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THE CLASSICAL CONTINUOUS OPTIMAL CONTROL OF COUPLED FOURTH ORDER LINEAR PARABOLIC EQUATIONS

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ABSTRACT. In this study, the finite element method based on piecewise cubic Hermite basis functions is employed to establish the existence and uniqueness of a coupled state vector solution for a system of fourth-order linear parabolic partial differential equations with Neumann boundary conditions. An existence theorem for a continuous classical optimal control vector associated with fourth-order linear parabolic partial differential equations is formulated and proved under appropriate conditions. The study also investigates the existence and uniqueness of the solution to the corresponding adjoint system associated with the state vector for a given classical coupled optimal control. Finally, the Fréchet derivative of the quadratic cost functional is derived to establish the necessary optimality condition for the control problem.

Keywords. Coupled fourth-order linear parabolic equations, Coupled adjoint equations, Quadratic cost function, Necessary condition.
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1. INTRODUCTION

Optimal control problems have attracted considerable attention due to their wide range of applications in real-life systems, including physics [12], robotics [11], medicine [7], and many other scientific fields. From a mathematical perspective, these problems are typically modeled by either ordinary differential equations (ODEs) or partial differential equations (PDEs), depending on the nature of the modeled phenomenon. Many researchers have studied optimal control problems associated with second- or fourth- order elliptic [2, 6], parabolic [4, 9], and hyperbolic [1, 3, 5, 10] PDEs. Furthermore, optimal control problems involving systems of coupled fourth order PDEs of elliptic type have been investigated [5], the present work is devoted to the theoretical analysis and study of the continuous classical coupled optimal control problem (CCLCPOCP) associated with fourth-order linear parabolic equations. The finite element (FE) method combined with piecewise cubic Hermite functions is employed to establish the

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existence and uniqueness of the state vector solution when the continuous control vector is specified. An existence theorem for a continuous classical coupled optimal control (CCLCPOC) vector is formulated and proved. The paper further investigates the existence and uniqueness of the solution to the coupled adjoint equations associated with the coupled state vector, assuming the availability of a classical optimal control vector. Furthermore, the Fréchet (FR) derivative of the quadratic cost functional (QCF) is formulated to provide the necessary optimality condition of the proposed control problem.

2. System Description

Consider the CCLCPOCP represented by the QCF

$$J(\overrightarrow{W}) = \frac{\lambda}{2}[\| Y - Y_d \|_S^2 + \| Z - Z_d \|_S^2] + \frac{K}{2}[\| W_1 \|_S^2 + \| W_2 \|_S^2] \tag{1}$$

subject to

$$Y_t + \Delta^2 Y + \Delta Y + Y - BZ = F(\overrightarrow{X}, t) + W_1(\overrightarrow{X}, t) \text{ in } S \tag{2}$$

$$Z_t + \Delta^2 Z + \Delta Z + Z + BY = G(\overrightarrow{X}, t) + W_2(\overrightarrow{X}, t) \text{ in } S \tag{3}$$

With the Neumann boundary conditions

$$\frac{\partial Y}{\partial n} = 0 \text{ on } \quad \partial S \tag{4}$$

$$\frac{\partial \Delta Y}{\partial n} = 0 \text{ on } \quad \partial S \tag{5}$$

$$\frac{\partial Z}{\partial n} = 0 \text{ on } \quad \partial S \tag{6}$$

$$\frac{\partial \Delta Z}{\partial n} = 0 \text{ on } \quad \partial S \tag{7}$$

And the initial conditions

$$Y(\overrightarrow{X}, 0) = Y^0(\overrightarrow{X}) \text{ on } \partial D \tag{8}$$

$$Z(\overrightarrow{X}, 0) = Z^0(\overrightarrow{X}) \text{ on } \partial D \tag{9}$$

Where D is a bounded Lipschitz domain in $\mathbb{R}^2$ having boundary $\partial D, B \in L^\infty(D), S = [0,T] \times D, T < \infty, \partial S = [0,T] \times \partial D, \overrightarrow{X} = (X_1, X_2) \in D, I = [0,T]$ and $\overrightarrow{\Upsilon} = (Y, Z) = (Y(\overrightarrow{X}, t), Z(\overrightarrow{X}, t)) \in (H^4(S))^2$ is the coupled state vector relating to the classical coupled control (CLCPC) vector $\overrightarrow{W} = (W_1, W_2) \in (L^2(S))^2$, additionally $\lambda, K > 0$, (Y_d, Z_d) represents the desired data, and $F, G \in L^2(S)$ are given functions. The set of admissible control is
$\overrightarrow{V}_a \subset (L^2(S))^2 = \{\overrightarrow{W} = (W_1(\overrightarrow{X}, t), W_2(\overrightarrow{X}, t)) \in (L^2(S))^2 : (W_1(\overrightarrow{X}, t), W_2(\overrightarrow{x}, t)) \in \overrightarrow{V} = V_1 \times V_2$ a.e. in $S\}$ where $\overrightarrow{V}$ is a convex set.
The CCLCPOCP is to minimize the QCF subject to $\overrightarrow{W} \in \overrightarrow{V}_a$.
Let $\overrightarrow{Q} = Q_1 \times Q_2$,
$\overrightarrow{Q} = \{\overrightarrow{q} : \overrightarrow{q} = (q_1, q_2) \in (H^2(D))^2, \frac{\partial q_1}{\partial n} = 0, \frac{\partial q_2}{\partial n} = 0 \text{ on } \partial D\}$

The weak formulation (WF) of ((2)-(9)) is achieved by employing the Green's theorem for equations (2) and (3), then for $(Y, Z) \in (H^2(D))^2$, we get

$$< Y_t, q_1 > +(\Delta Y, \Delta q_1) + (\nabla Y, \nabla q_1) + (Y, q_1) - (BZ, q_1) = (F, q_1) + (W_1, q_1) \quad (10)$$

$$< Z_t, q_2 > +(\Delta Z, \Delta q_2) + (\nabla Z, \nabla q_2) + (Z, q_2) + (BY, q_2) = (G, q_2) + (W_2, q_2) \quad (11)$$

$$(Y^0(\overrightarrow{X}), q_1) = (Y(0), q_1) \quad (12)$$

$$(Z^0(\overrightarrow{X}), q_2) = (Z(0), q_2) \quad (13)$$

To determine the solution of (10)-(13), the Galerkin FE method is employed by selecting an approximating subspace $\overrightarrow{Q}_n$ (which possesses a finite dimension) of $\overrightarrow{Q}$. Consequently (10)-(13) are transformed in the discrete WF;
Find $(Y_n, Z_n) \in Q_n \times Q_n$ such that

$$< Y_{nt}, q_1 > +(\Delta Y_n, \Delta q_1) + (\nabla Y_n, \nabla q_1) + (Y_n, q_1) - (BZ_n, q_1) = (F + W_1, q_1) \quad (14)$$

$$< Z_{nt}, q_2 > +(\Delta Z_n, \Delta q_2) + (\nabla Z_n, \nabla q_2) + (Z_n, q_2) + (BY_n, q_2) = (G + W_2, q_2) \quad (15)$$

$$(Y_n^0(\overrightarrow{X}), q_1) = (Y_n(0), q_1) \quad (16)$$

$$(Z_n^0(\overrightarrow{X}), q_2) = (Z_n(0), q_2) \quad (17)$$

3. Solution of the Problem

Theorem 3.1. *For each specified CLCPC vector guarantees that the WF possesses a unique solution* (Y, Z) *with* $(Y, Z) \in (L^2(I, Q))^2$ *and* $(Y_t, Z_t) \in (L^2(I, Q^*))^2$.

Proof. Let $\overrightarrow{Q}_n \subset \overrightarrow{Q}$ be the set of continuous piecewise cubic Hermite polynomials in D. Let $\overrightarrow{\Psi}$ and $\overrightarrow{\bar{\Psi}}$ be a finite Hermite basis of $\overrightarrow{Q}_n$ defined by
$\{\overrightarrow{\Psi}_1, ..., \overrightarrow{\Psi}_1, \overrightarrow{\bar{\Psi}}_2, ..., \overrightarrow{\bar{\Psi}}_2\}$then (Y_n, Z_n) is approximated by
$(Y_n, Z_n) = \sum_{j=1}^{n}(\overrightarrow{d}_j(t)\overrightarrow{\Psi}_j(\overrightarrow{X}), \overrightarrow{e}_j(t)\overrightarrow{\bar{\Psi}}_j(\overrightarrow{X})) =$
$(\sum_{j=1}^{n} d_{1j}(t)\Psi_{1j}(\overrightarrow{X}) + \sum_{j=1}^{n} e_{1j}(t)\bar{\Psi}_{1j}(\overrightarrow{X}), \sum_{j=1}^{n} d_{2j}(t)\Psi_{2j}(\overrightarrow{X}) + \sum_{j=1}^{n} e_{2j}(t)\bar{\Psi}_{2j}(\overrightarrow{X}))$
Where $d_{lj}(t)$ and $e_{lj}(t)$ are functions of t, $\forall l = 1, 2$. Substituting the approximation solution (Y_n, Z_n) in (14)-(17) with $\overrightarrow{q} = (q_1, q_2) \in \overrightarrow{Q}_n$, then the obtained system transformed into an equivalent system of ODEs which has a unique solution.

$$K_1 D_1'(t) + K_2 D_1(t) - K_3 D_2(t) = B_1 \quad (18)$$

$$H_1 D_2'(t) + H_2 D_2(t) - H_3 D_1(t) = B_2 \quad (19)$$

$$K_1 D_1(0) = B_1^0 \quad (20)$$

$$H_1 D_2(0) = B_2^0 \quad (21)$$

Where K_1, K_2, K_3, H_1, H_2 and H_3 are stiffness matrices with $D_1(t) = [d_{11}, \ldots, d_{1n}, e_{11}, \ldots, e_{1n}]^T$and $D_2(t) = [d_{21}, \ldots, d_{2n}, e_{21}, \ldots, e_{2n}]^T$, and the matrices of (18)-(21) are defined as follows
$K_1 = \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{bmatrix}$, $A_1(i, j) = (\Psi_{1j}, \Psi_{1i})$, $A_2(i, j) = (\bar{\Psi}_{1j}, \Psi_{1i})$, $A_3(i, j) = (\bar{\Psi}_{1j}, \bar{\Psi}_{1i})$
$K_2 = \begin{bmatrix} A_4 & A_5 \\ A_5^T & A_6 \end{bmatrix}$, $A_4(i, j) = (\Delta\Psi_{1j}, \Delta\Psi_{1i}) + (\Delta\Psi_{1j}, \Delta\Psi_{1i}) + (\Psi_{1j}, \Psi_{1i})$, $A_5(i, j) =$
$(\Delta\bar{\Psi}_{1j}, \Delta\Psi_{1i}) + (\nabla\bar{\Psi}_{1j}, \nabla\Psi_{1i}) + (\bar{\Psi}_{1j}, \Psi_{1i})$, $A_6(i, j) = (\Delta\bar{\Psi}_{1j}, \Delta\bar{\Psi}_{1i}) + (\nabla\bar{\Psi}_{1j}, \nabla\bar{\Psi}_{1i}) +$

$(\bar{\Psi}_{1j}, \bar{\Psi}_{1i})$, $K_3 = \begin{bmatrix} A_7 & A_8 \\ A_8^T & A_9 \end{bmatrix}$, $A_7(i,j) = (B\Psi_{2j}, \Psi_{1i})$, $A_8(i,j) = (B\bar{\Psi}_{2j}, \Psi_{1i})$, $A_9(i,j) = (B\bar{\Psi}_{2j}, \bar{\Psi}_{1i})$, $B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}$, $B_{11}(i) = (F, \Psi_{1i}) + (W_1, \Psi_{1i})$, $B_{12}(i) = (F, \bar{\Psi}_{1i}) + (W_1, \bar{\Psi}_{1i})$, $B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}$, $B_{21}(i) = (F, \Psi_{2i}) + (W_2, \Psi_{2i})$, $B_{22}(i) = (G, \bar{\Psi}_{2i}) + (W_2, \bar{\Psi}_{2i})$, $H_1 = \begin{bmatrix} C_1 & C_2 \\ C_2^T & C_3 \end{bmatrix}$, $C_1(i,j) = (\Psi_{2j}, \Psi_{2i})$, $C_2(i,j) = (\bar{\Psi}_{2j}, \Psi_{2i})$, $C_3(i,j) = (\bar{\Psi}_{2j}, \bar{\Psi}_{2i})$, $H_2 = \begin{bmatrix} C_4 & C_5 \\ C_5^T & C_6 \end{bmatrix}$, $C_4(i,j) = (\Delta\Psi_{2j}, \Delta\Psi_{2i}) + (\Delta\Psi_{2j}, \Delta\Psi_{2i}) + (\Psi_{2j}, \Psi_{2i})$, $C_5(i,j) = (\Delta\bar{\Psi}_{2j}, \Delta\Psi_{2i}) + (\nabla\bar{\Psi}_{2j}, \nabla\Psi_{2i}) + (\bar{\Psi}_{2j}, \Psi_{2i})$, $C_6(i,j) = (\Delta\bar{\Psi}_{2j}, \Delta\bar{\Psi}_{2i}) + (\nabla\bar{\Psi}_{2j}, \nabla\bar{\Psi}_{2i}) + (\bar{\Psi}_{2j}, \bar{\Psi}_{2i})$, $H_3 = \begin{bmatrix} C_7 & C_8 \\ C_8^T & C_9 \end{bmatrix}$,$C_7(i,j) = (B\Psi_{1j}, \Psi_{2i})$, $C_8(i,j) = (B\bar{\Psi}_{1j}, \Psi_{2i})$, $C_9(i,j) = (B\bar{\Psi}_{1j}, \bar{\Psi}_{2i})$, $B_1^0 = \begin{bmatrix} B_{11}^0 \\ B_{12}^0 \end{bmatrix}$, $B_{11}^0(i) = (Y_i^0, \Psi_{1i})$, $B_{12}(i)^0 = (Y_i^0, \bar{\Psi}_{1i})$, $B_2^0 = \begin{bmatrix} B_{21}^0 \\ B_{22}^0 \end{bmatrix}$, $B_{21}(i)^0 = (Z_i^0, \Psi_{2i})$, $B_{22}^0(i) = (Z_i^0, \bar{\Psi}_{2i})$.

To prove $\| \overrightarrow{\Upsilon}_n^0 \|^2_{(L^2(D))^2}$ is bounded.

Since $(Y^0, Z^0) \in (L^2(D))^2$, then there exist $\{q_{1n}^0\}$ and $\{q_{2n}^0\}$ in $\overrightarrow{Q}_n$ such that $q_{1n}^0 \longrightarrow Y^0$ and $q_{2n}^0 \longrightarrow Z^0$ strong convergence in $L^2(D)$, then according to the projection theorem and (16)-(17), one gets

$\overrightarrow{\Upsilon}_n^0 \longrightarrow \overrightarrow{\Upsilon}_n$ strong convergence in $L^2(D)$ and $\| \overrightarrow{\Upsilon}_n^0 \|_{(L^2(D))^2} \leq d$.

Now, to demonstrate that $\| \overrightarrow{\Upsilon}_n \|^2_{(L^\infty(I \times L^2(D)))^2}$, $\| \overrightarrow{\Upsilon}_n \|^2_{(L^2(S))^2}$ and $\| \overrightarrow{\Upsilon}_n \|_{(L^2(I \times Q))^2}$ are bounded, putting $q_1 = Y_n$ and $q_2 = Z_n$ in (14)-(15) , integrating both sides of the obtained equations over $0 \leq t \leq T$ and then adding the resulting equations, we obtain

$$\begin{aligned} \int_0^T < \overrightarrow{\Upsilon}_{nt}, \overrightarrow{\Upsilon}_n > dt + \int_0^T \| \overrightarrow{\Upsilon}_n \|^2_{(H^2(D))^2} dt \\ = \int_0^n [(F + W_1, Y_n) + (G + W_2, Z_n)] dt \end{aligned} \tag{22}$$

Applying Lemma (1.2) [13] to the first term of (22), and noting that the second term is positive, we conclude that

$\int_0^T \frac{d}{dt} \| \overrightarrow{\Upsilon}_n \|_{(L^2(D)^2} \leq \int_0^T \int_D [|FY_n| + |W_1||Y_n| + |GZ_n| + |W_2||Z_n|] d\overrightarrow{X} dt$

Which gives that

$\int_0^T \frac{d}{dt} \| \overrightarrow{\Upsilon}_n \|^2_{(L^2(D))^2} \leq [\| F \|^2_S + \| W_1 \|^2_S + \| G \|^2_S + \| W_2 \|^2_S + \int_0^T [2 \| Y_n \|^2_D$
$+2 \| Z_n \|^2_D] dt$

Since $\| F \|^2_S \leq \rho_1, \| G \|^2_S \leq \rho_2, \| W_l \|^2_S \leq \omega_l$ for $l = 1, 2.$, and $\| \overrightarrow{\Upsilon}_n \|_{(L^2(D))^2}$ is bounded, then we get

$\| \overrightarrow{\Upsilon}_n \|_{(L^2(D)^2} \leq N + 2 \int_0^T \| \overrightarrow{\Upsilon}_n \|_{(L^2(D)^2} dt$, with $N = (\rho_1 + \rho_2 + \omega_1 + \omega_2 + d)$

By utilizing Bellman-Gronwall inequality [10], we have

$\| \overrightarrow{\Upsilon}_n \|_{(L^2(D))^2} \leq Ne^{2T} = d^2$, where $\| \overrightarrow{\Upsilon}_n \|_{(L^\infty(I \times L^2(D))^2} \leq d$ and $\| \overrightarrow{\Upsilon}_n \|_{(L^2(S))^2} \leq d$

Now, repeating the previous argument in (22) but evaluating it at $t = T$,we

obtain
$\parallel \overrightarrow{\Upsilon}_n(T) \parallel^2_{(L^2(D))^2} - \parallel \overrightarrow{\Upsilon}_n(0) \parallel^2_{(L^2(D))^2} + \int_0^T \parallel \overrightarrow{\Upsilon}_n \parallel^2_{(H^2(D))^2} \leq \parallel \rho_1 \parallel^2_S + \parallel W_1 \parallel^2_S$
$+ \parallel \rho_2 \parallel^2_S + \parallel W_2 \parallel^2_S + \parallel \overrightarrow{\Upsilon}_n \parallel^2_{(L^2(S))^2}$
Since $\parallel \overrightarrow{\Upsilon}_n(T) \parallel^2_{(L^2(D))^2}$ is positive, then
$\parallel \overrightarrow{\Upsilon}_n(T) \parallel^2_{(L^2(D))^2} = \int_0^T \parallel \overrightarrow{\Upsilon}_n \parallel^2_{(H^2(D))^2} \leq \frac{(\rho_1+\rho_2+\omega_1+\omega_2+2d}{2} = \bar{C}$
Consider a sequence $\{\overrightarrow{Q}_n\} \subset \overrightarrow{Q}$, where for each $(q_1, q_2) \in \overrightarrow{Q}$, there is an associated sequence $\{\overrightarrow{q}_n\}$,$\forall n$ such that $\overrightarrow{q}_n \longrightarrow \overrightarrow{q}$ strong convergence in $\overrightarrow{S}$ and $\overrightarrow{q}_n \longrightarrow \overrightarrow{q}$ strong convergence in $(L^2(D))^2$. Then (14)-(17) yield a unique solution (Y_n, Z_n). By setting $q_1 = q_{1n}$ and $q_2 = q_{2n}$, $\forall n$ in (14)-(17), we get

$$\begin{aligned} < Y_{nt}, q_{1n} > +(\Delta Y_n, \Delta q_{1n}) + (\nabla Y_n, \nabla q_{1n}) + (Y_n, q_{1n}) - (BZ_n, q_{1n}) \\ = (F + W_1, q_{1n}) \end{aligned} \tag{23}$$

$$\begin{aligned} < Z_{nt}, q_{2n} > +(\Delta Z_n, \Delta q_{2n}) + (\nabla Z_n, \nabla q_{2n}) + (Z_n, q_{2n}) + (BY_n, q_{2n}) \\ = (G + W_2, q_{2n}) \end{aligned} \tag{24}$$

$$(Y_n^0(\overrightarrow{X}), q_{1n}) = (Y_n(0), q_{1n}) \tag{25}$$

$$(Z_n^0(\overrightarrow{X}), q_{2n}) = (Z_n(0), q_{2n}) \tag{26}$$

Since $\parallel \overrightarrow{\Upsilon}_n \parallel^2_{(L^2(I\times Q))^2}$ and $\parallel \overrightarrow{\Upsilon}_n \parallel^2_{(L^2(S))^2}$ are bounded, by using Alaoglu theorem, there exists $\{\overrightarrow{\Upsilon}_n\}$, such that $\overrightarrow{\Upsilon}_n \longrightarrow \overrightarrow{\Upsilon}$ weak convergence in $(L^2(S))^2$ and $\overrightarrow{\Upsilon}_n \longrightarrow \overrightarrow{\Upsilon}$ weak convergence in$(L^2(I \times Q))^2$.
Multiplying equations (23) and (25) by $\zeta_1(t) \in C^1(I)$ and equations (24) and (26) by $\zeta_2(t) \in C^1(I)$, and subsequently integrating both sides w.r.t (t) over $[0, T]$, followed by integrating by parts to the first terms of each resulting equation, one obtains

$$\begin{aligned} -\int_0^T (Y_n, q_{1n})\zeta_1^{'}(t)dt + \int_0^T [(\Delta Y_n, \Delta q_{1n}) + (\nabla Y_n, \nabla q_{1n}) + (Y_n, q_{1n}) - \\ (BZ_n, q_{1n})]\zeta_1(t)dt = \int_0^T [(F, q_{1n}) + (W_1, q_{1n})]\zeta_1(t)dt + (Y_n^0, q_{1n})\zeta_1(0) \end{aligned} \tag{27}$$

$$\begin{aligned} -\int_0^T (Z_n, q_{2n})\zeta_2^{'}(t)dt + \int_0^T [(\Delta Z_n, \Delta q_{2n}) + (\nabla Z_n, \nabla q_{2n}) + (Z_n, q_{2n}) + \\ (BY_n, q_{2n})]\zeta_2(t)dt = \int_0^T [(G, q_{2n}) + (W_2, q_{2n})]\zeta_2(t)dt + (Z_n^0, q_{2n})\zeta_2(0) \end{aligned} \tag{28}$$

Since $\overrightarrow{q}_n \longrightarrow \overrightarrow{q}$ strong convergence in $(L^2(D))^2$, then we get
$q_{ln}\zeta_l^{'} \longrightarrow q_l\zeta_l^{'}$ strong convergence in $L^2(S), \forall l = 1, 2$ then $q_{ln}\zeta_l^{'} \longrightarrow q_l\zeta_l^{'}$ strong convergence in $L^2(I \times Q), \forall l = 1, 2$ and $\overrightarrow{\Upsilon}_n \longrightarrow \overrightarrow{\Upsilon}$ weak convergence in$(L^2(I \times Q))^2$, we get
The left hand side of (27) converges to

$$\int_0^T [(Y, q_1)\zeta_1^{'}(t) + [(\Delta Y, \Delta q_1) + (\nabla Y, \nabla q_1) + (Y, q_1) - (BZ, q_1)]\zeta_1(t)]dt \tag{29}$$

and the left hand side of (28) converges to

$$\int_0^T [(Z,q_2)\zeta_2'(t)+[(\Delta Z,\Delta q_2)+(\nabla Z,\nabla q_2)+(Z,q_2)+(BY,q_2)]\zeta_2(t)]dt \tag{30}$$

$$(Y_n^0,q_{1n})\zeta_1(0)\ \text{ converges to}(Y^0,q_1)\zeta_1(0) \tag{31}$$

$$(Z_n^0,q_{2n})\zeta_2(0)\ \text{ converges to}(Z^0,q_2)\zeta_2(0) \tag{32}$$

Since $q_{ln} \longrightarrow q_l$ weak convergence in $L^2(D), \forall l = 1, 2$, we get

$$\int_0^T [(F,q_{1n})+(W_1,q_{1n})]\zeta_1(t)dt \text{ converges to } \int_0^T [(F,q_1)+(W_1,q_1)]\zeta_1(t)dt \tag{33}$$

$$\int_0^T [(G,q_{2n})+(W_2,q_{2n})]\zeta_2(t)dt \text{ converges to } \int_0^T [(G,q_2)+(W_2,q_2)]\zeta_2(t)dt \tag{34}$$

From the above convergences, we get

$$\begin{aligned} -\int_0^T (Y,q_1)\zeta_1'(t)dt+\int_0^T[(\Delta Y,\Delta q_1)+(\nabla Y,\nabla q_1)+(Y,q_{1n})-\\ (BZ,q_1)]\zeta_1(t)dt=\int_0^T[(F,q_1)+(W_1,q_1)]\zeta_1(t)dt+(Y^0,q_1)\zeta_1(0)\end{aligned} \tag{35}$$

$$\begin{aligned} -\int_0^T (Z,q_2)\zeta_2'(t)dt+\int_0^T[(\Delta Z,\Delta q_2)+(\nabla Z,\nabla q_2)+(Z,q_2)+\\ (BY,q_2)]\zeta_2(t)dt=\int_0^T[(G,q_2)+(W_2,q_2)]\zeta_2(t)dt+(Z^0,q_2)\zeta_2(0)\end{aligned} \tag{36}$$

Now, take $\zeta_l, \zeta_l(0) = \zeta_l(T) = 0, \forall l = 1, 2$ in (35) and (36), then applying integration by parts to the first terms of (35) and (36), we get

$$\begin{aligned} \int_0^T < Y_t,q_1 > \zeta_1(t)dt+\int_0^T[(\Delta Y,\Delta q_1)+(\nabla Y,\nabla q_1)+(Y,q_{1n})-\\ (BZ,q_1)]\zeta_1(t)dt=\int_0^T[(F,q_1)+(W_1,q_1)]\zeta_1(t)dt\end{aligned} \tag{37}$$

$$\begin{aligned} \int_0^T < Z_t,q_2 > \zeta_2(t)dt+\int_0^T[(\Delta Z,\Delta q_2)+(\nabla Z,\nabla q_2)+(Z,q_2)+\\ (BY,q_2)]\zeta_2(t)dt=\int_0^T[(G,q_2)+(W_2,q_2)]\zeta_2(t)dt\end{aligned} \tag{38}$$

Hence (Y, Z) is a solution vector of ((10)-(11)).
Also, take ζ_l, with $\zeta_l(0) \neq 0$ and $\zeta_l(T) = 0, \forall l = 1, 2$, by applying integration by parts to the first terms of (37) and(38), we obtain

$$\begin{aligned} -\int_0^T (Y,q_1)\zeta_1'(t)dt+\int_0^T[(\Delta Y,\Delta q_1)+(\nabla Y,\nabla q_1)+(Y,q_{1n})-\\ (BZ,q_1)]\zeta_1(t)dt=\int_0^T[(F,q_1)+(W_1,q_1)]\zeta_1(t)dt+(Y(0),q_1)\zeta_1(0)\end{aligned} \tag{39}$$

and

$$\begin{aligned}-\int_0^T (Z,q_2)\zeta_2'(t)dt + \int_0^T [(\Delta Z,\Delta q_2)+(\nabla Z,\nabla q_2)+(Z,q_2)+\\ (BY,q_2)]\zeta_2(t)dt = \int_0^T [(G,q_2)+(W_2,q_2)]\zeta_2(t)dt + (Z(0),q_2)\zeta_2(0)\end{aligned} \tag{40}$$

Subtracting (39)-(40) from (35)-(36) respectively, we get (12)-(13).
To demonstrate the strong convergence for (Y,Z) in $(L^2(I\times Q))^2$:
By putting $q_1 = Y$ and $q_2 = Z$ in (10) and (11) respectively, adding the two obtained equations. Subsequently by setting $q_1 = Y_n$ and $q_2 = Z_n$ in (14) and (15) respectively and adding the two obtained equations then integrating over $0 \le t \le T$ to get

$$\int_0^T [< \overrightarrow{\Upsilon}_t, \overrightarrow{\Upsilon} > + \mathcal{B}(\overrightarrow{\Upsilon}, \overrightarrow{\Upsilon})]dt = \int_0^T (F,Y)+(W_1,Y)+(G,Z)+(W_2,Z)dt \tag{41}$$

where $\mathcal{B}(\overrightarrow{\Upsilon}, \overrightarrow{\Upsilon}) = (\Delta Y,\Delta Y)+(\nabla Y,\nabla Y)+(Y,Y)+(\Delta Z,\Delta Z)+(\nabla Z,\nabla Z)+(Z,Z)$ is a bilinear form. And

$$\begin{aligned}\int_0^T [< \overrightarrow{\Upsilon}_{nt}, \overrightarrow{\Upsilon}_n > + \mathcal{B}(\overrightarrow{\Upsilon}_n, \overrightarrow{\Upsilon}_n)]dt\\ = \int_0^T [(F,Y_n)+(W_1,Y_n)+(G,Z_n)+(W_2,Z_n)]dt\end{aligned} \tag{42}$$

where $\mathcal{B}(\overrightarrow{\Upsilon}_n, \overrightarrow{\Upsilon}_n) = (\Delta Y_n,\Delta Y_n)+(\nabla Y_n,\nabla Y_n)+(Y_n,Y_n)+(\Delta Z,\Delta Z_n)+(\nabla Z_n,\nabla Z_n)+(Z_n,Z_n)$
Applying lemma(1.2) [13] to the first terms of (41) and (42), we get

$$\begin{aligned}\frac{1}{2}\parallel \overrightarrow{\Upsilon}(T)\parallel_{(L^2(D))^2} - \frac{1}{2}\parallel \overrightarrow{\Upsilon}(0)\parallel_{(L^2(D))^2} + \int_0^T \mathcal{B}(\overrightarrow{\Upsilon}, \overrightarrow{\Upsilon})dt\\ = \int_0^T [(F,Y)+(W_1,Y)+(G,Z)+(W_2,Z)]dt\end{aligned} \tag{43}$$

$$\begin{aligned}\frac{1}{2}\parallel \overrightarrow{\Upsilon}_n(T)\parallel_{(L^2(D))^2} - \frac{1}{2}\parallel \overrightarrow{\Upsilon}_n(0)\parallel_{(L^2(D))^2} + \int_0^T \mathcal{B}(\overrightarrow{\Upsilon}_n, \overrightarrow{\Upsilon}_n)dt\\ = \int_0^T [(F,Y_n)+(W_1,Y_n)+(G,Z_n)+(W_2,Z_n)]dt\end{aligned} \tag{44}$$

$$\begin{aligned}\frac{1}{2}\parallel \overrightarrow{\Upsilon}_n(T) - \overrightarrow{\Upsilon}(T)\parallel_{(L^2(D))^2} - \frac{1}{2}\parallel \overrightarrow{\Upsilon}_n(0) - \overrightarrow{\Upsilon}(0)\parallel_{(L^2(D))^2}\\ + \int_0^T \mathcal{B}(\overrightarrow{\Upsilon}_n - \overrightarrow{\Upsilon}, \overrightarrow{\Upsilon}_n - \overrightarrow{\Upsilon})dt = L_1 + L_2 + L_3\end{aligned} \tag{45}$$

Where $L_1 = \frac{1}{2}[\parallel \overrightarrow{\Upsilon}_n(T)\parallel_{(L^2(D))^2} - \parallel \overrightarrow{\Upsilon}_n(0)\parallel_{(L^2(D))^2}] + \int_0^T \mathcal{B}(\overrightarrow{\Upsilon}_n, \overrightarrow{\Upsilon}_n)dt$
$L_2 = \frac{1}{2}[(\overrightarrow{\Upsilon}_n(T), \overrightarrow{\Upsilon}(T)) - (\overrightarrow{\Upsilon}_n(0), \overrightarrow{\Upsilon}(0))] + \int_0^T \mathcal{B}(\overrightarrow{\Upsilon}_n(t), \overrightarrow{\Upsilon}(t))dt$
$L_3 = \frac{1}{2}[(\overrightarrow{\Upsilon}(T), \overrightarrow{\Upsilon}_n(T) - \overrightarrow{\Upsilon}(T)) - (\overrightarrow{\Upsilon}(0), \overrightarrow{\Upsilon}_n(0) - \overrightarrow{\Upsilon}(0))] + \int_0^T \mathcal{B}(\overrightarrow{\Upsilon}(t), \overrightarrow{\Upsilon}_n(t) -$

$\overrightarrow{\Upsilon}(t))dt$, since

$$\overrightarrow{\Upsilon}_n \longrightarrow \overrightarrow{\Upsilon}_n^0 \text{ strong convergence in}(L^2(D))^2 \tag{45a}$$

$$\text{and}\overrightarrow{\Upsilon}_n(T) \longrightarrow \overrightarrow{\Upsilon}(T) \text{ strong convergence in}(L^2(D))^2 \tag{45b}$$

$$\text{then}(\overrightarrow{\Upsilon}(T), \overrightarrow{\Upsilon}_n(T) - \overrightarrow{\Upsilon}(T)) \longrightarrow 0, \text{with}(\overrightarrow{\Upsilon}(0), \overrightarrow{\Upsilon}_n(0) - \overrightarrow{\Upsilon}(0)) \longrightarrow 0 \tag{45c}$$

$$\| \overrightarrow{\Upsilon}_n(T) - \overrightarrow{\Upsilon}(T) \|_{(L^2(D))^2} \longrightarrow 0, \text{with} \| \overrightarrow{\Upsilon}_n(0) - \overrightarrow{\Upsilon}(0) \|_{(L^2(D))^2} \longrightarrow 0 \tag{45d}$$

And since $\overrightarrow{\Upsilon}_n \longrightarrow \overrightarrow{\Upsilon}$ weak convergence in $(L^2(I \times Q))^2$, then

$$\int_0^T \mathcal{B}(\overrightarrow{\Upsilon}(t), \overrightarrow{\Upsilon}_n(t) - \overrightarrow{\Upsilon}(t))dt \longrightarrow 0 \tag{45e}$$

Since $\overrightarrow{\Upsilon}_n$ converges weakly to $\overrightarrow{\Upsilon}$ in $(L^2(S))^2$ then

$$\text{The right hand side of (44)} \longrightarrow \int_0^T [(F, Y) + (W_1, Y) + (G, Z) + (W_2, Z)]dt \tag{45f}$$

Hence (45) gives
$\| \overrightarrow{\Upsilon}_n - \overrightarrow{\Upsilon} \|_{(L^2(D))^2} = \int_0^T \mathcal{B}(\overrightarrow{\Upsilon}_n(t) - \overrightarrow{\Upsilon}(t), \overrightarrow{\Upsilon}_n(t) - \overrightarrow{\Upsilon}(t))dt \longrightarrow 0$, then $\overrightarrow{\Upsilon}_n \longrightarrow \overrightarrow{\Upsilon}_n^0$ in$(L^2(I \times Q))^2$
Let $\overrightarrow{\Upsilon} = (Y, Z)$ and $\overrightarrow{\Upsilon}_1 = (Y_1, Z_1)$ be two solutions of (10) and (11), subtracting the obtained equations from the other, and then putting $q_1 = Y - Y_1$ and $q_2 = Z - Z_1$, we get

$$< (Y - Y_1)_t, Y - Y_1 > +B(Y - Y_1, Y - Y_1) - (B(Y - Y_1), Z - Z_1) = 0 \tag{46}$$

$$< (Z - Z_1)_t, Z - Z_1 > +B(Z - Z_1, Z - Z_1) - (B(Z - Z_1), Y - Y_1) = 0 \tag{47}$$

Adding (46) and (47), then applying lemma (1.2) [13] to the first term, we obtain

$$\frac{1}{2}\frac{d}{dt} \| \overrightarrow{\Upsilon} - \overrightarrow{\Upsilon}_1 \|_{(L^2(D))^2} + \int_0^T \frac{1}{2} \| \overrightarrow{\Upsilon} - \overrightarrow{\Upsilon}_1 \|_{(H^2(D))^2} dt = 0 \tag{48}$$

The second term of (48) is positive, Integrating (48) over $[0, T]$ w.r.t. (t), we get

$$\| \overrightarrow{\Upsilon} - \overrightarrow{\Upsilon}_1 \|_{(L^2(D))^2} \leq 0 \text{ implies } \| \overrightarrow{\Upsilon} - \overrightarrow{\Upsilon}_1 \|_{(L^2(D))^2} = 0 \tag{49}$$

Integrating (48) over $[0, T]$, with using the initial conditions and (49), we get $\int_0^T \frac{1}{2} \| \overrightarrow{\Upsilon} - \overrightarrow{\Upsilon}_1 \|_{(H^2(D))^2} \, dt = 0$ implies $\int_0^T \frac{1}{2} \| \overrightarrow{\Upsilon} - \overrightarrow{\Upsilon}_1 \|_{(L^2(I \times Q))^2} \, dt = 0$ then $\overrightarrow{\Upsilon} = \overrightarrow{\Upsilon}_1$ □

4. EXISTENCE OF A CCLCPOC

Theorem 4.1. *If* $\overrightarrow{\Upsilon}$ *and* $\overrightarrow{\Upsilon} + \delta\overrightarrow{\Upsilon}$ *represent the pair of state vectors that corresponding to the pair of control vectors* $\overrightarrow{W}$ *and* $\overrightarrow{W} + \delta\overrightarrow{W}$ *in* $(L^2(D))^2$ *respectively, then* $\| \delta\overrightarrow{\Upsilon} \|_{(L^\infty(I \times L^2(D))^2} \leq m \| \delta\overrightarrow{W} \|_{(L^2(S))^2}$, $\| \delta\overrightarrow{\Upsilon} \|_{(L^2(S))^2} \leq m_1 \| \delta\overrightarrow{W} \|_{(L^2(S))^2}$, *and* $\| \delta\overrightarrow{\Upsilon} \|_{(L^2(I \times Q))^2} \leq \bar{m}_1 \| \delta\overrightarrow{W} \|_{(L^2(S))^2}$

Proof. According to theorem 3.1, since (Y_{W_1}, Z_{W_2}) is a solution of ((10)-(13)), let $(\bar{Y}_{\bar{W}_1}, \bar{Z}_{\bar{W}_2})$ be a solution of ((10)-(13)) corresponding to $(\bar{W}_1, \bar{W}_1) \in (L^2(S))^2$, we have

$$< \bar{Y}_t, q_1 > +(\Delta \bar{Y}, \Delta q_1) + (\nabla \bar{Y}, \nabla q_1) + (\bar{Y}, q_1) - (B\bar{Z}, q_1) = (F + \bar{W}_1, q_1) \tag{50}$$

$$< \bar{Z}_t, q_2 > +(\Delta \bar{Z}, \Delta q_2) + (\nabla \bar{Z}, \nabla q_2) + (\bar{Z}, q_2) + (B\bar{Y}, q_2) = (G + \bar{W}_2, q_2) \tag{51}$$

$$(\bar{Y}^0(\overrightarrow{X}), q_1) = (\bar{Y}(0), q_1) \tag{52}$$

$$(\bar{Z}^0(\overrightarrow{X}), q_2) = (\bar{Z}(0), q_2) \tag{53}$$

Subtracting ((10)-(13)) from ((50)-(53)) respectively, and setting $\delta Y = \bar{Y} - Y, \delta Z = \bar{Z} - Z, \delta W_1 = \bar{W}_1 - W_1$ and $\delta W_2 = \bar{W}_2 - W_2$ in the above resulting equations, we obtain

$$\begin{aligned} < \delta Y_t, q_1 > +(\Delta \delta Y, \Delta q_1) + (\nabla \delta Y, \nabla q_1) + (\delta Y, q_1) - (B\delta Z, q_1) \\ = (F + \delta W_1, q_1) \end{aligned} \tag{54}$$

$$\begin{aligned} < \delta Z_t, q_2 > +(\Delta \delta Z, \Delta q_2) + (\nabla \delta Z, \nabla q_2) + (\delta Z, q_2) + (B\delta Y, q_2) \\ = (G + \delta W_2, q_2) \end{aligned} \tag{55}$$

$$(\delta Y(0), q_1) = 0 \tag{56}$$

$$(\delta Z(0), q_2) = 0 \tag{57}$$

By substituting $\delta Y = q_1$ and $\delta Z = q_2$ in (54) and (55) respectively, adding the resulting equations and using Lemma (1.2)[13], we have
$\frac{1}{2}\frac{d}{dt} \parallel \delta \overrightarrow{\Upsilon} \parallel^2_{(L^2(D))^2} + \parallel \delta \overrightarrow{\Upsilon} \parallel^2_{(H^2(D))^2} = (\delta W_1, \delta Y) + (\delta W_2, \delta Z)$
Since $\parallel \delta \overrightarrow{\Upsilon} \parallel^2_{(H^2(D))^2}$ is positive, and integrating over $[0, T]$ w.r.t. (t), then applying Cauchy-Schwarz inequality [8], it follows that
$\int_0^T \frac{d}{dt} \parallel \delta \overrightarrow{\Upsilon} \parallel^2_{(L^2(D))^2} \leq 2 \int_0^T \int_D [|\delta W_1||\delta Y| + |\delta W_2||\delta Z|] d\overrightarrow{X} dt$
$\leq 2 \int_0^T \parallel \delta \overrightarrow{W} \parallel^2_{(L^2(D))^2} + 2 \int_0^T \parallel \delta \overrightarrow{\Upsilon} \parallel^2_{(L^2(D))^2}$
By applying the Bellman-Gronwall inequality, it follows that
$\parallel \delta \overrightarrow{\Upsilon} \parallel^2_{(L^2(D))^2} \leq 2e^{\int_0^T dt} \parallel \delta \overrightarrow{W} \parallel^2_{(L^2(S))^2}$, then $\parallel \delta \overrightarrow{\Upsilon} \parallel^2_{(L^2(D))^2} \leq m^2 \parallel \delta \overrightarrow{W} \parallel^2_{(L^2(S))^2}$,
where $m = 2e^T$, $\Longrightarrow \parallel \delta \overrightarrow{\Upsilon} \parallel_{(L^2(D))^2} \leq m \parallel \delta \overrightarrow{W} \parallel_{(L^2(S))^2}$, and
$\parallel \delta \overrightarrow{\Upsilon} \parallel_{(L^\infty(I \times L^2(D)))^2} \leq m \parallel \delta \overrightarrow{W} \parallel_{(L^2(S))^2}$,
since $\parallel \delta \overrightarrow{\Upsilon} \parallel^2_{(L^2(S)))^2} \leq Tm^2 \parallel \delta \overrightarrow{W} \parallel^2_{(L^2(S))^2}$, then
$\parallel \delta \overrightarrow{\Upsilon} \parallel_{(L^2(S))^2} \leq m_1 \parallel \delta \overrightarrow{W} \parallel_{(L^2(S))^2}$, $m_1^2 = Tm^2$, since
$\int_0^T \frac{d}{dt} \parallel \delta \overrightarrow{\Upsilon} \parallel^2_{(L^2(D))^2} + 2 \int_0^T \parallel \delta \overrightarrow{\Upsilon} \parallel^2_{(H^2(D))^2} \leq 2 \parallel \delta \overrightarrow{W} \parallel^2_{(L^2(S))^2} + 2 \parallel \delta \overrightarrow{\Upsilon} \parallel^2_{(L^2(S))^2}$
$\leq m_1^2 \parallel \delta \overrightarrow{W} \parallel^2_{(L^2(S))^2} \Longrightarrow \parallel \delta \overrightarrow{\Upsilon} \parallel^2_{(L^2(I \times Q))^2} \leq \bar{m}_1^2 \parallel \delta \overrightarrow{W} \parallel^2_{(L^2(S))^2}$, with $\bar{m}_1^2 = 1 + m^2$,
then $\parallel \delta \overrightarrow{\Upsilon} \parallel_{(L^2(I \times Q))^2} \leq \bar{m}_1 \parallel \delta \overrightarrow{W} \parallel_{(L^2(S))^2}$ □

Theorem 4.2. *The operator* $\overrightarrow{W} \longrightarrow \overrightarrow{\Upsilon}_{\overrightarrow{W}}$ *is continuous from* $(L^2(S))^2$ *into* $(L^2(I \times Q))^2$ *or* $(L^\infty(I \times L^2(D)))^2$.

Proof. By using the inequality of theorem 4.1 and since $\overrightarrow{\bar{\Upsilon}}$and $\overrightarrow{\Upsilon}$ are two solution vectors that corresponding to $\overrightarrow{\bar{W}}$ and $\overrightarrow{W}$ with $\delta\overrightarrow{\Upsilon} = \overrightarrow{\bar{\Upsilon}} - \overrightarrow{\Upsilon}$ and $\delta\overrightarrow{W} = \overrightarrow{\bar{W}} - \overrightarrow{W}$ one has $\| \overrightarrow{\bar{\Upsilon}} - \overrightarrow{\Upsilon} \|_{(L^\infty(I\times L^2(D)))^2} \leq m \| \overrightarrow{\bar{W}} - \overrightarrow{W} \|_{(L^2(S))^2}$
Thus, the above operator is Lipschitz continuous from $(L^2(S))^2$ into $(L^\infty(I \times L^2(D)))^2$. Moreover, by applying theorem 4.1 we get that the above operator is Lipschitz continuous from $(L^2(S))^2$ into$(L^2(I \times Q))^2$. □

Lemma 4.3. [6] *The QCF which is defined by* (1) *is weakly lower semicontinuous.*

Theorem 4.4. *Consider the QCF* (1)*, if* $J(\overrightarrow{W})$ *is coercive, then the there exists a CCLCPOC.*

Proof. Since $J(\overrightarrow{W})$ is coercive and $J(\overrightarrow{W}) \geq 0$, there exists a minimizing $\{\overrightarrow{W}_k\} \in \overrightarrow{V}_a, \forall k$ such that
$lim_{k\to\infty} J(\overrightarrow{W}_k) = \inf J(\overrightarrow{W}), \overrightarrow{\bar{W}} \in \overrightarrow{V}_a$ with $\| \overrightarrow{W}_k \|_{(L^2(S))^2} \leq d$, according to Alaoglu theorem there exists $\{\overrightarrow{W}_k\}$ such that $\overrightarrow{W}_k \longrightarrow \overrightarrow{W}$ weak convergence in $(L^2(S))^2, k \to \infty$ by using theorem 3.1 we have a unique solution $\overrightarrow{\Upsilon}_k$, then there exists a sequence $\{\overrightarrow{\Upsilon}_k\}$ that corresponding to the control $\{\overrightarrow{W}_k\}$ such that $\| \overrightarrow{\Upsilon}_k \|_{(L^2(I\times Q))^2}, \| \overrightarrow{\Upsilon}_k \|_{(L^\infty(I\times L^2(D)))^2}$ and $\| \overrightarrow{\Upsilon}_k \|_{(L^2(S))^2}$ are bounded, then we get $\overrightarrow{\Upsilon}_k \longrightarrow \overrightarrow{\Upsilon}$ weak convergence in $(L^2(I \times Q))^2$, $\overrightarrow{\Upsilon}_k \longrightarrow \overrightarrow{\Upsilon}$ weak convergence in $(L^\infty(I \times L^2(D)))^2$ and $\overrightarrow{\Upsilon}_k \longrightarrow \overrightarrow{\Upsilon}$ weak convergence in $(L^2(S))^2$(by Alaoglu theorem).
To find that the norm $\| \overrightarrow{\Upsilon}_k \|^2_{(L^2(I\times Q^*))^2}$is bounded, let (10) and (11) be expressed as

$$< Y_{kt}, q_1 >= -(\Delta Y_k, \Delta q_1) - (\nabla Y_k, \nabla q_1) - (Y_k, q_1) + (BZ_k, q_1) + (F + W_{1k}, q_1) \tag{58}$$

$$< Z_{kt}, q_2 >= -(\Delta Z_k, \Delta q_2) - (\nabla Z_k, \nabla q_2) - (Z_k, q_2) - (BY_k, q_2) + (G + W_{2k}, q_2) \tag{59}$$

Adding (58) and (59), integrating both sides of the resulting equation over $0 \leq t \leq T$, taking the absolute value and then applying Cauchy-Schwarz inequality, we obtain
$\int_0^T | < \overrightarrow{\Upsilon}_{kt}, \overrightarrow{q} > |dt \leq \| \Delta Y_k \|_s \| \Delta q_1 \|_s + \| \nabla Y_k \|_s \| \nabla q_1 \|_s + \| Y_k \|_s \| q_1 \|_s$
$+\epsilon \| Z_k \|_s \| q_1 \|_s + \| F \|_s \| q_1 \|_s + \| W_{1k} \|_s \| q_1 \|_s + \| \Delta Z_k \|_s \| \Delta q_2 \|_s +$
$\| \nabla Z_k \|_s \| \nabla q_2 \|_s + \| Z_k \|_s \| q_2 \|_s +\epsilon \| Y_k \|_s \| q_2 \|_s + \| G \|_s \| q_2 \|_s +$
$\| W_{2k} \|_s \| q_2 \|_s$, since
$\| Y_k \|_S \leq \| \overrightarrow{\Upsilon}_k \|_{(L^2(S))^2}$, $\| Z_k \|_S \leq \| \overrightarrow{\Upsilon}_k \|_{(L^2(S))^2}$, $\| \Delta Y_k \|_S \leq \| \Delta\overrightarrow{\Upsilon}_k \|_{(L^2(S))^2}$,
$\| \nabla Y_k \|_S \leq \| \nabla\overrightarrow{\Upsilon}_k \|_{(L^2(S))^2}$, $\| \Delta Z_k \|_S \leq \| \Delta\overrightarrow{\Upsilon}_k \|_{(L^2(S))^2}$, $\| \nabla Y_k \|_S \leq \| \nabla\overrightarrow{\Upsilon}_k \|_{(L^2(S))^2}$,
$\| q_{lk} \|_S \leq \| \overrightarrow{q}_k \|_{(L^2(S))^2}$,$\| \Delta q_{lk} \|_S \leq \| \Delta\overrightarrow{q}_k \|_{(L^2(S))^2}$, $\| \nabla q_{lk} \|_S \leq \| \nabla\overrightarrow{q}_k \|_{(L^2(S))^2}$,
$\| W_{lk} \|_S \leq \| \overrightarrow{W}_k \|_{(L^2(S))^2}$, $\| \overrightarrow{\Upsilon}_k \|_{(L^2(S))^2} \leq \| \overrightarrow{\Upsilon}_k \|_{(L^2(I\times Q))^2}$,
$\| \Delta\overrightarrow{\Upsilon}_k \|_{(L^2(S))^2} \leq \| \overrightarrow{\Upsilon}_k \|_{(L^2(I\times Q))^2}$, $\| \nabla\overrightarrow{\Upsilon}_k \|_{(L^2(S))^2} \leq \| \overrightarrow{\Upsilon}_k \|_{(L^2(I\times Q))^2}, \forall l = 1, 2,$
then
$| \int_0^T < \overrightarrow{\Upsilon}_{kt}, \overrightarrow{q} > dt| \leq [6 \| Y_k \|_{(L^2(I\times Q))^2} +2\epsilon \| Y_k \|_{(L^2(I\times Q))^2} +\bar{L}_1 + \bar{L}_2 + 2d]$

$\| \overrightarrow{q} \|_{(L^2(I\times Q))^2} \leq (\bar{K}(C)+\bar{L}) \| \overrightarrow{q} \|_{(L^2(I\times Q))^2}$, $\Longrightarrow \frac{|\int_0^T < \overrightarrow{\Upsilon}_{kt}, \overrightarrow{q} > dt|}{\| \overrightarrow{q} \|_{(L^2(I\times Q))^2}} \leq \bar{\bar{K}}$, then $\| \overrightarrow{\Upsilon}_k \|_{(L^2(I\times Q))^2} \leq \bar{\bar{K}}$.

Since $\overrightarrow{\Upsilon}_k$ is a solution of ((2)-(9)), we have

$$\begin{aligned} < Y_{kt}, q_1 > +(\Delta Y_k, \Delta q_1) + (\nabla Y_k, \nabla q_1) + (Y_k, q_1) - (BZ_k, q_1) \\ = (F + W_{1k}, q_1) \end{aligned} \tag{60}$$

$$\begin{aligned} < Z_{kt}, q_2 > +(\Delta Z_k, \Delta q_2) + (\nabla Z_k, \nabla q_2) + (Z_k, q_2) + (BY_k, q_2) \\ = (G + W_{2k}, q_2) \end{aligned} \tag{61}$$

Let $\zeta_1(t), \zeta_2(t) \in C^1[0,T]$ such that $\zeta_1(T) = \zeta_2(T) = 0$, multiplying (60) and (61) by $\zeta_1(T)$ and $\zeta_2(t)$ respectively, integrating both sides over $0 \leq t \leq T$ and then integrating by parts to the first terms of the resulting equations, we get

$$\begin{aligned} -\int_0^T (Y_k, q_1)\zeta_1'(t)dt + \int_0^T [(\Delta Y_k, \Delta q_1) + (\nabla Y_k, \nabla q_1) + (Y_k, q_1) - \\ (BZ_k, q_1)]\zeta_1(t)dt = \int_0^T [(F, q_1) + (W_{1k}, q_1)]\zeta_1(t)dt + (Y_k(0), q_1)\zeta_1(0) \end{aligned} \tag{62}$$

$$\begin{aligned} -\int_0^T (Z_k, q_2)\zeta_2'(t)dt + \int_0^T [(\Delta Z_k, \Delta q_2) + (\nabla Z_k, \nabla q_2) + (Z_k, q_2) + \\ (BY_k, q_2)]\zeta_2(t)dt = \int_0^T [(G, q_2) + (W_{2k}, q_2)]\zeta_2(t)dt + (Z_k(0), q_2)\zeta_2(0) \end{aligned} \tag{63}$$

Since $\overrightarrow{\Upsilon}_k \longrightarrow \overrightarrow{\Upsilon}$ weak convergence in $(L^2(S))^2$ and $\overrightarrow{\Upsilon}_k \longrightarrow \overrightarrow{\Upsilon}$ weak convergence in $(L^2(I\times Q))^2$, then we get the following convergence:
$-\int_0^T (Y_k, q_1)\zeta_1'(t)dt + \int_0^T [(\Delta Y_k, \Delta q_1) + (\nabla Y_k, \nabla q_1) + (Y_k, q_1) - (BZ_k, q_1)]\zeta_1(t)dt$ converges to

$$\int_0^T [-(Y, q_1)\zeta_1'(t)dt + [(\Delta Y, \Delta q_1) + (\nabla Y, \nabla q_1) + (Y, q_1) - (BZ, q_1)]\zeta_1(t)]dt \tag{64a}$$

and $-\int_0^T (Z_k, q_2)\zeta_2'(t)dt + \int_0^T [(\Delta Z_k, \Delta q_2) + (\nabla Z_k, \nabla q_2) + (Z_k, q_2) - (BY_k, q_2)]\zeta_2(t)dt$ converges to

$$-\int_0^T (Z, q_2)\zeta_2'(t)dt + \int_0^T [(\Delta Z, \Delta q_2) + (\nabla Z, \nabla q_2) + (Z, q_2) + (BY, q_2)]\zeta_2(t)dt \tag{64b}$$

Since $(Y_k(0), Z_k(0))$ is bounded in $(L^2(D))^2$ and according to the projection theorem, we have

$$(Y_k(0), q_1)\zeta_1(0) \text{ converges to } (Y(0), q_1)\zeta_1(0) \tag{65a}$$

$$(Z_k(0), q_2)\zeta_2(0) \text{ converges to} (Z(0), q_2)\zeta_2(0) \tag{65b}$$

And since $\overrightarrow{W}_k \longrightarrow \overrightarrow{W}$ weak convergence in $(L^2(S))^2$, then

$$\int_0^T [(F,q_1)+(W_{1k},q_1)]\zeta_1(t)dt \text{ converges to } \int_0^T [(F,q_1)+(W_1,q_1)]\zeta_1(t)dt \tag{66a}$$

$$\int_0^T [(G,q_2)+(W_{2k},q_2)]\zeta_2(t)dt \text{ converges to } \int_0^T [(G,q_2)+(W_2,q_2)]\zeta_2(t)dt \tag{66b}$$

By using ((64a)-(64b)), ((65a)-(65b)) and ((66a)-(66b)), we get

$$\begin{aligned} -\int_0^T (Y,q_1)\zeta_1'(t)dt + \int_0^T [(\Delta Y,\Delta q_1)+(\nabla Y,\nabla q_1)+(Y,q_1)- \\ (BZ,q_1)]\zeta_1(t)dt = \int_0^T [(F,q_1)+(W_1,q_1)]\zeta_1(t)dt + (Y(0),q_1)\zeta_1(0) \end{aligned} \tag{67a}$$

$$\begin{aligned} -\int_0^T (Z,q_2)\zeta_2'(t)dt + \int_0^T [(\Delta Z,\Delta q_2)+(\nabla Z,\nabla q_2)+(Z,q_2)+ \\ (BY,q_2)]\zeta_2(t)dt = \int_0^T [(G,q_2)+(W_2,q_2)]\zeta_2(t)dt + (Z(0),q_2)\zeta_2(0) \end{aligned} \tag{67b}$$

Now, take $\zeta_1(t),\zeta_2(t) \in C[0.T], \zeta_1(0)=\zeta_1(T)=0$, and $\zeta_2(0)=\zeta_2(T)=0$ and applying integration by parts to the first terms of (67a) and (67b), we obtain

$$\begin{aligned} \int_0^T < Y_t,q_1 > \zeta_1(t)dt + \int_0^T [(\Delta Y,\Delta q_1)+(\nabla Y,\nabla q_1)+(Y,q_{1n})- \\ (BZ,q_1)]\zeta_1(t)dt = \int_0^T [(F,q_1)+(W_1,q_1)]\zeta_1(t)dt \end{aligned} \tag{68a}$$

and

$$\begin{aligned} \int_0^T < Z_t,q_2 > \zeta_2(t)dt + \int_0^T [(\Delta Z,\Delta q_2)+(\nabla Z,\nabla q_2)+(Z,q_2)+ \\ (BY,q_2)]\zeta_2(t)dt = \int_0^T [(G,q_2)+(W_2,q_2)]\zeta_2(t)dt \end{aligned} \tag{68b}$$

Then $< Y_t,q_1 > +(\Delta Y,\Delta q_1)+(\nabla Y,\nabla q_1)+(Y,q_1)-(BZ,q_1) = (F,q_1)+(W_1,q_1)$ and$< Z_t,q_2 > +(\Delta Z,\Delta q_2)+(\nabla Z,\nabla q_2)+(Z,q_2)+(BY,q_2) = (G,q_2)+(W_2,q_2)$ Also, take $\zeta_1(t),\zeta_2(t) \in C^1[0,T], \zeta_1(T)=\zeta_2(T)=0$, and $\zeta_1(0)\neq 0, \zeta_2(0)\neq 0$ and using integration by parts to the first terms of (68a) and (68b), we obtain

$$\begin{aligned} -\int_0^T (Y,q_1)\zeta_1'(t)dt + \int_0^T [(\Delta Y,\Delta q_1)+(\nabla Y,\nabla q_1)+(Y,q_{1n})- \\ (BZ,q_1)]\zeta_1(t)dt = \int_0^T [(F,q_1)+(W_1,q_1)]\zeta_1(t)dt + (Y(0),q_1)\zeta_1(0) \end{aligned} \tag{69a}$$

and

$$\begin{aligned} -\int_0^T (Z,q_2)\zeta_2'(t)dt + \int_0^T [(\Delta Z,\Delta q_2)+(\nabla Z,\nabla q_2)+(Z,q_2)+ \\ (BY,q_2)]\zeta_2(t)dt = \int_0^T [(G,q_2)+(W_2,q_2)]\zeta_2(t)dt + (Z(0),q_2)\zeta_2(0) \end{aligned} \tag{69b}$$

By subtracting ((69a)-(69b)) from ((67a)-(67b)) respectively, we have $(Y^0, q_1)\zeta_1(0) = (Y(0), q_1)\zeta_1(0)$, and $(Z^0, q_2)\zeta_2(0) = (Z(0), q_2)\zeta_2(0)$, then $Y^0 = Y(0) = Y^0(\overrightarrow{X}), Z^0 = Z(0) = Z^0(\overrightarrow{X})$

Since by lemma 4.3 and $\overrightarrow{W}_k \longrightarrow \overrightarrow{W}$ weak convergence in $(L^2(D))^2$

$J(\overrightarrow{W}) \leq \lim_{k\to\infty} \inf J(\overrightarrow{W}_k) = \lim_{k\to\infty} J(\overrightarrow{W}_k) = \inf J(\overrightarrow{\overrightarrow{W}})$

$J(\overrightarrow{W}) \leq \inf J(\overrightarrow{\overrightarrow{W}}) = \min J(\overrightarrow{W})$.Hence $\overrightarrow{W}$ is a CCLCPOC □

5. Necessary condition (NEC) of optimality

Theorem 5.1. *The QCF is given in equation (1) and the coupled adjoint equations corresponding to (2)-(9) are derived by*

$$-P_{1t} + \Delta^2 P_1 + \Delta P_1 + P_1 - BP_2 = \lambda(Y - Y_d) \text{ in } S \tag{70}$$

$$-P_{2t} + \Delta^2 P_2 + \Delta P_2 + P_2 + BP_1 = \lambda(Z - Z_d) \text{ in } S \tag{71}$$

$$P_1 = 0 \text{ on } \partial S \tag{72}$$

$$\frac{\partial P_1}{\partial n} = 0 \text{ on } \partial S \tag{73}$$

$$P_2 = 0 \text{ on } \partial S \tag{74}$$

$$\frac{\partial P_2}{\partial n} = 0 \text{ on } \partial S \tag{75}$$

For $\overrightarrow{P} = (P_1, P_2)$, then the FR derivative of the QCF is obtained by $(\nabla J(\overrightarrow{W}), \delta\overrightarrow{W}) = (\overrightarrow{P} + K\overrightarrow{W}, \delta\overrightarrow{W}), (W_1, W_2) \in \overrightarrow{V}_a$

Proof. The WF of (70)-(75) is given by

$$\begin{aligned} - < P_{1t}, q_1 > +(\Delta P_1, \Delta q_1) + (\nabla P_1, \nabla q_1) + (P_1, q_1) - (BP_2, q_1) \\ = \lambda(Y - Y_d, q_1) \end{aligned} \tag{76}$$

$$\begin{aligned} - < P_{2t}, q_2 > +(\Delta P_2, \Delta q_2) + (\nabla P_2, \nabla q_2) + (P_2, q_2) + (BP_1, q_2) \\ = \lambda(Z - Z_d, q_2) \end{aligned} \tag{77}$$

The existence of a unique solution for (76) and (77) can be established using the method outlined in theorem 3.1.

By substituting $q_1 = P_1$ and $q_2 = P_2$ in (54) and (55) respectively, we obtain

$$\begin{aligned} < \delta Y_t, P_1 > +(\Delta\delta Y, \Delta P_1) + (\nabla\delta Y, \nabla P_1) + (\delta Y, P_1) - (B\delta Z, P_1) \\ = (F, P_1) + (\delta W_1, P_1) \end{aligned} \tag{78}$$

$$\begin{aligned} < \delta Z_t, P_2 > +(\Delta\delta Z, \Delta P_2) + (\nabla\delta Z, \nabla P_2) + (\delta Z, P_2) + (B\delta Y, P_2) \\ = (G, P_2) + (\delta W_2, P_2) \end{aligned} \tag{79}$$

Also, by setting $q_1 = \delta Y$ and $q_2 = \delta Z$ in (76) and (77) respectively, to get

$$\begin{aligned} - < P_{1t}, \delta Y > +(\Delta P_1, \Delta\delta Y) + (\nabla P_1, \nabla\delta Y) + (P_1, \delta Y) - (BP_2, \delta Y) \\ = \lambda(Y - Y_d, \delta Y) \end{aligned} \tag{80}$$

$$\begin{aligned} - < P_{2t}, \delta Z > +(\Delta P_2, \Delta\delta Z) + (\nabla P_2, \nabla\delta Z) + (P_2, \delta Z) + (BP_1, \delta Z) \\ = \lambda(Z - Z_d, \delta Z) \end{aligned} \tag{81}$$

By integrating both sides of ((78)-(81)) over $0 \leq t \leq T$ applying integrating by parts to the first terms of each resulting equations, subsequently subtracting each of these resulting equations from its corresponding original equation ((78)-(81)), and finally adding the two resulting equations, we obtain

$$(\delta W_1, P_1) + (\delta W_2, P_2) = \lambda(Y - Y_d, \delta Y) + \lambda(Z - Z_d, \delta Z) \tag{82}$$

Since
$J(\overrightarrow{W} + \delta\overrightarrow{W}) - J(\overrightarrow{W}) = (\delta W_1, P_1) + (\delta W_2, P_2) + (KW_1, \delta W_1)$
$+ \frac{\lambda}{2} \parallel \delta\overrightarrow{\Upsilon} \parallel^2_{(L^2(S))^2} + \frac{K}{2} \parallel \delta\overrightarrow{W} \parallel^2_{(L^2(S))^2}$ Or
$J(\overrightarrow{W} + \delta\overrightarrow{W}) - J(\overrightarrow{W}) = (\overrightarrow{P} + K\overrightarrow{W}, \delta\overrightarrow{W}) + \frac{\lambda}{2} \parallel \delta\overrightarrow{\Upsilon} \parallel^2_{(L^2(S))^2} + \frac{K}{2} \parallel \delta\overrightarrow{W} \parallel^2_{(L^2(S))^2}$
By using the second inequality of theorem 4.1, we get
$\frac{\lambda}{2} \parallel \delta\overrightarrow{\Upsilon} \parallel^2_{(L^2(S))^2} = \epsilon_1(\delta\overrightarrow{W}) \parallel \delta\overrightarrow{W} \parallel_{(L^2(S))^2}$, with $\epsilon_1(\delta\overrightarrow{W}) = \frac{m_1^2}{2} \parallel \delta\overrightarrow{W} \parallel_{(L^2(S))^2}$ and
$\frac{K}{2} \parallel \delta\overrightarrow{W} \parallel^2_{(L^2(S))^2} = \epsilon_2(\delta\overrightarrow{W}) \parallel \delta\overrightarrow{W} \parallel_{(L^2(S))^2}$
Then we obtain
$J(\overrightarrow{W} + \delta\overrightarrow{W}) - J(\overrightarrow{W}) = (\overrightarrow{P} + K\overrightarrow{W}, \delta\overrightarrow{W}) + \bar{\epsilon}(\delta\overrightarrow{W}) \parallel \delta\overrightarrow{W} \parallel_{(L^2(S))^2}$, where $\bar{\epsilon}(\delta\overrightarrow{W}) = \epsilon_1(\delta\overrightarrow{W}) + \epsilon_2(\delta\overrightarrow{W})$, and $\bar{\epsilon}(\delta\overrightarrow{W}) \longrightarrow 0$, as $\parallel \delta\overrightarrow{W} \parallel_{(L^2(S))^2} \longrightarrow 0$
From the FR derivative, we obtain
$(\nabla J(\overrightarrow{W}), \delta\overrightarrow{W}) = (\overrightarrow{P} + K\overrightarrow{W}, \delta\overrightarrow{W}), (W_1, W_2) \in \overrightarrow{V}_a$ $\square$

Theorem 5.2. *The CCLCPOC of the above problem is* $\overrightarrow{P} + K\overrightarrow{W} = 0$ *with* $\overrightarrow{\Upsilon} = \overrightarrow{\Upsilon}_{\overrightarrow{W}}$ *and* $\overrightarrow{P} = \overrightarrow{P}_{\overrightarrow{W}}$

Proof. if $\overrightarrow{W}$ is an CCLCPOC of the problem, then
$J(\overrightarrow{W}) = \min J(\overrightarrow{\bar{W}}), \forall\overrightarrow{\bar{W}} \in (L^2(S))^2$ i.e., $\nabla J(\overrightarrow{W}) = 0$ implies $\overrightarrow{P} + K\overrightarrow{W} = 0$
Then the NEC is $(\overrightarrow{P} + K\overrightarrow{W}, \delta\overrightarrow{W}) \geq 0, \forall\delta\overrightarrow{W} = \overrightarrow{\bar{W}} - \overrightarrow{W}$, then
$(\overrightarrow{P} + K\overrightarrow{W}, \overrightarrow{\bar{W}}) \geq (\overrightarrow{P} + K\overrightarrow{W}, \overrightarrow{W}), \forall\overrightarrow{\bar{W}} \in (L^2(S))^2$ $\square$

6. Conclusion

The finite element method based on piecewise cubic Hermite basis functions is applied to establish the existence and uniqueness of the state vector solution for the coupled fourth-order linear parabolic equations with a fixed control vector. The existence of an associated coupled optimal control vector is established rigorously.In addition, the existence and uniqueness of the adjoint system solution is also demonstrated. The Fréchet derivative of the quadratic cost functional is provided. Finally, yielding the necessary optimality condition.

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