forall t \geq r \implies w = 0. \tag{3.42}$$

(see [8], Remark 4.3).

(5) We emphasize that the obtained result in Theorem 3.5 generalizes the corresponding result of [9] to the setting of neutral systems.

(6) In the context of Hilbert space, analogues results to Theorem 3.5 have been established in [19].

When $X_u = X$ and $X_s = \{0\}$, the following corollary holds as an immediate consequence of Theorem 3.5.

Corollary 3.7. *Let $S(t)$ be a contraction semigroup on X, and suppose that there exist $T > r$ and $\mu > 0$ satisfying*

$$\int_r^T |\langle \mathcal{F}(S(\sigma + .)w), \mathcal{J}(S(\sigma)w)\rangle| d\sigma \geq \mu\|w\|^2, \ \forall w \in X. \tag{3.43}$$

Furthermore, there exists $\gamma > 0$ such that

$$|\langle w, \mathcal{J}(S(r)z)\rangle| \leq \gamma |\langle w, \mathcal{J}(z)\rangle|, \text{ for all } w, z \in X. \tag{3.44}$$

Then, the feedback control

$$v(t) = -\langle \mathcal{F}(w_t), \mathcal{J}(\mathcal{D}w_t)\rangle, \, \forall t \geq 0, \tag{3.45}$$

provides strong stabilization of system (1.1), together with the explicit decay estimate:

$$\|w(t)\| = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \text{ as } t \longrightarrow +\infty.$$

Example 3.8. Let $\Omega = (0,1)$ and $1 < p < \infty$. We consider the following coupled neutral systems with distributed delay described by

$$\begin{cases} \dfrac{\partial}{\partial t}\big[y(x,t) - qy(x,t-r)\big] = v(t)\left(y(x,t-r) + \displaystyle\int_{-r}^{0} y(x,t+\theta)d\theta\right), & x \in \Omega, \ t \geq 0, \\ \dfrac{\partial}{\partial t}\big[z(x,t) - qz(x,t-r)\big] = \Delta\big[z(x,t) - qz(x,t-r)\big] & \\ +v(t)\left(\displaystyle\int_{-r}^{0} \Big(y(x,t+\theta) + z(x,t+\theta)\Big)d\theta\right), & x \in \Omega, \ t \geq 0, \\ z(0,t) = z(1,t) = 0, & t \geq 0, \\ y(x,t) = \phi_1(x,t), & x \in \Omega, \ t \in [-r,0], \\ z(x,t) = \phi_2(x,t), & x \in \Omega, \ t \in [-r,0], \end{cases} \tag{3.46}$$

where $-1 < q < 1$, $r > 0$ and $(\phi_1, \phi_2) \in \big(C([-r,0], L^p(\Omega))\big)^2$. In the sequel, we take $X = (L^p(\Omega))^2$ as state space. The system (3.46) may be expressed in the form (1.1) by setting

$$\mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix} \text{ with the domain } \mathrm{D}(\mathcal{A}) = L^p(\Omega) \times \big(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\big).$$

The control operator $\mathcal{F} : C\big([-r,0], X\big) \longrightarrow X$ is defined by

$$\mathcal{F}(\psi) = \begin{pmatrix} \psi_1(-r) + \displaystyle\int_{-r}^{0} \psi_1(\theta)d\theta \\ \displaystyle\int_{-r}^{0} \big(\psi_1(\theta) + \psi_2(\theta)\big)d\theta \end{pmatrix}, \tag{3.47}$$

for each $\psi \in C\big([-r,0], X\big)$ $\big($i.e. $\psi(t) = \big(\psi_1(t), \psi_2(t)\big) \in (L^p(\Omega))^2\big)$. The operator $\mathcal{A}$ generates the semigroup $S(t) = \begin{pmatrix} \mathrm{Id}_{L^p(\Omega)} & 0 \\ 0 & U(t) \end{pmatrix}$, where $U(t)$ is the heat semigroup generated by the operator Δ which is defined for all $t > 0$ by

$$(U(t)z)(x) = \int_0^1 K(t,x,s)z(s)ds, \ \forall z \in L^p(\Omega),$$

where K is the heat kernel given by $K(t,x,s) = \displaystyle\sum_{k=1}^{\infty} 2e^{-k^2\pi^2 t}\sin(k\pi x)\sin(k\pi s)$.

The heat semigroup $U(t)$ generated by the Laplace operator with Dirichlet boundary conditions in $L^p(\Omega)$ is exponentially stable (see [13]). In this case, the state space $X = (L^p(\Omega))^2$ admits a decomposition into $X = X_u \oplus X_s$, where $X_u = L^p(\Omega) \times \{0\}$ and $X_s = \{0\} \times L^p(\Omega)$, from where it comes that $S_u(t) = \mathrm{Id}_{X_u}$, which implies that the condition (3.27) is satisfied. On the other hand, the duality mapping $\mathcal{J}$ defined on $(L^p(\Omega))^2$ is Lipschitz and is given by

$$\mathcal{J}(y,z) = \Big\{ \frac{|y|^{p-2}}{\|y\|^{p-2}} y, \frac{|z|^{p-2}}{\|z\|^{p-2}} z \Big\}, \tag{3.48}$$

for all $(y,z) \in (L^p(\Omega))^2$, (see [3], p. 82). Thus, for all $(y_2,z_2) \in (L^p(\Omega))^2$ and $(y_1,z_1) \in (L^q(\Omega))^2$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\big\langle (y_1,z_1), \mathcal{J}(y_2,z_2) \big\rangle_X = \int_0^1 y_1(x) \frac{|y_2|^{p-2}(x)}{\|y_2\|^{p-2}} y_2(x) dx + \int_0^1 z_1(x) \frac{|z_2|^{p-2}(x)}{\|z_2\|^{p-2}} z_2(x) dx.$$

Now, for all $w = \begin{pmatrix} y \\ 0 \end{pmatrix} \in X_u$ and $t \geq r$, we have

$$\begin{aligned} \big|\big\langle \mathcal{F}(S_u(t+.)w), \mathcal{J}\big(S_u(t)w\big) \big\rangle\big| &= \Big| \Big\langle y + \int_{-r}^0 y d\theta, J(y) \Big\rangle \Big| \\ &= (1+r) |\langle y, J(y) \rangle| \\ &= (1+r) \|y\|^2, \end{aligned} \tag{3.49}$$

where $J(y) = \frac{|y|^{p-2}}{\|y\|^{p-2}} y$, $\forall y \in L^p(\Omega)$. Hence, it comes for all $T > r$ that

$$\int_r^T \big|\big\langle \mathcal{F}(S_u(t+.)w), \mathcal{J}\big(S_u(t)w)\big) \big\rangle\big| \, dt \geq \mu \|w\|^2,$$

where $\mu = (T-r)(1+r)$. Thus, the assumption (3.2) holds. Applying Theorem 3.5, we conclude that the system (3.46) can be strongly stabilized by means of the feedback control

$$v(t) = -\int_0^1 \int_{-r}^0 y(x,t+\theta) d\theta \frac{|y(x,t) - qy(x,t-r)|^{p-2}}{\|y(t) - qy(t-r)\|^{p-2}} \big(y(x,t) - qy(x,t-r) \big) dx, \tag{3.50}$$

satisfying the following decay estimate

$$\Big(\int_0^1 y^p(x,t) dx \Big)^{\frac{2}{p}} + \Big(\int_0^1 z^p(x,t) dx \Big)^{\frac{2}{p}} = \mathcal{O}\Big(\frac{1}{t} \Big), \text{ as } \longrightarrow +\infty. \tag{3.51}$$

Example 3.9. Consider the following uncoupled neutral systems with distributed delay, defined by:

$$\begin{cases} \dfrac{\partial}{\partial t}\big[y(x,t)-qy(x,t-r)\big]=\mathcal{A}_0\big[y(x,t)-qy(x,t-r)\big] & \\ +v(t)\Big(\alpha y(x,t-r)+\displaystyle\int_{-r}^{0}y(x,t+\theta)d\theta\Big), & x\in(-\pi,\pi),\ t\geq 0,\\ \dfrac{\partial}{\partial t}\big[z(x,t)-qz(x,t-r)\big]=-\beta\big[z(x,t)-qz(x,t-r)\big] & \\ +v(t)\Big(z(x,t-r)+\displaystyle\int_{-r}^{0}z(x,t+\theta)d\theta\Big), & x\in(0,1),\ t\geq 0,\\ y(x,t)=\phi_1(x,t), & x\in(-\pi,\pi),\ t\in[-r,0],\\ z(x,t)=\phi_2(x,t), & x\in(0,1),\ t\in[-r,0], \end{cases} \tag{3.52}$$

where $-1<q<1$ and $\alpha,\beta,r>0$. The system (3.52) takes the form of (1.1) if we set $w(t)=\big(y(.,t),z(.,t)\big)\in X=L^2(-\pi,\pi)\times L^p(0,1)$ with $p>1$. In this case, we have

$$\mathcal{A}=\begin{pmatrix}\mathcal{A}_0 & 0\\ 0 & -\beta\,\mathrm{Id}_{L^p(0,1)}\end{pmatrix}\text{ with domain } \mathrm{D}(\mathcal{A})=\mathrm{D}(\mathcal{A}_0)\times L^p(0,1),$$

where the operator $\mathcal{A}_0$ is defined by $\mathcal{A}_0=-\dfrac{\partial^4}{\partial x^4}-\dfrac{\partial^2}{\partial x^2}$ such that $\mathrm{D}(\mathcal{A}_0)=\Big\{y\in H^4(-\pi,\pi):\dfrac{\partial^n y}{\partial x^n}(-\pi)=\dfrac{\partial^n y}{\partial x^n}(\pi),\ n=0,...,3\Big\}$. The linear operator $\mathcal{F}:C([-r,0],X)\longrightarrow X$ is given by

$$\mathcal{F}(\psi)=\begin{pmatrix}\alpha\psi_1(-r)+\displaystyle\int_{-r}^{0}\psi_1(\theta)d\theta\\ \psi_2(-r)+\displaystyle\int_{-r}^{0}\psi_2(\theta)d\theta\end{pmatrix},$$

for every $\psi\in C\big([-r,0],X\big)$ $\big($i.e. $\psi(t)=\big(\psi_1(t),\psi_2(t)\big)$, where $\psi_1(t)\in L^2(-\pi,\pi)$ and $\psi_2(t)\in L^p(0,1)$ for all $t\in[-r,0]\big)$. The spectrum of $\mathcal{A}_0$ is given by the simple eigenvalues $\delta_n=-n^4+n^2$, $\forall n\geq 1$, with its corresponding eigenfunctions defined by $\varphi_n(x)=\dfrac{1}{\sqrt{\pi}}\sin(nx)$. This leads to

$$w(x,t)=\begin{pmatrix}\displaystyle\sum_{n=1}^{+\infty}\langle y(t),\varphi_n\rangle_{L^2(-\pi,\pi)}\varphi_n(x)\\ z(x,t)\end{pmatrix},$$

Since the operator $\mathcal{A}_0$ is an infinitesimal generator of the contraction semigroup $S_1(t)$ in $L^2(-\pi,\pi)$ defined by $S_1(t)y=\displaystyle\sum_{n=1}^{+\infty}e^{\delta_n t}\langle y,\varphi_n\rangle\varphi_n$, it yields that the operator $\mathcal{A}$ generates the semigroup $S(t)=\begin{pmatrix}S_1(t) & 0\\ 0 & S_2(t)\end{pmatrix}$, where $S_2(t)z=e^{-\beta t}z$, $\forall z\in L^p(0,1)$. Remarking that $L^2(-\pi,\pi)=vect(\varphi_1)\oplus vect\big(\{\varphi_n,n\geq 2\}\big)$, it comes that the space $X=L^2(-\pi,\pi)\times L^p(0,1)$ can be decomposed according to

$X = X_u \oplus X_s$, where $X_u = vect(\varphi_1) \times \{0\}$ and $X_s = vect\big(\{\varphi_n, n \geq 2\}\big) \times L^p(0,1)$. Thus, we have

$$w^u(t) = \begin{pmatrix} \langle y(t), \varphi_1 \rangle \varphi_1 \\ 0 \end{pmatrix}.$$

Moreover, $S_u(t) = \begin{pmatrix} \mathrm{Id}_{vect(\varphi_1)} & 0 \\ 0 & 0 \end{pmatrix} = \mathrm{Id}_{X_u}$ and $S_s(t)$ is given by $S_s(t)\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{n=2}^{+\infty} e^{\delta_n t} \langle y, \varphi_n \rangle \varphi_n \\ e^{-\beta t} z \end{pmatrix}$, $\forall (y,z) \in X_s$, $(\delta_n < 0, \forall n \geq 2)$. Now, let $\begin{pmatrix} \zeta \\ 0 \end{pmatrix} \in X_u$ where $\zeta \in vect(\varphi_1)$. For all $t \geq r$, we have

$$\begin{aligned} \left\langle \mathcal{F}\left(S_u(t+.) \begin{pmatrix} \zeta \\ 0 \end{pmatrix} \right), S_u(t) \begin{pmatrix} \zeta \\ 0 \end{pmatrix} \right\rangle_{X_u} &= (\alpha + r)\|\zeta\|^2 \\ &= (\alpha + r)\|(\zeta, 0)\|^2_{X_u}. \end{aligned} \tag{3.53}$$

Thus, the condition (3.40) holds for $\mu = (T - r)(\alpha + r) > 0$, as well as (3.2) since X_u is a Hilbert space. In addition, since $S_u(t) = \mathrm{Id}_{X_u}$, it yields that the assumption (3.27) is satisfied where $\gamma = 1$. Consequently, we conclude according to Theorem 3.5 that the feedback control

$$\begin{aligned} v(t) = -\frac{1}{\pi} \int_{-\pi}^{\pi} &\left(\alpha y(x,t) + \int_{-r}^{0} y(x, t+\theta) d\theta \right) \sin(x) dx \\ &\times \int_{-\pi}^{\pi} \big(y(x,t) - q y(x, t-r) \big) \sin(x) dx, \end{aligned} \tag{3.54}$$

ensures strong stabilization of the system (3.52) with the decay estimate:

$$\int_{-\pi}^{\pi} y^2(x,t) dx + \left(\int_0^1 z^p(x,t) dx \right)^{\frac{2}{p}} = \mathcal{O}\left(\frac{1}{t} \right), \text{ as } \longrightarrow +\infty. \tag{3.55}$$

To highlight the stabilizing action of the feedback control (3.54), we perform numerical simulations using the following parameters: $X = L^2(-\pi, \pi) \times L^4(0,1)$, $q = 0.9$, $r = 0.2$, $\alpha = 4$, $\beta = 1$, $\phi_1(x,t) = 0.1 + \sin(x) + 0.3\sin(2x)$, for all $(x,t) \in (-\pi, \pi) \times [-0.2, 0]$, and $\phi_2(x,t) = 0.1 + \sin(\pi x) + 0.3 \sin(2\pi x)$, for all $(x,t) \in (0,1) \times [-0.2, 0]$. Figure 1 illustrates the evolution of the free state norm (i.e. with control $v(t) = 0$). When the feedback control (3.54) is applied, we obtain Figure 2 which illustrate the evolution of the stabilized state. Figure 3 depicts the profile of the stabilizing feedback control (3.54).

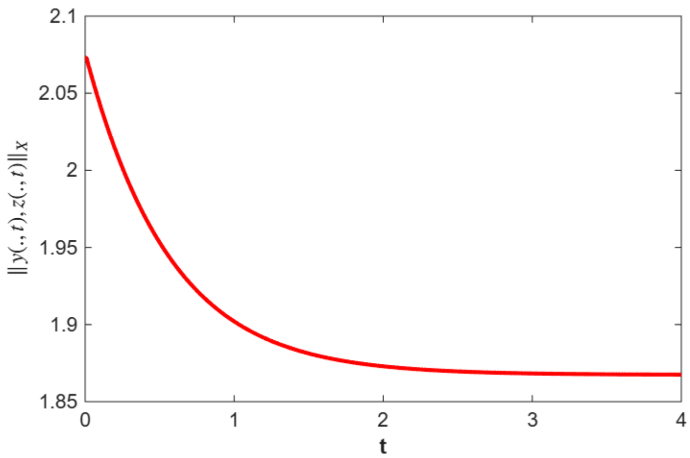

FIGURE 1. The norm of the free state norm

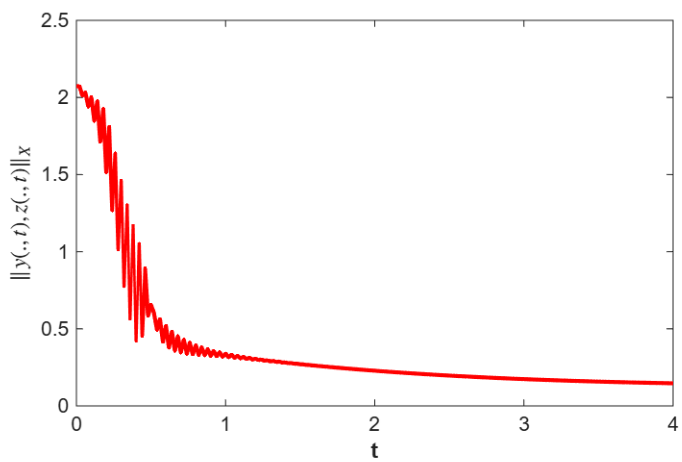

FIGURE 2. The evolution of the stabilised state norm

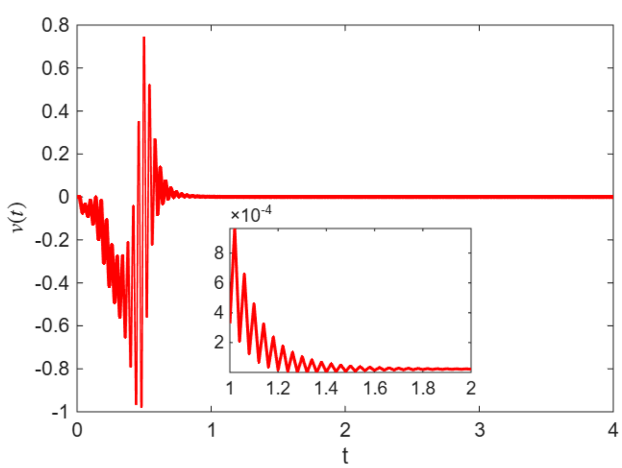

FIGURE 3. The evolution of the stabilising control (3.54)

Example 3.10. Consider the following neutral functional differential equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\big[w(x,t) - qw(x,t-r)\big] = i\beta\big[w(x,t) - qw(x,t-r)\big] \\ +v(t)\Big(\displaystyle\int_{-r}^{0} w(x,t+\theta)d\theta\Big), & x \in (0,1),\ t \geq 0, \\ w(x,t) = \phi(x,t), & x \in (0,1),\ t \in [-r,0], \end{cases} \tag{3.56}$$

where $-1 < q < 1$, $r > 0$, $\beta \in \mathbb{R}^*$ and $\phi \in C\big([-r,0],\, L^p((0,1);\mathbb{C})\big)$ with $p > 1$. Let us take $X = L^p((0,1);\mathbb{C})$ as a state space. The system (3.56) has the form of (1.1) if we set $\mathcal{A} = i\beta\,\mathrm{Id}_{L^p((0,1);\mathbb{C})}$, $(i \in \mathbb{C})$ on the domain $\mathrm{D}(\mathcal{A}) = L^p((0,1);\mathbb{C})$. Moreover, the control operator is defined by $\mathcal{F} : C\big([-r,0],X\big) \longrightarrow X$

$$\mathcal{F}(\psi) = \int_{-r}^{0} \psi(\theta)d\theta, \tag{3.57}$$

for every $\psi \in C\big([-r,0],X\big)$. The operator $\mathcal{A}$ is the generator of the semigroup of isometries $S(t)$ defined as follows:

$$S(t)w = e^{i\beta t}w,\ \forall w \in L^p((0,1);\mathbb{C}). \tag{3.58}$$

Now, for all $(w,z) \in X^2$, we have

$$\begin{aligned} \big|\big\langle w, \mathcal{J}(S(r)z)\big\rangle\big| &= \left|\int_0^1 w(x)\frac{|S(r)z(x)|^{p-2}}{\|S(r)z\|^{p-2}}\overline{S(r)z(x)}dx\right| \\ &= \left|\int_0^1 w(x)\frac{|e^{i\beta r}z(x)|^{p-2}}{\|e^{i\beta r}z\|^{p-2}}e^{-i\beta r}\overline{z(x)}dx\right| \\ &= \left|\int_0^1 w(x)\frac{|z(x)|^{p-2}}{\|z\|^{p-2}}\overline{z(x)}dx\right| \quad (|e^{i\beta r}| = |e^{-i\beta r}| = 1) \\ &= \big|\big\langle w, \mathcal{J}(z)\big\rangle\big|. \end{aligned}$$

Consequently, the assumption (3.44) is satisfied. Furthermore, for all $w \in L^p((0,1);\mathbb{C})$ and $t \geq r$, we have

$$\begin{aligned} \big|\big\langle \mathcal{F}(S(t+.)w), \mathcal{J}\big(S(t)w\big)\big\rangle\big| &= \left|\Big\langle \int_{-r}^{0} e^{i\beta(t+\theta)}d\theta w, \mathcal{J}\left(e^{i\beta t}w\right)\Big\rangle\right| \\ &= \left|\frac{e^{i\beta t}(1-e^{-i\beta r})}{i\beta}\right|\left|\int_0^1 w(x)\frac{|e^{i\beta t}w(x)|^{p-2}}{\|e^{i\beta t}w\|^{p-2}}e^{-i\beta t}\overline{w(x)}dx\right| \\ &= \frac{|1-e^{-i\beta r}|}{|\beta|}\left|\int_0^1 w(x)\frac{|w(x)|^{p-2}}{\|w\|^{p-2}}\overline{w(x)}dx\right| \quad (|e^{i\beta t}| = |e^{-i\beta t}| = 1) \\ &= \frac{|1-e^{-i\beta r}|}{|\beta|}\int_0^1 \frac{|w(x)|^p}{\|w\|^{p-2}}dx \quad (w(x)\overline{w(x)} = |w(x)|^2) \\ &= \frac{|1-e^{-i\beta r}|}{|\beta|}\|w\|^2. \end{aligned} \tag{3.59}$$

Therefore, it comes for all $T > r$ that

$$\int_r^T \left|\left\langle \mathcal{F}(S(t+.)w), \mathcal{J}\big(S(t)w\big)\right\rangle\right| dt \geq \mu\|w\|^2,$$

where $\mu = (T-r)\dfrac{|1-e^{-i\beta r}|}{|\beta|} > 0$ ($1-e^{-i\beta r} \neq 0$ since $r > 0$ and $\beta \in \mathbb{R}^*$). Thus, the condition (3.43) holds. Applying Corollary 3.7, It follows that the solution of (3.56) satisfies

$$\left(\int_0^1 |w(x,t)|^p dx\right)^{\frac{2}{p}} = \mathcal{O}\left(\frac{1}{t}\right), \text{ as } \longrightarrow +\infty, \tag{3.60}$$

by employing the following feedback control law:

$$v(t) = -\int_0^1 \int_{-r}^0 w(x,t+\theta)d\theta \frac{|w(x,t)-qw(x,t-r)|^{p-2}}{\|w(t)-qw(t-r)\|^{p-2}}\big(\overline{w(x,t)} - q\overline{w(x,t-r)}\big)dx. \tag{3.61}$$

4. Conclusion

In this work, we introduce a bounded feedback control relying only on the projection of the state onto an appropriate subspace of a reflexive Banach state space, to investigate strong and polynomial stabilization for a class of infinite dimensional bilinear neutral systems with distributed delays under the observability like assumption (3.2). Several applications are provided to demonstrate the effectiveness of the theoretical results. As a future perspective, it would be worthwhile to study the exponential stabilization of systems similar to (1.1).

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