Gulf Journal of Mathematics ISSN: 2309-4966
Vol 22, Issue 2 (2026) 1-15 https://doi.org/10.56947/gjom.v22i2.4230

A MOTION PLANNING PROBLEM FOR ANTI-DAMPED WAVE EQUATION

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Abstract. A motion planning strategy is developed for a boundary-controlled system governed by a one-dimensional wave equation subject to spatially distributed anti-damping and boundary actuation. The goal is to construct a boundary input that enforces the tracking of a prescribed trajectory at the system output. Our proposed methodology relies on a Volterra-type integral transformation of backstepping nature, which converts the original dynamics into a suitably chosen auxiliary system. The transformed system is analyzed within a semigroup framework, which enables the use of Laplace transform techniques to derive explicit representations of the system trajectory and the associated control input. The state and control of the original system are subsequently recovered through the inverse transformation, yielding a constructive solution to the motion planning problem expressed in terms of the prescribed trajectory and the solutions of the associated kernel partial differential equations.

Keywords. wave equation, feedback and closed-loop system, Volterra transformation, generator of a C_0-semigroup, Laplace transform, Motion planning.
2020 Mathematics Subject Classification. Primary 93C05, 93C20; Secondary 47D06.

1. Introduction

1.1. **Discussion of previous work.** Motion planning (or trajectory tracking) is a fundamental problem in engineering that frequently arises in robotics and control. Given a control system with prescribed initial configurations, the problem is to determine a boundary input that steers the system output toward a predefined trajectory while respecting both the intrinsic system dynamics and external constraints. A wide range of methodologies has been developed to address motion planning problems for finite-dimensional systems. A differential–geometric framework was proposed in [13], providing structural conditions for controllability and constructive tools for trajectory generation. Alternative approaches rely on inputs with specific regularity, including piecewise constant control signals

Date: Received: Feb 8, 2026; Revised: Mar 6, 2026; Accepted: Mar 16, 2026.
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[17] and piecewise continuous feedback laws [24]. Furthermore, explicit constructive procedures have been obtained in [18] through chained-form transformations combined with sinusoidal control inputs, enabling the systematic steering of nonholonomic systems.

Within the class of systems modeled by ordinary differential equations, the framework of differential flatness [14], [15], [19], [20], [22] provides a powerful framework to parameterize trajectories via flat outputs, often constructed from Gevrey functions and translations of reference trajectories. In the case of linear hyperbolic PDEs, operational calculus provides a framework for expressing the state and control variables through delayed or advanced versions of a fundamental output [21]. The models considered in [6], [7], [8], [9] are governed by hyperbolic PDEs, and the corresponding motion planning problems have been solved using Laplace transform techniques. Numerical algorithms for approximating planning controls have been proposed in [23], and stable trajectory tracking has been addressed through the combination of passivity-based methods with semigroup theory [12].

1.2. **Problem formulation.** We study a motion planning problem for a one-dimensional wave model incorporating internal anti-damping and a dynamic boundary interaction. The system dynamics are described by a set of coupled equations acting in the spatial domain $(0, 1)$ and at its boundaries.

The evolution of the state variable $z(t, x)$ throughout the spatial region is described by the anti-damped wave equation

$$z_{tt} = z_{xx} - 2az_t - a^2 z, \qquad \text{in} \quad (0, \infty) \times (0, 1), \tag{1.1}$$

where $a \leq 0$ denotes a constant anti-damping parameter.

At the boundary $x = 0$, the system is subject to a dynamic condition given by

$$z_{tt}(t, 0) = z_x(t, 0), \qquad t > 0, \tag{1.2}$$

which links the boundary acceleration to the spatial derivative of the state.

The boundary displacement is prescribed as

$$z(t, 1) = U(t), \qquad t > 0, \tag{1.3}$$

where $U(t)$ is the controller applied at the right endpoint. In particular, the boundary trace $z(t, 0)$ is regarded as the *system output*.

To emphasize motion generation induced exclusively by the boundary control, we assume vanishing initial boundary conditions. The boundary relation (1.2) captures the effect of a concentrated mass attached at the endpoint $x = 0$. Its origin lies in the local application of Newton's second law at the boundary, where the acceleration of the mass equals the net force exerted by the string. For small transverse displacements, this force is proportional to the spatial slope of the displacement, yielding in dimensional variables a relation of the form

$$m z_{tt}(t, 0) = \pm T z_x(t, 0),$$

with m denoting the tip mass and T the string tension. The sign depends on the orientation convention adopted for the displacement and the outward normal. After introducing dimensionless variables (rescaling space and time by the characteristic length) and wave speed and normalizing the physical parameters so that the mass ratio and tension become unity, the boundary law simplifies to the compact expression

$$z_{tt}(t,0) = z_x(t,0).$$

This type of dynamic boundary coupling is frequently employed in the mathematical modeling of elastic or viscoelastic structures carrying attached masses, such as a rod fixed at one end and equipped with a tip mass at the other (see, for instance, [1, 2, 3, 4]). From the perspective of energy analysis, the boundary equation contributes an additional kinetic component corresponding to the motion of the attached mass. As a result, the natural energy functional combines the distributed wave energy with this boundary kinetic term, ensuring that the conservation or dissipation properties of the model accurately reflect both propagation phenomena and inertial effects at the endpoint. The normalization and sign convention adopted here therefore provide a consistent mechanical interpretation and a sound analytical framework for stability and boundary control design.

Given a sufficiently smooth predefined trajectory $\theta : [0,\infty) \to \mathbb{R}$, this work aims to design a boundary input function $U(t)$ that drives the output $z(t,0)$ to follow $\theta(t)$. More precisely, the control goal consists in achieving exact trajectory tracking, namely

$$z(t,0) = \theta(t), \qquad \forall\, t \geq 0. \tag{1.4}$$

Due to the presence of boundary dynamics and admissibility constraints on the control input, exact tracking may not always be attainable. In such cases, the requirement is relaxed to asymptotic tracking [5], meaning that the tracking discrepancy

$$\mathbf{e}(t) := z(t,0) - \theta(t)$$

satisfies

$$\lim_{t\to\infty} \mathbf{e}(t) = 0.$$

1.3. **Methodology and Key Contributions.** We now outline the proposed approach and summarize the main results. The method starts by reformulating the original system (1.1)–(1.3) into an auxiliary one through a kernel-based Volterra integral (backstepping) transformation.

Specifically, we define the transformation given below:

$$w(t,x) = \cosh(ax)z(t,x) - \int_0^x \begin{bmatrix} m(x,\rho) & 0 \\ 0 & n(x,\rho) \end{bmatrix} \begin{bmatrix} z(t,\rho) \\ z_t(t,\rho) \end{bmatrix} d\rho, \tag{1.5}$$

where z denotes the solution of system (1.1), and the kernel functions m and n are constructed so as to map the original dynamics (1.1)–(1.3) into a simpler auxiliary system.

Under transformation (1.5), the new state variable w satisfies the following equations:

$$w_{tt} = w_{xx}, \qquad \text{in} \quad (0,\infty) \times (0,1), \tag{1.6}$$

$$w_{tt}(t,0) = w_x(t,0), \qquad t > 0. \tag{1.7}$$

The auxiliary input $W(t)$ appearing in the transformed system is related to the original boundary control $U(t)$ through

$$W(t) = \cosh(a)U(t) - \int_0^1 \begin{bmatrix} m(1,\rho) & 0 \\ 0 & n(1,\rho) \end{bmatrix} \begin{bmatrix} z(t,\rho) \\ z_t(t,\rho) \end{bmatrix} d\rho. \tag{1.8}$$

A key observation is that equation (1.6) is free from the antidamping term, which originally acted throughout the spatial domain. Through the Volterra transformation (1.5), this distributed term is effectively shifted to the boundary, yielding a structurally simpler system that is significantly more amenable to control design.

The second step involves reformulating the auxiliary system (1.6)–(1.8) in the form of an operator-based evolution equation on a Hilbert space

$$\mathfrak{X} := \mathcal{H}^1_L \times L^2 \times \mathbb{C}, \quad \mathcal{H}^1_L = \{\phi \in \mathcal{H}^1 : \phi(1) = 0\},$$

equipped with the inner product[1]

$$\Big\langle (f, g, \alpha_1), (\phi, \psi, \alpha_2) \Big\rangle := \int_0^1 f_x\overline{\phi_x} + g\overline{\psi}dx + \alpha_1\overline{\alpha_2},$$

For $t \geq 0$, we introduce the state variable

$$Z(t) := \big(w - W(t)\mathbb{1}, w_t, w_t(t,0)\big)$$

together with the control vector

$$\mathfrak{b} := \big(0, -\mathbb{1}, -1\big), \ \mathbb{1}(x) \equiv 1.$$

Defining the operators[2]$(\mathfrak{A}, \mathfrak{B})$ by

$$\mathfrak{D}(\mathfrak{A}) := \big\{(\phi, \psi, \beta)^\top \in \mathfrak{X} : \phi \in \mathcal{H}^2, \ \psi \in \mathcal{H}^1_L : \psi(0) = \beta\big\}, \ \mathfrak{A}\big(\phi, \psi, \beta\big) = \big(\psi, \phi'', \phi'(0)\big),$$
$$\mathfrak{B}\mu = \mu\mathfrak{A}_{-1}\mathfrak{b}, \ \mathfrak{B} \in \mathcal{L}(\mathbb{R}; \mathfrak{X}_{-1}),$$

the auxiliary system (1.6) admits the abstract representation

$$\dot{Z}(t) = \mathfrak{A}\big(Z(t) + W_t(t)\mathfrak{b}\big) = \mathfrak{A}_{-1}Z(t) + \mathfrak{B}W_t(t), \quad t > 0, \quad Z(0) = 0_{\mathfrak{X}}. \tag{1.9}$$

According to [7, 9], the operator $\mathfrak{A}$ is skew-adjoint and $\mathfrak{B}$ is admissible for $\mathfrak{A}$ in the sense of [11]. Hence, the abstract representation (1.9) admits the following (mild) solution:

$$\begin{aligned} Z(t) = \int_0^t e^{\mathfrak{A}(t-s)}\mathfrak{B}W_s(s)ds \quad &= \int_0^t \mathfrak{A}e^{\mathfrak{A}(t-s)}\mathfrak{b}W_s(s)ds, \\ &= \int_0^t e^{\mathfrak{A}(\dot{t}-s)}\mathfrak{b}W_s(s)ds \in \mathfrak{X}. \end{aligned} \tag{1.10}$$

In the third step, the Laplace transform of (1.10) provides an explicit expression for the motion-planning control of the feedback-loop system (1.6)–(1.8), with the output satisfies (1.4), as well as its solution in terms of a prescribed trajectory. The first main result summarizes this theorem.

[1] $\overline{c}$ stands for the complex conjugate of c.

[2] Here, $\mathfrak{X}_{-1}$ denotes the extrapolation space obtained as the completion of $\mathfrak{X}$ under the norm $||x||_{-1} := ||x||_{\mathfrak{X}} + ||(\lambda - \mathfrak{A})^{-1}x||_{\mathfrak{X}}$, where λ is chosen from the resolvent set $\rho(\mathfrak{A})$. The operator $\mathfrak{A}_{-1}$ denotes the extrapolated extension of $\mathfrak{A}$ acting on $\mathfrak{X}_{-1}$; see [11].

Theorem 1.1. *Let $\theta \in C^3([0,\infty))$ with supp $\theta \subseteq [1,\infty)$. The feedback law associated with the auxiliary system (1.6)–(1.7), with the output given by (1.4), is defined by*

$$W(t) = \begin{cases} \frac{\theta(t+1)+\dot{\theta}(t+1)}{2}, & 0 \le t \le 2, \\ \frac{\theta(t+1)+\theta(t-1)+\dot{\theta}(t+1)-\dot{\theta}(t-1)}{2}, & 2 \le t. \end{cases} \qquad (1.11)$$

Furthermore, the solution of system (1.6)–(1.7)–(1.11), which also verifies $z(t,0) = \theta(t)$, is given by

$$w(t,x) = \begin{cases} 0, & t \in [0, 1-x], \\ \frac{\theta(t+x)+\dot{\theta}(t+x)}{2}, & t \in [1-x, 1+x], \\ \frac{\theta(t+x)+\theta(t-x)+\dot{\theta}(t+x)-\dot{\theta}(t-x)}{2}, & t \in [1+x, \infty), \end{cases} \qquad (1.12)$$

for all $t > 0$ and $x \in [0,1]$.

The final step consists in reconstructing the solution of initial dynamics by means of the inverse Volterra transformation:

$$z(t,x) = \mathfrak{S} \begin{bmatrix} w(t,\cdot) \\ w_t(t,\cdot) \end{bmatrix}(x) := \cosh(ax)w(t,x) - \int_0^x \begin{bmatrix} l(x,\rho) & 0 \\ 0 & k(x,\rho) \end{bmatrix} \begin{bmatrix} w(t,\rho) \\ w_t(t,\rho) \end{bmatrix} d\rho, \qquad (1.13)$$

where w denotes the solution of the auxiliary system (1.6)–(1.7)–(1.8), and the kernels l and k are suitably designed. This transformation allows one to recover explicitly the original state entirely in terms of the auxiliary dynamics. The subsequent theorem states the second principal result.

Theorem 1.2. *Let $\theta \in C^3([0,\infty))$ with supp $\theta \subseteq [1,\infty)$. The solution of system (1.1)–(1.4) is then*

$$z(t,x) = \begin{cases} 0, & t \in [0,1-x], \\ \frac{1}{2}\mathfrak{S} \begin{bmatrix} \theta(t+\cdot) + \dot{\theta}(t+\cdot) \\ \theta(t+\cdot) + \dot{\theta}(t+\cdot) \end{bmatrix}(x), & t \in [1-x, 1+x], \\ \frac{1}{2}\mathfrak{S} \begin{bmatrix} \theta(t+\cdot) + \theta(t-\cdot) + \dot{\theta}(t+\cdot) - \dot{\theta}(t-\cdot) \\ \dot{\theta}(t+\cdot) + \dot{\theta}(t-\cdot) + \ddot{\theta}(t+\cdot) - \ddot{\theta}(t-\cdot) \end{bmatrix}(x), & t \in [1+x, \infty), \end{cases} \qquad (1.14)$$

for all $t > 0$ and $x \in [0,1]$, where l and k are solutions of PDEs (4.1) and (4.2), respectively.

By evaluating (1.14) at $x = 1$, we get the final main result

Theorem 1.3. *Let $\theta \in C^3([0,\infty))$ with supp $\theta \subseteq [1,\infty)$. The motion-planning control for system (1.1)–(1.4) has the form:*

$$U(t) = \begin{cases} \frac{1}{2}\mathfrak{S}\begin{bmatrix} \theta(t+\cdot)+\dot{\theta}(t+\cdot) \\ \theta(t+\cdot)+\dot{\theta}(t+\cdot) \end{bmatrix}(1), & \mathbf{t} \in [\mathbf{0},\mathbf{2}], \\ \frac{1}{2}\mathfrak{S}\begin{bmatrix} \theta(t+\cdot)+\theta(t-\cdot)+\dot{\theta}(t+\cdot)-\dot{\theta}(t-\cdot) \\ \dot{\theta}(t+\cdot)+\dot{\theta}(t-\cdot)+\ddot{\theta}(t+\cdot)-\ddot{\theta}(t-\cdot) \end{bmatrix}(1), & \mathbf{t} \in [\mathbf{2},\infty), \end{cases} \tag{1.15}$$

where l and k are solutions of PDEs (4.1) and (4.2), respectively.

Formula (1.15) provides an explicit motion–planning control law for system (1.1), formulated using the prescribed trajectory θ and the kernel functions l and k. This main result shows that, for sufficiently regular reference trajectories, the proposed control drives the output of anti-damped wave system along the desired motion, thereby solving the motion planning objective for the original system.

1.4. **Organization.** The subsequent sections are arranged as described below.

Section (2) presents the transformation of system (1.1)–(1.3) into an auxiliary target one via a Volterra integral transformation. The associated kernel functions are shown to meet a system of coupled hyperbolic PDEs posed on a triangular domain. Solvability of the kernel systems is proved through an appropriately designed iterative procedure, yielding existence and regularity results that ensure the validity and invertibility of the transformation.

In Section (3), the Laplace transform in time is used to derive explicit expressions for both the control law and the associated auxiliary state. Based on these results, Theorem 1.1 is established.

The main contributions of this work, namely Theorems 1.2 and 1.3, are demonstrated in the section (4). The inverse transformation (1.13) provides explicit formulas for both the state of system (1.1)–(1.4) and the associated controller, formulated based on a prescribed trajectory θ and the solutions of corresponding kernel PDEs

2. Derivation of kernel equations.

2.1. **Transformation to a target system.** This section outlines the procedure used to construct the kernel functions m and n, which enable the transformation of our original hyperbolic system (1.1)–(1.3) into the auxiliary (1.6)–(1.8). To this end, we apply the transformation (1.5). Assuming that z satisfies the dynamics described by (1.1)–(1.3), a straightforward computation yields

$$\begin{aligned} w_{tt}(t,x) - w_{xx}(t,x) &= \big[2\frac{d}{dx}n(x,x) - 2a\cosh(ax)\big]z_t(t,x) + \big[2an(x,x) - 2a\sinh(ax)\big]z_x(t,x) \\ &\quad + \big[2\frac{d}{dx}m(x,x) - 2a^2\cosh(ax) - 2an_\rho(x,x)\big]z(t,x) \\ &+ \int_0^x \big[m_{xx}(x,\rho) - m_{\rho\rho}(x,\rho) + 2an_{\rho\rho}(x,\rho) + a^2 m(x,y) - 2a^3 n(x,y)\big]z(t,\rho)d\rho \\ &\quad + \int_0^x \big[n_{xx}(x,\rho) - n_{\rho\rho}(x,\rho) + 2am(x,\rho) - 3a^2 n(x,\rho)\big]z_t(t,\rho)d\rho \\ &\quad - n_\rho(x,0)z_t(t,0) + [2an_\rho(x,0) - m_\rho(x,0)]z(t,0) \\ &\quad + n(x,0)z_{tx}(t,0) + [m(x,0) - 2an(x,0)]z_x(t,0). \end{aligned}$$

Consequently, equation (1.6) is valid precisely when the kernel functions m and n solve the system of equations described as follows:

$$m_{xx}(x,\rho) - m_{\rho\rho}(x,\rho) = 2a^3 n(x,\rho) - 2an_{\rho\rho}(x,\rho) - a^2 m(x,\rho), \tag{2.1}$$

$$\frac{d}{dx}m(x,x) = a^2\cosh(ax) + an_\rho(x,x), \tag{2.2}$$

$$m(x,0) = 0, \quad m_\rho(x,0) = 0, \tag{2.3}$$

$$n_{xx}(x,\rho) - n_{\rho\rho}(x,\rho) = 3a^2 n(x,\rho) - 2am(x,\rho), \tag{2.4}$$

$$\frac{d}{dx}n(x,x) = a\cosh(ax), \tag{2.5}$$

$$n(x,x) = \sinh(ax), \tag{2.6}$$

$$n(x,0) = 0, \quad n_\rho(x,0) = 0. \tag{2.7}$$

Our next objective is to derive an explicit expression for the diagonal value $m(x,x)$. From equation (2.2), we deduce

$$n_\rho(x,x) = -a\cosh(ax) + \frac{1}{a}\frac{d}{dx}m(x,x). \tag{2.8}$$

Consequently, determining $m(x,x)$ requires the evaluation of $n_\rho(x,x)$. From relation (2.5), one obtains

$$n_x(x,x) = -n_\rho(x,x) + a\cosh(ax). \tag{2.9}$$

Combining (2.4), (2.8), and the above identity leads to the following sequence of equalities:

$$\begin{aligned} n_{xx}(x,x) - n_{\rho\rho}(x,x) &= 3a^2 n(x,x) - 2am(x,x) \\ [n_x(x,x) - n_\rho(x,x)]' &= 3a^2 n(x,x) - 2am(x,x) \\ \big[-2n_\rho(x,x) + a\cosh(ax)\big]' &= 3a^2 n(x,x) - 2am(x,x), \\ \big[\frac{-2}{a}\frac{d}{dx}m(x,x) + 3a\cosh(ax)\big]' &= 3a^2 n(x,x) - 2am(x,x), \\ \frac{-2}{a}\frac{d^2}{dx^2}m(x,x) + 2am(x,x) &= 3a^2\big[n(x,x) - \sinh(ax)\big], \end{aligned}$$

this computation shows that $m(x,x)$ satisfies the following equation:

$$\begin{cases} \frac{d^2}{dx^2}m(x,x) - a^2 m(x,x) = 0, \\ m(0,0) = 0. \end{cases} \tag{2.10}$$

A solution of (2.10) is therefore given by

$$m(x,x) = \sinh(ax). \tag{2.11}$$

On the other hand, using (1.2) together with the transformation (1.5), one recovers the boundary condition (1.7) under the conditions $m(0,0) = n(0,0) = 0$.

As a result, the system (1.6)-(1.8) holds if and only if the kernel functions m and n are characterized in domain $\Delta := \{(x,\rho) \in \mathbb{R}^2 \mid 0 \le \rho \le x \le 1\}$ by the following PDEs

$$\begin{cases} m_{xx} - m_{\rho\rho} = 2a^3 n - 2a n_{\rho\rho} - a^2 m, \\ m(x,x) = \sinh(ax), \\ m(x,0) = m_\rho(x,0) = 0, \end{cases} \tag{2.12}$$

and

$$\begin{cases} n_{xx} - n_{\rho\rho} = 3a^2 n - 2am, \\ n(x,x) = \sinh(ax), \\ n(x,0) = n_\rho(x,0) = 0. \end{cases} \tag{2.13}$$

2.2. **Existence of the kernels.** In this section, we employ a successive approximation scheme, inspired by the approach in [6, 10, 16], to prove the existence of sufficiently smooth solutions to the kernel equations (2.12) and (2.13) posed on domain Δ. Since uniqueness is not essential for the subsequent construction of the backstepping transformation, its proof is not detailed here.

To prove the existence of m and n, we introduce the characteristic coordinates $\eta = x + \rho$ and $\zeta = x - \rho$, under which the second-order operator satisfies

$$m_{xx} - m_{\rho\rho} = 4\, m_{\eta\zeta}, \qquad n_{xx} - n_{\rho\rho} = 4\, n_{\eta\zeta}.$$

By incorporating the boundary conditions along $x = \rho$ and $\rho = 0$, systems (2.12)–(2.13) may be rewritten as an equivalent system of Volterra integral equations over Δ:

$$m(x,\rho) = \sinh(a\rho) + \int_\rho^x \int_0^\rho \big(2a\, n_{\rho\rho} + a^2 m + 2a^3 n\big)(s,r)\, dr\, ds,$$

$$n(x,\rho) = \sinh(a\rho) + \int_\rho^x \int_0^\rho \big(5a^2 n + 2a\, m\big)(s,r)\, dr\, ds.$$

Let $(m^{(0)}, n^{(0)}) = (0,0)$ and define recursively, for $k \ge 0$,

$$(m^{(k+1)}, n^{(k+1)}) = \mathcal{V}(m^{(k)}, n^{(k)}),$$

where $\mathcal{V}$ denotes the Volterra operators appearing above. Because the integration domain satisfies $0 \le r \le \rho \le s \le x$, standard Volterra estimates imply that $\mathcal{V}$ maps $C^2(\Delta) \times C^2(\Delta)$ into itself and is locally Lipschitz. As a consequence, the sequence $(m^{(k)}, n^{(k)})$ converges uniformly on Δ, together with its derivatives up to second order.

Passing to the limit yields a classical solution (m, n) of (2.12)–(2.13) on Δ. Uniqueness follows from linearity and a Grönwall-type estimate applied to the difference of two solutions. This leads to the following result.

Lemma 2.1. *For any fixed constant a, the coupled kernel systems* (2.12)–(2.13) *admit a unique solution*

$$(m, n) \in C^2(\Delta) \times C^2(\Delta).$$

3. Explicit expressions of $W(t)$ and $w(t, x)$.

This section provides explicit expressions for the control $W(t)$ and the state $w(t, x)$ associated with a given reference trajectory. Since $\mathfrak{A}$ is skew-adjoint on $\mathfrak{X}$, the spectrum of $\mathfrak{A}$ is confined to the imaginary axis, that is, $\sigma(\mathfrak{A}) \subset i\mathbb{R}$, the associated semigroup satisfies the growth bound $\omega_0(e^{\mathfrak{A}\cdot}) \le 0$. Consequently, for any $Z_0 \in \mathcal{X}$, the Laplace transform[3] of $e^{\mathfrak{A}\cdot}Z_0$ exists for each $\lambda > 0$, and $\widehat{e^{\mathfrak{A}\cdot}Z_0}(\lambda) = R(\lambda, \mathfrak{A})Z_0$. Applying the Laplace transform to (1.10) yields

$$\widehat{Z}(\lambda) = [\lambda^2 R(\lambda, \mathfrak{A})\mathfrak{b} - \lambda\mathfrak{b}]\widehat{W}(\lambda), \;\; \lambda > 0. \tag{3.1}$$

For notational convenience, define

$$R(\lambda, \mathfrak{A})\mathfrak{b} = (f, h, \alpha), \quad \lambda > 0. \tag{3.2}$$

Comparing the first components of (3.1) and (3.2) gives

$$\widehat{w}(\lambda, x) = [1 + \lambda^2 f(x)]\widehat{W}(\lambda). \tag{3.3}$$

Observing that $w(t, 0) = z(t, 0) = \theta(t)$, relation (3.3) implies

$$\theta(t) = [1 + \lambda^2 f(0)]\widehat{W}(\lambda). \tag{3.4}$$

In order to determine $f(0)$, we should solve (3.2). Indeed, equation (3.2), which may be written as,

$$(\lambda - \mathfrak{A})(f, h, \alpha) = \mathfrak{b}, \tag{3.5}$$

is equivalent to, in an expanded form,

$$f \in H^2, \;\; g = \lambda f, \;\; g(0) = \alpha,$$
$$\begin{cases} f'' - \lambda^2 f = 1, \quad \text{in } [0, 1], \\ f'(0) - \lambda f(0) = 1, \\ f(1) = 0. \end{cases} \tag{3.6}$$

It is worth noting that the function f, and consequently h, depends on the parameter λ, that is, $f(x) := f(\lambda, x)$. For notational convenience, we write $f(x)$

[3] Let $\mathcal{X}$ be a Hilbert space. For a locally Bochner integrable function $f : \mathbb{R}^+ \to \mathcal{X}$, we denote by $\widehat{f}(\cdot)$ its Laplace transform, defined by

$$\widehat{f}(\zeta) := \int_0^\infty e^{-\zeta t} f(t)\, dt = \lim_{s\to\infty} \int_0^s e^{-\zeta t} f(t)\, dt,$$

whenever the limit exists.

whenever no ambiguity arises. The solution of system (3.6) can be expressed in the following general form:

$$f(x) = Ae^{\lambda x} + Be^{-\lambda x} - \lambda^{-2}, \qquad \lambda > 0 \tag{3.7}$$

in which A and B denote two constants that depend on λ, to be determined. Applying the boundary condition associated with (3.6) yields

$$\begin{bmatrix} e^{\lambda} & e^{-\lambda} \\ \lambda - 1 & \lambda + 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} s^{-2} \\ 0 \end{bmatrix}. \tag{3.8}$$

Let us define $\Delta(\lambda) := (\lambda+1)e^{\lambda} - (\lambda-1)e^{-\lambda} > 0$ for $\lambda > 0$. Then, from system (3.8) we obtain

$$A = \frac{\lambda+1}{\lambda^2\Delta(\lambda)}, \qquad B = \frac{1-\lambda}{\lambda^2\Delta(\lambda)},$$

Evaluating (3.7) at $x = 0$, we get

$$1 + \lambda^2 f(0) = \frac{2}{\lambda^2\Delta(\lambda)}. \tag{3.9}$$

By (3.4) we find,

$$\widehat{W}(\lambda) = \frac{1}{2}\Delta(\lambda)\widehat{\theta}(\lambda). \tag{3.10}$$

Since $\Delta(\lambda) \to \infty$ as $\lambda \to \infty$, $\Delta(\lambda)$ cannot be the Laplace transform of any distribution with support contained in $[0,\infty)$. However, observing that, when $\theta \in C^3([0,\infty))$ with $\operatorname{supp}\theta \subseteq [1,\infty)$, the maps $\lambda \longrightarrow \lambda^2 e^{\lambda}\theta(\lambda)$ and $\lambda \longrightarrow \lambda^2 e^{-\lambda}\theta(\lambda)$ are respectively the Laplace transform of $\ddot{\theta}_1(t)$ and $\ddot{\theta}_2(t)$ where

$$\theta_1(t) := \theta(t+1), \quad t \geq 0, \qquad \theta_2(t) := \begin{cases} 0, & t \in [0,1] \\ \theta(t-1), & t \in [1,\infty). \end{cases}$$

If we define for $t \geq 0$,

$$h_1(t) = 1 + t, \qquad h_2(t) := 1 - t,$$

then

$$\begin{aligned} \widehat{W}(\lambda) =& \frac{\Delta(\lambda)\widehat{\theta}(\lambda)}{2} \\ =& \frac{\left(\lambda^{-1} + \lambda^{-2}\right)\lambda^2 e^{\lambda}\widehat{\theta}(\lambda) - \left(\lambda^{-1} + \lambda^{-2}\right)\lambda^2 e^{-\lambda}\widehat{\theta}(\lambda)}{2} \\ =& \frac{\widehat{h}_1(\lambda)\widehat{\ddot{\theta}}_1(\lambda) - \widehat{h}_2(\lambda)\widehat{\ddot{\theta}}_2(\lambda)}{2} \\ =& \frac{\widehat{h_1 \circ \ddot{\theta}_1}(\lambda) - \widehat{h_2 \circ \ddot{\theta}_2}(\lambda)}{2}, \end{aligned}$$

for all $\lambda > 0$, where the symbol $(\circ)$ stands for the convolution product. Thus, we can choose the control W of the form

$$W(t) = \frac{h_1 \circ \ddot{\theta}_1(t) - h_2 \circ \ddot{\theta}_2(t)}{2} = \frac{1}{2}\int_0^t h_1(t-s)\ddot{\theta}_1(s) - h_2(t-s)\ddot{\theta}_2(s)ds,$$

for all $t \geq 0$. Of course $W(\cdot) \in H^1_{loc}(0,\infty)$. Moreover, an integration by parts leads to the following explicit expression of $W(t)$:

$$W(t) = \begin{cases} \frac{\theta(t+1)+\dot{\theta}(t+1)}{2}, & 0 \leq t \leq 2 \\ \frac{\theta(t+1)+\theta(t-1)+\dot{\theta}(t+1)-\dot{\theta}(t-1)}{2}, & 2 \leq t. \end{cases} \tag{3.11}$$

Now, we derive the explicit expression of $w(t,x)$ at any position (t,x). Indeed, from (3.3), (3.7) and (3.9) we have

$$\begin{aligned}\widehat{w}(\lambda,x) &= [1+\lambda^2 f(x)]\widehat{W}(\lambda) \\ &= \left[(\lambda+1)e^{\lambda x} - (\lambda-1)e^{-\lambda x}\right]\frac{\widehat{W}(\lambda)}{\Delta(\lambda)} \\ &= \frac{(\lambda+1)e^{\lambda x} - (\lambda-1)e^{-\lambda x}}{2}\widehat{\theta}(\lambda)\end{aligned}$$

for all $\lambda > 0$. Let $\theta \in C^3([0,\infty))$ with supp $\theta \subseteq [1,\infty)$ and define for $x \in [0,1]$ the functions

$$\theta_3(t) := \theta(t+x), \quad t \geq 0, \qquad \theta_4(t) := \begin{cases} 0, & t \in [0,1+x] \\ \theta(t-x), & t \in [1+x,\infty). \end{cases}$$

Then

$$\lambda^2 e^{\lambda x}\widehat{\theta}(\lambda) = \widehat{\ddot{\theta}_3}(\lambda) \qquad \text{and} \qquad \lambda^2 e^{-\lambda x}\widehat{\theta}(\lambda) = \widehat{\ddot{\theta}_4}(\lambda).$$

Whence

$$\begin{aligned}\widehat{w}(\lambda,x) &= \frac{(\lambda+1)e^{\lambda x} - (\lambda-1)e^{-\lambda x}}{2}\widehat{\theta}(\lambda) \\ &= \frac{\left(\lambda^{-1}+\lambda^{-2}\right)\lambda^2 e^{\lambda x}\widehat{\theta}(\lambda) - \left(\lambda^{-1}+\lambda^{-2}\right)\lambda^2 e^{-\lambda x}\widehat{\theta}(\lambda)}{2} \\ &= \frac{\widehat{h_1}(\lambda)\widehat{\ddot{\theta}_3}(\lambda) - \widehat{h_2}(\lambda)\widehat{\ddot{\theta}_4}(\lambda)}{2} \\ &= \frac{\widehat{h_1 \circ \ddot{\theta}_3}(\lambda) - \widehat{h_2 \circ \ddot{\theta}_4}(\lambda)}{2}.\end{aligned}$$

Therefore,

$$
\begin{aligned}
w(t,x) &= \frac{h_1 \circ \ddot{\theta}_3(t) - h_2 \circ \ddot{\theta}_4(t)}{2} \\
&= \begin{cases} 0, & t \in [0, 1-x], \\ \frac{1}{2}\int_0^t h_1(t-s)\ddot{\theta}(s+x)ds, & t \in [1-x, 1+x], \\ \frac{1}{2}\int_0^t h_1(t-s)\ddot{\theta}(s+x) - h_2(t-s)\ddot{\theta}(s-x)ds, & t \in [1+x, \infty), \end{cases}
\end{aligned}
$$

By integrating by parts we find the following explicit expression of $w(t,x)$:

$$
w(t,x) = \begin{cases} 0, & t \in [0, 1-x], \\ \frac{\theta(t+x)+\dot{\theta}(t+x)}{2}, & t \in [1-x, 1+x], \\ \frac{\theta(t+x)+\theta(t-x)+\dot{\theta}(t+x)-\dot{\theta}(t-x)}{2}, & t \in [1+x, \infty), \end{cases}
$$

for all $t > 0$ and $x \in [0,1]$. Accordingly, Theorem 1.1 follows.

4. Motion planning problem for system (1.6)–(1.3) (Demonstration of Theorems 1.2 and 1.3)

In this section, we prove Theorems 1.2 and 1.3. More precisely, we derive constructive formulas for the state variable $z(t,x)$ and the boundary input $U(t)$. These representations are expressed in terms of a prescribed reference trajectory and the solutions of the corresponding kernel partial differential equations. The inverse backstepping transformation (1.13) is used to obtain these representations.

Using arguments analogous to those presented in Section 2, the inverse transformation (1.13) maps the auxiliary system (1.6)–(1.8) onto hyperbolic system (1.1)–(1.3). This mapping is well-defined precisely when the kernel functions l and k satisfy the PDEs stated below:

$$
\begin{cases} l_{xx} - l_{\rho\rho} = 2ak_{\rho\rho} + a^2 k, \\ l(x,x) = \frac{a^2}{2\sqrt{b}(a^2+b)}\left[\sqrt{b}\sinh(ax) - a\sinh(\sqrt{b}x)\right], \\ l(x,0) = l_\rho(x,0) = 0, \end{cases} \tag{4.1}
$$

where $b = \frac{a^2+2a}{2}$, and

$$
\begin{cases} k_{xx} - k_{\rho\rho} = (2a+a^2)l, \\ k(x,x) = -\sinh(ax), \\ k(x,0) = k_\rho(x,0) = 0. \end{cases} \tag{4.2}
$$

Therefore, as in section (2), it can be seen that the PDEs (4.1) and (4.2) admit unique solutions $l, k \in C^2(\Delta)$, respectively. Then it results from (1.13) and (1.12) that

$$z(t,x)=\begin{cases}0, & t\in[0,1-x],\\ \frac{1}{2}\mathfrak{S}\begin{bmatrix}\theta(t+\cdot)+\dot{\theta}(t+\cdot)\\ \theta(t+\cdot)+\dot{\theta}(t+\cdot)\end{bmatrix}(x), & t\in[1-x,1+x],\\ \frac{1}{2}\mathfrak{S}\begin{bmatrix}\theta(t+\cdot)+\theta(t-\cdot)+\dot{\theta}(t+\cdot)-\dot{\theta}(t-\cdot)\\ \dot{\theta}(t+\cdot)+\dot{\theta}(t-\cdot)+\ddot{\theta}(t+\cdot)-\ddot{\theta}(t-\cdot)\end{bmatrix}(x), & t\in[1+x,\infty),\end{cases} \tag{4.3}$$

for all $t>0$ and $x\in[0,1]$, where l and k are solutions of PDEs (4.1) and (4.2), respectively. Accordingly, Theorem 1.2 follows. Substituting $x=1$ in (4.3), we get Theorem 1.3

5. Conclusion

This work develops a systematic methodology for addressing the motion planning problem associated with the hyperbolic system (1.1)–(1.3). The original PDE model is first mapped onto an auxiliary target system via the direct backstepping transformation (1.5). The resulting system is then analyzed within a semigroup framework, which enables the use of Laplace transform techniques to derive explicit expressions for both the solution of the auxiliary system (1.6)–(1.8) and the associated control input. Finally, the inverse transformation (1.13) is applied to reconstruct both the state trajectory and the corresponding boundary input of the original model (1.1)–(1.3), thereby achieving the proposed motion planning objective.

Acknowledgement. The author sincerely thanks the referees for their thorough efforts and for their constructive remarks and recommendations.

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