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OPTIMAL QUADRATURE RULE WITH DERIVATIVE IN SOBOLEV SPACE

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Abstract. As a result of extensive scientific research conducted on a global scale, approximating exact integrals and integral equations in the problems of the mechanics of liquids and gases with high accuracy leads to the construction of optimal quadrature formulas. Use of simple interpolation quadrature formulas in solving such problems requires a large amount of computational work. The effectiveness of quadrature formulas is characterized by their degree of accuracy and order of accuracy. In this work, the optimal quadrature formula with derivative is constructed using the first and second order derivatives of the function at the nodes in the real-valued Sobolev space.

Keywords. Sobolev space, quadrature formulas, extremal function, norm of the error functional, Wandermonde determinant, optimal coefficients.
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1. Introduction

Numerical analysis is a branch of mathematics and mechanics that deals with the development of effective methods for obtaining numerical solutions to complex problems. Due to the development of computing technology, numerical problem solving is developing. Constructing optimal quadrature formulas with derivatives is one of the current problems in computational mathematics. They are used to approximate definite integrals to finite sums. The effectiveness of quadrature formulas is characterized by their degree of accuracy and order of accuracy. Such formulas can be obtained using variational principles. The problem of optimization of quadrature formulas using the variational approach involves minimizing the norm of the error functional in the given space of functions [18, 19].

Among the quadrature formulas of the following form:

$$\int\limits_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N}\sum_{j=0}^{\alpha} K_{\beta j}\varphi^{(j)}(x_\beta)$$

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In [8], it is proved that, in the Sobolev space $W_2^{(m)}$, the quadrature formula under consideration is optimal for odd values of m, Jensikbayev [20] showed that the Euler-Maklauren formula is optimal in $L_2^{(m)}(0,1)$ space, Shadimetov [13] at work for $\alpha = 0$ optimal quadrature formula is constructed in $L_2^{(m)}(0,1)$ space and error functional norm calculated.

In the work of [1] $2n-1$ degree, all algebraic polynomials have been studied using the following Gaussian-type formula.

$$\int\limits_a^b f(t)dt \cong \sum_{k=1}^{n} a_k \frac{1}{h_k} \int\limits_{x_k}^{x_k+h_k} f(t)dt.$$

In [2] the authors aim to optimally express the integration of rational functions through nodes and weights, thereby obtaining the best constant estimates between norms. In [4], singular integrals approx count for derivative quadrature formulas received, and the Riemann Zeta function derivatives count for an efficient algorithm given. Paikaray et al. [12] introduce a numerical technique using Walsh function approximation and the corresponding operational matrix of integration to address the linear stochastic Volterra integral equation. Nasiri et al. [10] applied the homotopy perturbation method (HPM) and variational iteration method (VIM) for solving fractional Fredholm and Volterra integro differential equation.

In this work, the following derivative quadrature formula is constructed

$$\int\limits_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N} C_0\,[\beta]\varphi[\beta] + \frac{h^2}{12}(\varphi'(0) - \varphi'(1)) + \sum_{\beta=0}^{N} C_1\,[\beta]\varphi''[\beta] \tag{1.1}$$

with error functional

$$\ell_N(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_0\,[\beta]\delta\,(x-[\beta]) + \frac{h^2}{12}(\delta'(x) - \delta'(x-1)) - \sum_{\beta=0}^{N} C_1\,[\beta]\delta''\,(x-[\beta]) \tag{1.2}$$

where $\varepsilon_{[0,1]}(x)$- [0,1] is the characteristic function of the interval, $\delta(x)$- Dirac's delta function, $C_1[\beta]$, $\beta = \overline{0,N}$ (1) the unknown coefficients of the formula, $[\beta] = h\beta$, $h = \frac{1}{N}$, $\varphi \in L_2^{(m)}(0,1)$. $L_2^{(m)}(0,1)$ in space of the function norm as below determined

$$\|\varphi\|_{L_2^{(m)}} = \left\{ \int\limits_0^1 \left(\frac{d^m(\varphi)}{dx^m} \right)^2 dx \right\}^{\frac{1}{2}}$$

$\ell_N(x)$ error functional $L_2^{(m)*}(0,1)$ since it is defined in the conjunction space, it satisfies the following conditions[18]

$$(\ell_N, x^\alpha) = 0, \quad \alpha = 0, 1, 2, ..., m-1. \tag{1.3}$$

From the definition of the norm of a functional

$$\left\|\ell_N|L_2^{(m)*}(0,1)\right\| = \sup_{\left\|\varphi|L_2^{(m)}(0,1)\right\|\neq 0} \frac{|(\ell_N,\varphi)|}{\left\|\varphi|L_2^{(m)}(0,1)\right\|}$$

We have the Cauchy-Schwarz inequality

$$|(\ell_N,\varphi| \leq \| \varphi|L_2^{(m)}(0,1) \| \cdot \| \ell_N|L_2^{(m)*}(0,1) \|$$

It can be seen from this inequality that (1) the error of the quadrature formula is from above $\ell_N(x)$ is evaluated by the norm of the error functional. We minimize the norm of the error function only by the coefficients when the nodes are fixed.

$$\left\|\overset{\circ}{\ell}_N|L_2^{(m)*}\right\| = \inf_{C_1[\beta]} \left\|\ell_N|L_2^{(m)*}\right\| \tag{1.4}$$

that satisfy relation (1.4) $C_1[\beta]$ are called optimal coefficients [19, 14, 15].

Theorem 1.1. *In $L_2^{(m)}(0,1)$ space for $m \geq 3$ the coefficients of the optimal quadrature formula in the form (1.1) are follows*

$$C_1[0] = h^3 \sum_{k=1}^{m-3} a_k \frac{q_k^N - q_k}{1-q_k}, \qquad \beta = 0, N,$$

$$C_1[\beta] = h^3 \sum_{k=1}^{m-3} a_k \left(q_k^\beta + q_k^{N-\beta}\right), \qquad \beta = \overline{1, N-1}$$

here $a_k = \frac{\Delta_{k,m-3}(P)}{\Delta_{m-3}}$, $\Delta_{m-3} = \prod\limits_{i=1}^{m-3} \frac{q_i}{(1-q_i)^2}\left[W(-x_1,-x_2,...,-x_{m-3})\right.$

$$\left.+\sum_{j=1}^{m-3} \sum_{1\leq i_1<i_2<...<i_j\leq m-3} \left(q_{i_1}q_{i_2}\cdots q_{i_j}\right)^N W\left(-x_1,..,q_{i_1}x_{i_1},..,q_{i_j}x_{i_j},..,-x_{m-3}\right)\right],$$

$$\Delta_{k,m-3}(P) = \prod_{\substack{l=1\\ l\neq k}} \frac{q_l}{(1-q_l)^2} \sum_{i=1}^{m-3} (-1)^{i+k} P_i \left[W_{k,i}(-x_1,-x_2,...,-x_{m-3})\right.$$

$$\left.+\sum_{j=1}^{m-3} \sum_{\substack{1\leq i_1<i_2<...<i_j\leq m-3\\ i_j\neq k}} (q_{i_1}q_{i_2}\cdots q_{i_j})^N W_{k,i}\left(-x_1,..,x_{i_1}q_{i_1},..,x_{i_j}q_{i_j},..,-x_{m-3}\right)\right],$$

$$P_\alpha = \frac{1}{\alpha!}\left[\frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)} - \sum_{i=1}^{\alpha-1} P_i\,\Delta^i 0^\alpha\right], \quad \alpha = \overline{1, m-3}, \quad x_i = \frac{1}{1-q_i},$$

$W_{k,i}(-x_1,-x_2,...,-x_{m-3})$

$$= \sum_{\substack{1\leq j_1<j_2<...<j_i\leq m-3,\\ j_i\neq k}} x_{j_1}x_{j_2}\cdots x_{j_i} \cdot W(x_1,x_2,...,x_{k-1},x_{k+1},...,x_{m-3}).$$

The article is organized as follows: In the second part, certain definitions and theorems are given to prove theorems. The proof of the main theorem of the article is presented in the third section. The theoretical results obtained in the fourth section are numerically analysed. The conclusion is presented in the fifth section.

2. Preliminaries

In this section, we present certain definitions and formulas that will be needed in our proof of the main results obtained. Suppose that φ and ψ are functions of real variables and are $\mathbb{R}$ defined in space.

Definition 2.1. Function $\varphi[\beta]$ is called a function of discrete argument, if it is given on some set of integer values of β [18, 19].

Definition 2.2. This is the inner product of two discrete $\varphi[\beta]$ and functions $\psi[\beta]$

$$[\varphi, \psi] = \sum_{\beta=-\infty}^{\infty} \varphi[\beta] \cdot \psi[\beta]$$

is said to the expression defined by the equality.

Definition 2.3. $\varphi[\beta]$ and $\psi[\beta]$ the convolution of functions with discrete arguments is defined as follows

$$\varphi[\beta] * \psi[\beta] = \sum_{\gamma=-\infty}^{\infty} \varphi[\gamma] \cdot \psi\left([\beta] - [\gamma]\right).$$

The following formula is valid [7]

$$\sum_{\gamma=0}^{n-1} q^{\gamma}\gamma^k = \frac{1}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i \Delta^i 0^k - \frac{q^n}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^i \Delta^i \gamma^k|_{\gamma=n}, \tag{2.1}$$

where $\Delta^i\gamma^k$ is the finite difference of order i of γ^k, q is ratio of a geometric progression, $\Delta^i 0^k = \sum\limits_{\ell=1}^{i} (-1)^{i-\ell} C_i^{\ell} \ell^k$, $\Delta^{\alpha} x^k = \sum\limits_{p=0}^{k} C_k^p \Delta^{\alpha} 0^p x^{k-p}$.

In calculating some sums[5]

$$\sum_{\gamma=0}^{\beta-1} \gamma^k = \sum_{j=1}^{k+1} \frac{k! B_{k+1-j}}{j!(k+1-j)!}\beta^j, \tag{2.2}$$

where B_{k+1-j} are Bernoulli numbers.

3. Proof of Theorem 1.1.

In this section, we present the proof of the (1.1).

Proof. For proof (1.1), we need to solve the following problems in sequence:
a) find an overview of the norm of the error functional;
b) obtain a system of equations of the Wiener-Hopf type;
c) find optimal coefficients by solving a system of equations.

We use the concept of an extremal function to find a general representation of the norm of the error functional. In $L_2^{(m)}(0,1)$ space $\ell_N(x)$ of the error functional $U_\ell(x)$ the extremal function has the form[18]

$$U_\ell(x) = (-1)^m \ell_N(x) * G_m(x) + P_{m-1}(x), \tag{3.1}$$

where $G_m(x) = \frac{|x|^{2m-1}}{2\cdot(2m-1)!}$, $P_{m-1}(x)$ is a algebraic polynomial of degree $(m-1)$.

$$\left\| \ell_N \left| L_2^{(m)^*}(0,1) \right\|^2 = (\ell_N, U_\ell) = \int_{-\infty}^{+\infty} \ell_N(x) U_\ell(x) dx$$

$$= (-1)^m \left[\sum_{\beta=0}^{N} C_1[\beta] \sum_{\gamma=0}^{N} C_1[\gamma] \frac{|[\beta]-[\gamma]|^{2m-5}}{2(2m-5)!} - 2\sum_{\beta=0}^{N} C_1[\beta] \int_0^1 \frac{|x-[\beta]|^{2m-3}}{2(2m-3)!} dx \right.$$

$$- \frac{h^2}{6} \sum_{\beta=0}^{N} C_1[\beta] \left[\frac{[\beta]^{2m-4} + [\beta-1]^{2m-4}}{2(2m-4)!} \right] + 2\sum_{\beta=0}^{N} C_0[\beta] \sum_{\gamma=0}^{N} C_1[\gamma] \frac{|[\beta]-[\gamma]|^{2m-3}}{2(2m-3)!}$$

$$- \frac{h^2}{6} \sum_{\beta=0}^{N} C_0[\beta] \left[\frac{[\beta]^{2m-2} + ([\beta]-1)^{2m-2}}{2(2m-2)!} \right] - 2\sum_{\beta=0}^{N} C_0[\beta] \int_0^1 \frac{|x-[\beta]|^{2m-1}}{2(2m-1)!} dx$$

$$\left. + \sum_{\beta=0}^{N} C_0[\beta] \sum_{\gamma=0}^{N} C_0[\gamma] \frac{|[\beta]-[\gamma]|^{2m-1}}{2(2m-1)!} + \frac{h^2}{6(2m-1)!} + \frac{1}{(2m+1)!} + \frac{h^4}{144(2m-3)!} \right]. \tag{3.2}$$

We find the minimum of equality (3.2) by coefficients $C_1[\beta]$, $(\beta = 0, 1, ..., N)$ under conditions(1.3). For this, using the method of Lagrange unknown multipliers to find the conditional extremum of multivariable functions, we get the following system of equations

$$\sum_{\gamma=0}^{N} C_1[\gamma] G_{m-2}([\beta]-[\gamma]) + P_{m-3}[\beta] = F_m[\beta], \qquad \beta = \overline{0, N} \tag{3.3}$$

$$\sum_{\gamma=0}^{N} C_1[\gamma][\gamma]^\alpha = -\sum_{j=1}^{\alpha} \frac{\alpha! B_{\alpha+3-j} h^{\alpha+3-j}}{j!(\alpha+3-j)!}, \qquad \alpha = \overline{0, m-3} \tag{3.4}$$

here

$$F_m[\beta] = \sum_{i=0}^{2m-5} \frac{[\beta]^{2m-5-i}}{(2m-5-i)!} \left[-\frac{B_{i+3} h^{i+3}}{(i+3)!} + \sum_{j=1}^{i} \frac{(-1)^i B_{i+3-j} h^{i+3-j}}{2j!(i+3-j)!} \right] + \frac{B_{2m-2} h^{2m-2}}{(2m-2)!} \tag{3.5}$$

$G_{m-2}(x) = \frac{|x|^{2m-5}}{2(2m-5)!}$, B_{i+3} - Bernoulli numbers.

(3.3)-(3.4) are system of discrete equations of the Wiener-Hopf type. This system has a unique solution, and this solution gives a minimum value to $\| \ell_N \|^2$

(the square of the norm of the error function)[13]. To solve this system, we use the approach based on the d^{2m-2}/dx^{2m-2} discrete analogue of the $D_{m-2}[\beta]$ differential operator[16]. For this, we rewrite the equation (3.3) use **Definition 2.3**, taking $C_1[\beta] = 0$ for $\beta < 0$ and $\beta > N$

$$G_{m-2}[\beta] * C_1[\beta] + P_{m-3}[\beta] = F_m[\beta], \qquad \beta = 0, 1, ..., N \tag{3.6}$$

and following the substitution:

$$v[\beta] = G_{m-2}[\beta] * C_1[\beta], \tag{3.7}$$

$$u[\beta] = v[\beta] + P_{m-3}[\beta] \tag{3.8}$$

$C_1[\beta]$ coefficients can be found by applying $D_{m-2}[\beta]$ discrete analogues to both sides of the equation (3.8)

$$C_1[\beta] = hD_{m-2}[\beta] * u[\beta]. \tag{3.9}$$

To calculate the convolution of (3.9), we need to find the representation of the function $u[\beta]$ for all integer values of β. If $[\beta] \in [0, 1]$, then $u[\beta] = F_m[\beta]$. We need to find $u[\beta]$, which intervals in $\beta < 0$ and $\beta > N$.

$\beta < 0$

$$v[\beta] = C_1[\beta] * G_{m-2}[\beta] = C_1[\beta] * \frac{|[\beta]|^{2m-5}}{2(2m-5)!} = \sum_{\gamma=0}^{N} C_1[\gamma] \frac{|[\beta]-[\gamma]|^{2m-5}}{2(2m-5)!}$$

$$= \sum_{i=0}^{m-3} \frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)! \cdot i!} \sum_{j=1}^{i} \frac{i! B_{i+3-j} h^{i+3-j}}{j!(i+3-j)!} - \sum_{i=m-2}^{2m-5} \frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)! \cdot i!} \sum_{\gamma=0}^{N} C_1[\gamma][\gamma]^i,$$

$\beta > N$

$$v[\beta] = -\sum_{i=0}^{m-3} \frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)! \cdot i!} \sum_{j=1}^{i} \frac{i! B_{i+3-j} h^{i+3-j}}{j!(i+3-j)!} + \sum_{i=m-2}^{2m-5} \frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)! \cdot i!} \sum_{\gamma=0}^{N} C_1[\gamma][\gamma]^i$$

We introduce the following designations

$$R_{2m-5}[\beta] = \sum_{i=0}^{m-3} \frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)! \cdot i!} \sum_{j=1}^{i} \frac{i! B_{i+3-j} h^{i+3-j}}{j!(i+3-j)!},$$

$$Q_{m-3}[\beta] = \sum_{i=m-2}^{2m-5} \frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)! \cdot i!} \sum_{\gamma=0}^{N} C_1[\gamma][\gamma]^i, \tag{3.10}$$

after these designations

$$v[\beta] = \begin{cases} R_{2m-5}[\beta] - Q_{m-3}[\beta], & \beta < 0, \\ -R_{2m-5}[\beta] + Q_{m-3}[\beta], & \beta > N. \end{cases} \tag{3.11}$$

So,

$$u[\beta] = \begin{cases} R_{2m-5}[\beta] + Q^-_{m-3}[\beta], & \beta < 0, \\ F_m[\beta], & 0 \le \beta \le N, \\ -R_{2m-5}[\beta] + Q^+_{m-3}[\beta], & \beta < 0 \end{cases} \tag{3.12}$$

where

$$Q^-_{m-3}[\beta] = P_{m-3}[\beta] - Q_{m-3}[\beta], \tag{3.13}$$

$$Q^+_{m-3}[\beta] = P_{m-3}[\beta] + Q_{m-3}[\beta] \tag{3.14}$$

$Q^{-}_{m-3}[\beta]$ and $Q^{+}_{m-3}[\beta]$ are unknown polynomials of degree $(m-3)$. If we find $Q^{-}_{m-3}[\beta]$ and $Q^{+}_{m-3}[\beta]$, then from (3.13)-(3.14) we get

$$P_{m-3}[\beta] = \frac{1}{2}\left(Q^{+}_{m-3}[\beta] + Q^{-}_{m-3}[\beta]\right),$$
$$Q_{m-3}[\beta] = \frac{1}{2}\left(Q^{+}_{m-3}[\beta] - Q^{(-)}_{m-3}[\beta]\right).$$

We find $C_1[\beta]$ optimal coefficients when $\beta = 1, 2, ..., N-1$ using the exact form of functions with discrete arguments $D_{m-2}[\beta]$ and $u[\beta]$. For this, we introduce the following designations

$$a_k = \frac{(2m-5)!(1-q_k)^{2m-3}}{h^{2m-6} q_k E_{2m-5}(q_k)} \sum_{\gamma=1}^{\infty} q_k^{\gamma}\left(R_{2m-5}[-\gamma] + Q^{-}_{m-3}[-\gamma] - F_m[-\gamma]\right), \tag{3.15}$$

$$b_k = \frac{(2m-5)!(1-q_k)^{2m-3}}{h^{2m-6} q_k E_{2m-5}(q_k)} \sum_{\gamma=1}^{\infty} q_k^{\gamma}\left[-R_{2m-5}\left(1+[\gamma]\right) + Q^{+}_{m-3}\left(1+[\gamma]\right) - F_m\left(1+[\gamma]\right)\right]. \tag{3.16}$$

Theorem 3.1. *$C_1[\beta]$, $\beta = 1, 2, ..., N-1$ coefficients of the optimal quadrature formula of the form (1.1) for $m \geq 3$ in $L_2^{(m)}(0,1)$ space are as follows*

$$C_1[\beta] = h^3 \sum_{k=1}^{m-3}\left(a_k q_k^{\beta} + b_k q_k^{N-\beta}\right), \qquad \beta = 1, 2, ..., N-1 \tag{3.17}$$

where a_k, b_k are determined by (3.15)-(3.16).

Proof. When $\beta = 1, 2, ..., N-1$, using (3.5), (3.8), (3.9) and $D_{m-2}[\beta]$, we can write the following equation for $C_1[\beta]$

$$C_1[\beta] = hD_{m-2}[\beta] * u[\beta] = h\left\{D_{m-2}[\beta] * F_m[\beta]\right.$$
$$+ \sum_{k=1}^{m-3} q_k^{\beta} \frac{(2m-5)!}{h^{2m-4}} \frac{(1-q_k)^{2m-3}}{q_k E_{2m-5}(q_k)} \sum_{\gamma=1}^{\infty} q_k^{\gamma}\left(R_{2m-5}[-\gamma] + Q^{-}_{m-3}[-\gamma] - F_m[-\gamma]\right)$$
$$+ \sum_{k=1}^{m-3} q_k^{N-\beta} \frac{(2m-5)!}{h^{2m-4}} \frac{(1-q_k)^{2m-3}}{q_k E_{2m-5}(q_k)} \sum_{\gamma=1}^{\infty} q_k^{\gamma}\left(-R_{2m-5}\left(1+[\gamma]\right) + Q^{+}_{m-3}\left(1+[\gamma]\right)\right)$$
$$\left. - F_m\left(1+[\gamma]\right)\right\} = h^3 \sum_{k=1}^{m-3}\left(a_k q_k^{\beta} + b_k q_k^{N-\beta}\right).$$

□

Therefore, $C_1[\beta]$ $\beta = \overline{1, N-1}$ optimal coefficients depend on a_k and b_k unknowns, and $2m-6$ equations are needed to determine them. First, we find the expression of coefficients $C_1[0]$ and $C_1[N]$ from equation (3.4)

$$C_1[0] = \sum_{\beta=1}^{N-1} C_1[\beta][\beta] - \sum_{\beta=1}^{N-1} C_1[\beta], \tag{3.18}$$

$$C_1[N] = -\sum_{\beta=1}^{N-1} C_1[\beta][\beta]. \tag{3.19}$$

We calculate the following sum from (3.3)

$$S=\sum_{\gamma=0}^{N}C_1[\gamma]G_{m-2}\left([\beta]-[\gamma]\right)=\sum_{\gamma=0}^{N}C_1[\gamma]\frac{|[\beta]-[\gamma]|^{2m-5}}{2(2m-5)!}$$

$$=\sum_{\gamma=0}^{\beta}C_1[\gamma]\frac{([\beta]-[\gamma])^{2m-5}}{(2m-5)!}-\sum_{\gamma=0}^{N}C_1[\gamma]\frac{([\beta]-[\gamma])^{2m-5}}{2(2m-5)!},$$

from equations (2.1), (3.4), (3.17) and taking into account that q_k is the root of the Euler-Frobenius polynomial of degree $(2m-6)$, we get the following equation for S

$$S=\frac{[\beta]^{2m-5}}{(2m-5)!}C_1[0]-\sum_{i=0}^{2m-5}\frac{[\beta]^{2m-5-i}h^{i+3}}{(2m-5-i)!\cdot i!}\left[\sum_{k=1}^{m-3}a_k\frac{q_k}{q_k-1}\sum_{\alpha=0}^{i}\left(\frac{1}{q_k-1}\right)^{\alpha}\Delta^{\alpha}0^{i}\right.$$

$$\left.+\sum_{k=1}^{m-3}p_k\frac{q_k^N}{1-q_k}\sum_{\alpha=0}^{i}\left(\frac{q_k}{1-q_k}\right)^{\alpha}\Delta^{\alpha}0^{i}\right]+\sum_{i=0}^{m-3}\frac{[\beta]^{2m-5-i}(-1)^{i}}{2(2m-5-i)!\cdot i!}\sum_{j=1}^{i}\frac{i!B_{i+3-j}h^{i+3-j}}{j!(i+3-j)!}$$

$$-\sum_{i=m-2}^{2m-5}\frac{[\beta]^{2m-5-i}(-1)^{i}}{2(2m-5-i)!\cdot i!}\sum_{\gamma=0}^{N}C_1[\gamma][\gamma]^{i}. \tag{3.20}$$

Substituting the expression (3.20) into (3.3), we equate the corresponding of degree $[\beta]^{2m-5}$

$$C_1[0]=h^3\sum_{k=1}^{m-3}\frac{a_kq_k^N-b_kq_k}{1-q_k}, \tag{3.21}$$

$$\sum_{k=1}^{m-3}\sum_{i=1}^{\alpha}\frac{a_kq_k+b_kq_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^{i}0^{\alpha}=\frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)}. \tag{3.22}$$

We find $C_1\left[N\right]$ from equation (3.4), using (3.17) and (3.18)

$$C_1[N]=h^3\sum_{k=1}^{m-3}\frac{a_kq_k^N-b_kq_k}{1-q_k}.$$

We write the remainder of (3.4)

$$\sum_{\beta=0}^{N}C_1\left[\beta\right]\left[\beta\right]^{\alpha}=-\sum_{j=1}^{\alpha}\frac{\alpha!B_{\alpha+3-j}h^{\alpha+3-j}}{j!(\alpha+3-j)!},\qquad \alpha=\overline{1,\,m-3}, \tag{3.23}$$

using (2.1) and (3.17) for the left side of (3.23), we get the following equality

$$\sum_{\beta=0}^{N}C_1[\beta][\beta]^{\alpha}=\sum_{k=1}^{m-3}\sum_{\beta=1}^{N-1}h^3\left(a_kq_k^{\beta}+b_kq_k^{N-\beta}\right)[\beta]^{\alpha}+C_1\left[N\right]$$

$$=h^3\sum_{k=1}^{m-3}\left[a_k\sum_{\beta=1}^{N-1}q_k^{\beta}\beta^{\alpha}+b_k\sum_{\beta=1}^{N-1}q_k^{N-\beta}\beta^{\alpha}\right]\cdot h^{\alpha}+C_1\left[N\right]$$

$$=h^{\alpha+3}\sum_{k=1}^{m-2}\sum_{i=0}^{\alpha}\frac{a_kq_k^{i}+b_kq_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^{i}0^{\alpha}$$

$$-\sum_{t=0}^{\alpha}\frac{\alpha!h^{t+3}}{t!(\alpha-t)!}\sum_{k=1}^{m-3}\sum_{i=0}^{\alpha}\frac{a_kq_k^{N+i}+(-1)^{i+1}b_kq_k}{(1-q_k)^{i+1}}\Delta^{i}0^{t}+C_1[N]. \tag{3.24}$$

Substituting (3.24) into (3.23) we equate the corresponding degreess of $h^{\alpha+3}$

$$\sum_{k=1}^{m-3}\sum_{i=0}^{\alpha}\frac{a_kq_k^{N+i}+(-1)^{i+1}b_kq_k}{(1-q_k)^{i+1}}\Delta^i0^\alpha=\sum_{k=1}^{m-3}\sum_{i=0}^{\alpha}\frac{a_kq_k^{i}+(-1)^{i+1}b_kq_k^{N+1}}{(1-q_k)^{i+1}}\Delta^i0^\alpha,\ \alpha=\overline{1,m-3} \tag{3.25}$$

$$\sum_{k=1}^{m-3}\sum_{i=0}^{j}\frac{a_kq_k^{N+i}+(-1)^{i+1}b_kq_k}{(1-q_k)^{i+1}}\Delta^i0^j=\frac{B_{j+3}}{(j+1)(j+2)(j+3)},\quad j=\overline{1,\alpha-1},\ \alpha=\overline{1,m-3} \tag{3.26}$$

From (3.17), (3.22) and (3.25) we get the following new system of equations for a_k and b_k unknowns

$$\sum_{k=1}^{m-3}\sum_{i=0}^{\alpha}\frac{a_kq_k^{N+i}+(-1)^{i+1}b_kq_k}{(1-q_k)^{i+1}}\Delta^i0^\alpha=(-1)^{\alpha+1}\frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)}=0,\qquad \alpha=\overline{1,m-3} \tag{3.27}$$

Taking into account that $B_{2N+1}=0$, subtracting (3.27) from (3.22) gives $a_k=b_k$.

$$\sum_{k=1}^{m-3}a_k\sum_{i=1}^{\alpha}\frac{q_k^{N+i}+(-1)^{i+1}q_k}{(1-q_k)^{i+1}}\Delta^i0^\alpha=\frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)},\quad \alpha=\overline{1,m-3}. \tag{3.28}$$

We solve the system (3.28) using Cramer's method. For this purpose, we express it in the following form

$$\sum_{k=1}^{m-3}a_k\frac{q_k^{N+\alpha}+(-1)^{\alpha+1}q_k}{(1-q_k)^{\alpha+1}}=P_\alpha, \tag{3.29}$$

$$P_\alpha=\frac{1}{\alpha!}\left[\frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)}-\sum_{i=1}^{\alpha-1}P_i\,\Delta^i0^\alpha\right],\quad \alpha=\overline{1,m-3} \tag{3.30}$$

$$\Delta_{m-3}=\begin{vmatrix}\frac{q_1^{N+1}+q_1}{(1-q_1)^2} & \frac{q_2^{N+1}+q_2}{(1-q_2)^2} & \dots & \frac{q_{m-3}^{N+1}+q_{m-3}}{(1-q_{m-3})^2}\\ \frac{q_1^{N+2}-q_1}{(1-q_1)^3} & \frac{q_2^{N+2}-q_2}{(1-q_2)^3} & \dots & \frac{q_{m-3}^{N+2}-q_{m-3}}{(1-q_{m-3})^3}\\ \dots & \dots & \dots & \dots\\ \frac{q_1^{N+m-3}+(-1)^{m-2}q_1}{(1-q_1)^{m-2}} & \frac{q_2^{N+m-3}+(-1)^{m-2}q_2}{(1-q_2)^{m-2}} & \dots & \frac{q_{m-3}^{N+m-3}+(-1)^{m-2}q_{m-3}}{(1-q_{m-3})^{m-2}}\end{vmatrix} \tag{3.31}$$

$$\Delta_{k,m-3}(P)=\begin{vmatrix}\frac{q_1^{N+1}+q_1}{(1-q_1)^2} & P_1 & \dots & \frac{q_{m-3}^{N+1}+q_{m-3}}{(1-q_{m-3})^2}\\ \frac{q_1^{N+2}-q_1}{(1-q_1)^3} & P_2 & \dots & \frac{q_{m-3}^{N+2}-q_{m-3}}{(1-q_{m-3})^3}\\ \dots & \dots & \dots & \dots\\ \frac{q_1^{N+m-3}+(-1)^{m-2}q_1}{(1-q_1)^{m-2}} & P_{m-1} & \dots & \frac{q_{m-3}^{N+m-3}+(-1)^{m-2}q_{m-3}}{(1-q_{m-3})^{m-2}}\end{vmatrix} \tag{3.32}$$

We write (3.31) using the additivity property of the determinant

$$\Delta_{m-3}=\prod_{i=1}^{m-3}\frac{q_i}{(1-q_i)^2}\left[W\left(-x_1,-x_2,...,-x_{m-3}\right)\right.$$

$$\left.+\sum_{j=1}^{m-3}\sum_{1\le i_1<...<i_j\le m-3}\left(q_{i_1}q_{i_2}\cdot q_{i_j}\right)^NW\left(-x_1,.,q_{i_1}x_{i_1},.,q_{i_j}x_{i_j},.,-x_{m-3}\right)\right] \tag{3.33}$$

here $x_i = \frac{1}{1-q_i}$. From (3.33) it can be obtained that it is $\Delta_{m-3} \neq 0$. So, the system of equations (3.29) has a unique solution. We also write $\Delta_{k,m-3}$ using the above method

$$\Delta_{k,m-3}(P) = \prod_{\substack{l=1\\ l\neq k}} \frac{q_l}{(1-q_l)^2} \sum_{i=1}^{m-3} (-1)^{i+k} P_i \left[W_{k,i}\left(-x_1,-x_2,...,-x_{m-3}\right) \right.$$

$$\left. + \sum_{j=1}^{m-3} \sum_{\substack{1\leq i_j \leq m-3\\ i_j\neq k}} \left(q_{i_1}\cdot q_{i_j}\right)^N W_{k,i}\left(-x_1,.,x_{i_1}q_{i_1},.,x_{i_j}q_{i_j},.,-x_{m-3}\right)\right] \tag{3.34}$$

where

$$W_{k,i}\left(-x_1,-x_2,...,-x_{m-3}\right) = \begin{vmatrix} 1 & 1 & ... & 1 & 1 & ... & 1 \\ -x_1 & -x_2 & ... & -x_{k-1} & -x_{k+1} & ... & -x_{m-3} \\ ... & ... & ... & ... & ... & ... & ... \\ -x_1^{i-1} & -x_2^{i-1} & ... & -x_{k-1}^{i-1} & -x_{k+1}^{i-1} & ... & -x_{m-3}^{i-1} \\ -x_1^{i+1} & -x_2^{i+1} & ... & -x_{k-1}^{i+1} & -x_{k+1}^{i+1} & ... & -x_{m-3}^{i+1} \\ ... & ... & ... & ... & ... & ... & ... \\ -x_1^{m-4} & -x_2^{m-4} & ... & -x_{k-1}^{m-4} & -x_{k+1}^{m-4} & ... & -x_{m-3}^{m-4} \end{vmatrix}$$

$$= \sum_{\substack{1\leq j_1<j_2<...<j_i\leq m-3,\\ j_i\neq k}} x_{j_1}x_{j_2}\cdot x_{j_i}\cdot W\left(x_1,x_2,.,x_{k-1},x_{k+1},.,x_{m-3}\right). \tag{3.35}$$

Using (3.33) and (3.34), we find the solution of the system of equations (3.29)

$$a_k = \frac{\Delta_{k,m-3}(P)}{\Delta_{m-3}}, \qquad k = \overline{1, m-3}.$$

Theorem 1.1. is completely proved. □

4. Numerical results

We numerically analyze the analytical results obtained in this section and compare them with other works. Using optimal quadrature formulas, the definite integral of rapidly oscillating functions. In $L_2^{(m)}(0,1)$ in Sobolev space using functions $\varphi(x) = \sin x + e^x$ and $f(x) = \ln(1+x) - e^{-x}$, when $N = 10;\ 50;\ 100;\ 1\,50$, we analyze the quadrature formula error (E_1) constructed in works [17] and the quadrature formula error (E_2) constructed in this work in the tables below. $I_1 = \int\limits_0^1 \varphi(x)dx$ and $I_2 = \int\limits_0^1 f(x)dx$

Table 1. Error between the exact and numerical solution of $\varphi(x)$

N	I_1	$I_1 - E_1$	$I_1 - E_2$
10	2.177979522	$2.566524 * 10^{-8}$	$1.822356 * 10^{-8}$
50	2.177979522	$5.62759 * 10^{-10}$	$5.864527 * 10^{-12}$
100	2.177979522	$5.767234 * 10^{-12}$	$2.507102 * 10^{-13}$
150	2.177979522	$5.922840 * 10^{-13}$	$1.002754 * 10^{-14}$

Table 2. Error between the exact and numerical solution of $f(x)$

N	I_2	I_2-E_1	I_2-E_2
10	-0.245826197	$4.175665*10^{-8}$	$2.992376*10^{-8}$
50	-0.245826197	$1.214528*10^{-11}$	$9.874508*10^{-12}$
100	-0.245826197	$9.514037*10^{-12}$	$3.117117*10^{-13}$
150	-0.245826197	$1.094573*10^{-13}$	$4.202717*10^{-14}$

It is known that finding an analytical solution of integral equations is quite complicated. We consider here the numerical solution of Fredholm's integral equation of the second kind, and in this connection, we recall the following methods [9, 6, 11, 3].

We are given an integral equation

$$y(x)=\lambda\int\limits_0^1 K(x,t)y(t)dt+f(x),$$

we replace the integral with the following sum

$$\int\limits_0^1 K(x,t)y(t)dt\cong\sum_{j=0}^{N}\alpha_j K(x_i,t_j)y(t_j),\quad i=\overline{0,N},$$

where α_j are the coefficients of the quadrature formula. By inserting the last approximate equation into the given integral equation, we take $x=x_i, t=t_i$ $(i=0,1,2...,N)$ and following form a system of linear equations.

$$y_i-\lambda\sum_{j=0}^{N}\alpha_j K_{ij}y_j=f_i, i=\overline{0,N}$$

where $x_i=t_i,\ \ f(x_i)=f_i,\ \ K(x_i,t_j)=K_{ij}$. We find y_i, $(i=0,1,2...N)$ from the system of linear equations.

We apply the derived optimal quadrature formula constructed in this work to the numerical solution of Fredholm integral equation of the second kind. To do this, we first replace the first and second derivatives in the formula with fourth-order finite differences

$$\varphi^{'}(0)=\frac{-25\varphi(0)+48\varphi(h)-36\varphi(2h)+16\varphi(3h)-3\varphi(4h)}{12h}$$

$$\varphi^{'}(1)=\frac{25\varphi(1)-48\varphi((N-1)h)+36\varphi((N-2)h)-16\varphi((N-3)h)+3\varphi((N-4)h)}{12h}$$

$$\varphi''(x_i)=\frac{-\varphi(x_{i-2})+16\varphi(x_{i-1})-30\varphi(x_i)+16\varphi(x_{i+1})-\varphi(x_{i+2})}{12h^2},\qquad i=\overline{2,N-2}$$

$$\varphi''(0)=\frac{45\varphi(0)-154\varphi(h)+214\varphi(2h)-156\varphi(3h)+61\varphi(4h)-10\varphi(5h)}{12h^2}$$

$$\varphi''(1)=\frac{1}{12h^2}\left[45\varphi(1)-154\varphi((N-1)h)+214\varphi((N-2)h)-156\varphi((N-3)h)\right.$$
$$\left.+61\varphi((N-4)h)-10\varphi((N-5)h)\right]$$

.

In Table 3, the numerical solution of the Fredholm integral equation of the second kind error is analyzed using Simpson formula **(Error Simp)** and optimal quadrature formula (1.1) **(Error OQF)** with $N = 6$ nodes.

$$u(x) = e^x + 2\int_0^1 e^{x+t}u(t)dt$$

Table 3. Numerical solution of the integral equation

x_0	**Exact solution**	**Error Simp**	**Error OQF**
0	-0.1855612526	$1.485665 * 10^{-5}$	$1.332376 * 10^{-6}$
1/6	-0.2192147180	$1.754528 * 10^{-5}$	$1.584508 * 10^{-6}$
2/6	-0.2589715897	$2.074037 * 10^{-5}$	$1.867117 * 10^{-6}$
3/6	-0.3059387842	$2.454573 * 10^{-5}$	$2.202717 * 10^{-6}$
4/6	-0.3614239684	$2.904037 * 10^{-5}$	$2.607117 * 10^{-6}$
5/6	-0.4269719685	$3.424573 * 10^{-5}$	$3.072717 * 10^{-6}$
1	-0.5044077809	$4.044573 * 10^{-5}$	$3.632717 * 10^{-6}$

Figure 1 shows the absolute error of the numerical solution of the definite integrals of the functions $f_1(x) = e^{x^2} + \ln(1 + x^2)$ and $\varphi_1(x) = \cos x + e^{-x}$ using Midpoint, Trapezoid, Simpson 1/3, Simpson 3/8 and OQF(optimal quadrature formula) at N=20 nodes.

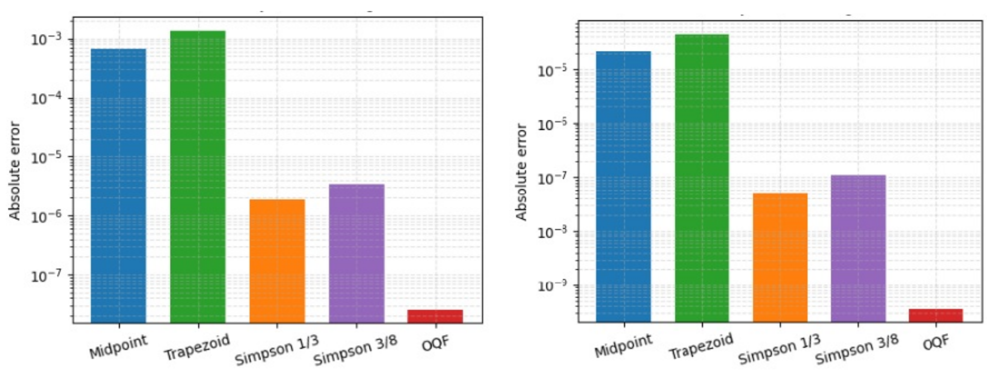

FIGURE 1. Absolute error of functions $f(x)_1$ and $\varphi_1(x)$

In Figure 2, $f_2(x) = x^{10} + \sin(2x)$, $\varphi_2(x) = x^5 + x^4$, $N = 10$ nodes.

5. Conclusion

In this research work, we have constructed derivative optimal quadrature formulas with derivatives for the approximation of definite integrals and the numerical solution of integral equations. Using the extremal function, we obtained an overview of the norm of the error function. Since the error function is a multivariate function, we found its extremum under certain conditions. By solving the system of equations, we found the analytical representation of the coefficients. We numerically analyzed that the obtained results are more effective than classical formulas.

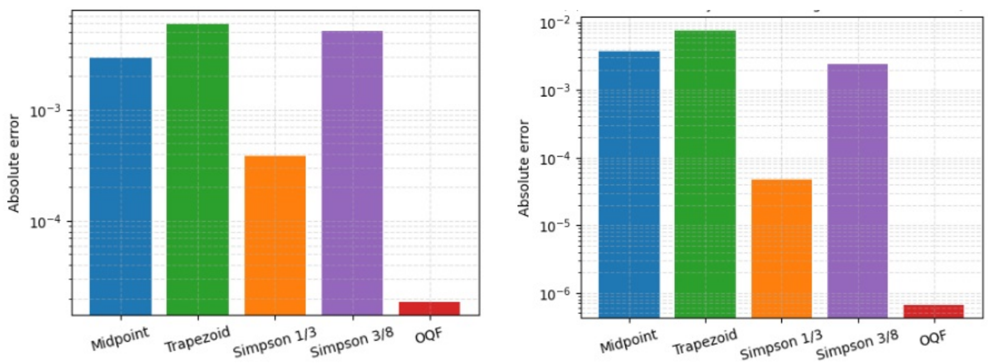

FIGURE 2. Absolute error of functions $f(x)_2$ and $\varphi_2(x)$

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