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SOLUTION TO VOLTERRA STOCHASTIC INTEGRAL EQUATIONS USING FUZZY FIXED-POINTS OF INTEGRAL TYPE CONTRACTIONS

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ABSTRACT. The main aim of this research work is first to establish the idea of integral type theta-contractions and then establish certain novel fuzzy fixed-point theorems for fuzzy mappings. The framework of b-metric spaces is used for this purpose. The results are demonstrated with adequate illustrations. Furthermore, an example is provided to justify the requirements of the proven results. Lastly, to reinforce the validity of our results, an application is provided that authenticates the existence of a solution to stochastic Volterra integral equations by employing the axioms of the theorem. These new findings will open the door for more study in this field and are unique in the literature on b-metric spaces.

Keywords. b-metric space; fuzzy mapping; fuzzy set; fuzzy Θ-type generalized almost contraction mapping;
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1. Introduction

In mathematics, a fixed point ($\mathcal{FP}$) is a value that does not change under a given transformation. Fixed point theory has evolved into a rich and multifaceted field comprising principles and methods from topology, analysis, algebra, geometry, as well as discrete and computational mathematics. Metric fixed point theory originates with the concept of Picard successive approximations for establishing the existence and uniqueness of solutions to nonlinear initial value problems, optimization problems, and dynamic system stability studies. The Polish mathematician Stefan Banach [12] is credited with placing the underlying ideas into an abstract framework suitable for broad applications well beyond the scope of elementary differential and integral equations, see [3, 11, 17, 18, 22, 23, 24, 25, 26, 27, 28, 31, 32]. Another development is fixed points for set-valued operators developed towards the mid-20th century with the celebrated extensions of the Brouwer and Lefschetz theorems by Kakutani and

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Eilenberg-Montgomery, respectively. The Banach contraction principle was later extended to multi-valued contractions by Nadler [21].
The notion of a b-metric space ($\mathcal{BMS}$) is initially proposed by Bakhtin [13] in 1989. Czerwick [15] used this idea to establish a generalization of Banach contraction principle in such spaces by weakening the triangular property of metric spaces. In 1965, L. A. Zadeh [7] introduced fuzzy sets in his ground breaking work, which served as the basis for the fuzzy set ($\mathcal{FS}$) theory. One of the most significant advances in fuzzy set theory is the discovery of the fuzzy equivalents of Banach contraction principle. Nadler's result [21] on a metric space for multi-valued mappings ($\mathcal{MM}$s) is equivalent to Heilpern's $\mathcal{FP}$ theorem [16] for fuzzy mappings ($\mathcal{FM}$s). Recent research on fuzzy mappings on $\mathcal{BMS}$ by Batul et al. [2] can be seen for better understanding in this regard.
Branciari [5] established a few $\mathcal{FP}$ results for mappings that meet integral-type contractive criteria. Branciari introduced a self-mapping $\mathcal{S}$ on U that meets the following condition:

$$\int_0^{\tilde{\sigma}(\mathcal{S}(\mathsf{c}_1),\mathcal{S}(\mathsf{c}_2))} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \lambda \int_0^{\tilde{\sigma}(\mathsf{c}_1,\mathsf{c}_2)} \triangle(\varsigma)\mathrm{d}(\varsigma),$$

where $\triangle : [0,\infty) \longrightarrow [0,\infty)$ is Lebesgue integrable and summable on each compact subset of $[0,\infty)$ and $\int_0^{\mathrm{t}} \triangle(\varsigma)\mathrm{d}(\varsigma) > 0$ for each $\mathrm{t} > 0$ with a metric space $(\mathsf{U},\tilde{\sigma})$, $\lambda \in (0,1)$, and $\mathsf{c}_1,\mathsf{c}_2 \in U$.

This research is focused on the concept of fuzzy Θ-type generalized almost contraction mappings ($\mathcal{F}_\Theta G\mathcal{AC}$). The main objective is to use $\mathcal{F}_\Theta G\mathcal{AC}$ in order to generalize the result of Kanwal et al. [30].

2. Preliminaries

Definition 2.1. Let Ξ be the class of functions $\triangle : [0,\infty) \longrightarrow [0,\infty)$ so that:

- (i): $\triangle$ is a Lebesgue integrable function and summable on each compact subset of $[0,\infty)$;
- (ii): $\int_0^{\mathrm{t}} \triangle(\varsigma)\mathrm{d}(\varsigma) > 0$ for each $\mathrm{t} > 0$.
- (iii): Δ is non-increasing on $[0,\infty)$.

Definition 2.2. [20] A mapping Λ is called an $\mathcal{FM}$ mapping if Λ maps U into $\mathcal{F}(\mathsf{U})$.

For a number $\alpha_{\mathrm{F}} \in (0,1]$, an alpha-level set of a fuzzy set F is defined as

$$[\mathrm{F}]_{\alpha_{\mathrm{F}}} = \Big\{\mathsf{c} \in \mathsf{U} : \mathrm{F}(\mathsf{c}) \geq \alpha_{\mathrm{F}}\Big\}, \text{ if } \alpha_{\mathrm{F}} \in (0,1],$$
$$[\mathrm{F}]_0 = \Big\{\overline{\mathsf{c} \in \mathsf{U} : \mathrm{F}(\mathsf{c}) > 0}\Big\},$$

where $\overline{A}$ means the closure of the set A.

Definition 2.3. [6] Let $(\mathsf{U},\tilde{\sigma}_b)$ be a complete $\mathcal{BMS}$ and $\Re,\Gamma$ be the pair of $\mathcal{FM}$s. Let $\alpha_{\Re(\mathrm{f})}$ and $\alpha_{\Gamma(\mathrm{f})} \in (0,1]$. Then, $\mathrm{f} \in \mathsf{U}$ is said to be a common $\alpha-$ fuzzy $\mathcal{FP}$ if

$\mathrm{f} \in [\Re(\mathrm{f})]_{\alpha_{\Re(\mathrm{f})}} \cap [\Gamma(\mathrm{f})]_{\alpha_{\Gamma(\mathrm{f})}}$, where $[\Re(\mathrm{f})]_{\alpha_{\Re(\mathrm{f})}}$ and $[\Gamma(\mathrm{f})]_{\alpha_{\Gamma(\mathrm{f})}}$ are the alpha-level sets of fuzzy sets $\Re(\mathrm{f})$ and $\Gamma(\mathrm{f})$, respectively.

Recently, Samet et al. [29] introduced a new type of contractions, which is called the Θ-contractions and established some new $\mathcal{FP}$ results.

Definition 2.4. Consider a mapping $\Theta : (0,\infty) \longrightarrow (1,\infty)$ so that:
(θ_1): Θ is nondecreasing;
(θ_2): for any sequence $\{\alpha_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n\to\infty} \alpha_n = 0^+$;
(θ_3): there exist $\mathsf{k} \in (0,1)$ and $l \in (0,\infty]$ so that $\lim_{\alpha\to 0^+} \dfrac{\Theta(\alpha)-1}{\alpha^{\mathsf{k}}} = l$.

A function $\mathcal{S} : (\mathsf{U}) \longrightarrow \mathsf{U}$ is known as a Θ contraction if there is Θ that satisfies the conditions $(\theta_1)-(\theta_3)$ and a number $\mathsf{k} \in (0,1)$ so that for all $\mathsf{c}_1, \mathsf{c}_2 \in \mathsf{U}$,
$\tilde{\sigma}_b(\mathcal{S}\mathsf{c}_1, \mathcal{S}\mathsf{c}_2) \neq 0 \Rightarrow \Theta(\tilde{\sigma}_b(\mathcal{S}\mathsf{c}_1, \mathcal{S}\mathsf{c}_2)) \leq [\Theta(\tilde{\sigma}_b(\mathsf{c}_1, \mathsf{c}_2))]^{\mathsf{k}}$.

Later on, motivated by the idea of Samet et al. [29], Ameer et al. [1] presented the following definition:

Definition 2.5. Let $s \geq 1$. Denote θ_s the set of all functions $\Theta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, which fulfills the following conditions:
(θ_{1s}): Θ is non-decreasing;
(θ_{2s}): for each sequence $\{\mathtt{t}_n\} \subset \mathbb{R}^+$, $\lim_{n\to\infty} \Theta(\mathtt{t}_n) = 1 \Leftrightarrow \lim_{n\to\infty} \mathtt{t}_n = 0^+$;
(θ_{3s}): there exist $\mathsf{k} \in (0,1)$ and $l \in (0,\infty]$ so that $\lim_{\mathtt{t}\to 0^+} \dfrac{\Theta(\mathtt{t})-1}{\mathtt{t}^{\mathsf{k}}} = l$;
(θ_{4s}): for each sequence $\{\mathtt{t}_n\} \subset \mathbb{R}^+$, such that $\Theta(s\mathtt{t}_n) \leq [\Theta(\mathtt{t}_{n-1})]^{\mathsf{k}}\ \forall\, n \in \mathbb{N}$ then $\Theta(s^n\mathtt{t}_n) \leq [\Theta(s^{n-1}t_{n-1})]^{\mathsf{k}}$ for some $\mathsf{k} \in (0,1)$ and $\forall n \in \mathbb{N}$.

Definition 2.6. [8] Let $\mathsf{CB}(\mathsf{U})$ be the family of closed and bounded subsets of the $\mathcal{BMS}$ $(\mathsf{U}, \tilde{\sigma}_b)$. For $\mathsf{c}_1 \in \mathsf{U}$ and $\mathcal{S}, T \in \mathsf{CB}(\mathsf{U})$, set

$$\tilde{\sigma}_b(\mathsf{c}_1, T) = \inf_{\mathsf{c}_2\in T}(\tilde{\sigma}_b(\mathsf{c}_1, \mathsf{c}_2)).$$

Define $\mathtt{H} : \mathsf{CB}(\mathsf{U}) \times \mathsf{CB}(\mathsf{U}) \longrightarrow [0,+\infty)$ as

$$\mathtt{H}(S,T) = \max\Big\{\sup_{\mathsf{c}_1\in\mathcal{S}} \tilde{\sigma}_b(\mathsf{c}_1, T), \sup_{\mathsf{c}_2\in T} \tilde{\sigma}_b(\mathsf{c}_2, \mathcal{S})\Big\},$$

then, $(\mathsf{CB}(\mathsf{U}), \mathtt{H})$ is named as a Pompeiu-Hausdorff metric space.

Lemma 2.7. [10] *Every sequence* $\{\mathsf{c}_n\}_{n\in\mathbb{N}}$ *of elements from a* $\mathcal{BMS}$ $(\mathsf{U}, \tilde{\sigma}_b)$ *satisfying the property*

$$\tilde{\sigma}_b(\mathsf{c}_{n+1}, \mathsf{c}_n) \leq K\tilde{\sigma}_b(\mathsf{c}_n, \mathsf{c}_{n-1}),\ \forall\ K \in [0,1),$$

is a Cauchy sequence.

Lemma 2.8. [8] *Let* $(\mathsf{U}, \tilde{\sigma}_b)$ *to be a complete* $\mathcal{BMS}$. *For* $\mathcal{S}, T \in \mathsf{CB}(\mathsf{U})$, $s \geq 0$ *and* $\mathrm{v} \in S$, *take*

$$\tilde{\sigma}_b(\mathrm{v}, T) \leq s\mathtt{H}(\mathcal{S}, T).$$

Definition 2.9. [19] Let ρ be the set of strictly increasing functions in a $\mathcal{BMS}$, $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

(ρ_1): $\sum_{n=0}^{\infty} s^n \varphi^n < +\infty$, where φ^n denotes the nth iterate of φ;
(ρ_2): $\varphi(\mathrm{t}) < \mathrm{t}\ \forall\ \mathrm{t} > 0$ and $\varphi(0) = 0\ \forall\ \mathrm{t} = 0$. Then, it is called a comparison function on $\mathcal{BMS}$.

Lemma 2.10. [4] *If* $\{\mathsf{P}_n\}$ *is a sequence in* $[0, \infty)$ *and* $\psi \in \Xi$, *then*
$\lim_{n\longrightarrow\infty} \int_0^{\mathrm{P}_n} \psi(\varsigma)\mathrm{d}(\varsigma) = 0 \ \Leftrightarrow\ \mathrm{P}_n \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Lemma 2.11. [9] *Let* $(\mathsf{U}, \tilde{\sigma}_b)$ *be a complete* $\mathcal{BMS}$. *For* $S, T \in \mathsf{CB}(\mathsf{U})$ *and* $\mathrm{v} \in \mathcal{S}$, *then, for any* $\mathrm{w} \in T$,

$$\tilde{\sigma}_b(\mathrm{v}, T) \leq \tilde{\sigma}_b(\mathrm{v}, \mathrm{w}).$$

In 2016, Anita et al. [14] proposed the following result in the context of a $\mathcal{FM}$ as follows:

Theorem 2.12. *Assume* $(\mathsf{U}, \tilde{\sigma}_b)$ *to be a complete* $\mathcal{BMS}$, *and let the* $\mathcal{FM}$ $\Re : \mathsf{U} \to \digamma(\mathsf{U})$, *such that* $\forall\ \mathsf{c}_1, \mathsf{c}_2 \in \mathsf{U}$ *and* $\exists\ \varphi \in \rho$,

$$\mathtt{H}([\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}}, [\Re(\mathsf{c}_2)]_{\alpha_{\Re(\mathsf{c}_2)}} \leq \varphi(\Pi(\mathsf{c}_1, \mathsf{c}_2))),$$

where

$$\Pi(\mathsf{c}_1, \mathsf{c}_2) = \max\Big\{\tilde{\sigma}(\mathsf{c}_1, \mathsf{c}_2), \tilde{\sigma}(\mathsf{c}_1, [\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}}), \tilde{\sigma}(\mathsf{c}_2, [\Re(\mathsf{c}_2)]_{\alpha_{\Re(\mathsf{c}_2)}}), \frac{\{\tilde{\sigma}(\mathsf{c}_1, [\Re(\mathsf{c}_2)]_{\alpha_{\Re(\mathsf{c}_2)}}) + \tilde{\sigma}(\mathsf{c}_2, [\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}})\}}{2s}\Big\},$$

where $[\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}} \in \mathsf{CB}(\mathsf{U})$ *is non-empty and* $\alpha_{\Re}(\mathsf{c}_1) \in (0, 1]$. *Then,* $\Re$ *has an* α-*fuzzy* $\mathcal{FP}$.

3. Main Results

Throughout the article, $\tilde{\sigma}_b$ is considered as a complete $\mathcal{BMS}$. Motivated by the ideas of Anita et al. [14] and Kanwal et al. [30], we will introduce fuzzy Θ-type generalized almost contraction mappings ($\mathcal{F}_\Theta GAC$) in $\mathcal{BMS}$ as follows:

Definition 3.1. Suppose that $(\mathsf{U}, \tilde{\sigma}_b)$ is a complete $\mathcal{BMS}$ with $s \geq 1$, then the pair $\Re, \Gamma : \mathsf{U} \to \digamma(\mathsf{U})$ is said to be $\mathcal{F}_\Theta GAC$ if $\forall\ \mathsf{c}_1, \mathsf{c}_2 \in \mathsf{U}$, $[\Gamma(\mathsf{c}_1)]_{\alpha_{\Gamma(\mathsf{c}_1)}}$, $[\Re(\mathsf{c}_2)]_{\alpha_{\Re(\mathsf{c}_2)}} \in \mathsf{CB}(\mathsf{U})$ are non-empty and $\alpha_{\Re(\mathsf{c}_1)}, \alpha_{\Gamma(\mathsf{c}_2)} \in (0, 1]$, such that

$$\mathtt{H}([\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}}, [\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}}) > 0,$$

$$\begin{aligned}
&\int_0^{[\Theta\left(s\mathtt{H}([\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}}, [\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}})\right)]} \triangle(\varsigma)\,\mathrm{d}\varsigma \\
&\leq \int_0^{[\Theta\left(\Pi(\mathsf{c}_1,\mathsf{c}_2)\right)]^k + L\mathfrak{D}(\mathsf{c}_1,\mathsf{c}_2)} \triangle(\varsigma)\,\mathrm{d}\varsigma \qquad (3.1)\\
&\leq \int_0^{\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2)} \triangle(\varsigma)\,\mathrm{d}\varsigma.
\end{aligned}$$

where $\Theta \in \theta_s$, $L \geq 0$,

$$\Pi(\mathsf{c}_1, \mathsf{c}_2) = \max\Big\{\tilde{\sigma}_b(\mathsf{c}_1, \mathsf{c}_2), \tilde{\sigma}_b(\mathsf{c}_1, [\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}}), \tilde{\sigma}_b(\mathsf{c}_2, [\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}}), \frac{\{\tilde{\sigma}_b(\mathsf{c}_1, [\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}}) + \tilde{\sigma}_b(\mathsf{c}_2, [\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}})\}}{2s}\Big\},$$

and

$$\mathfrak{D}(\mathsf{c}_1, \mathsf{c}_2) = \min\Big\{\tilde{\sigma}_b(\mathsf{c}_1, [\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}}), \tilde{\sigma}_b(\mathsf{c}_2, [\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}}), \tilde{\sigma}_b(\mathsf{c}_1, [\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}}), \tilde{\sigma}_b(\mathsf{c}_2, [\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}})\Big\}.$$

Theorem 3.2. *Let $(\mathsf{U}, \tilde{\sigma}_b)$ be a complete $\mathcal{BMS}$ with $s \geq 1$ and $\Re, \Gamma$ be the pair of $\mathcal{F}_\Theta GAC$. Then, there is $\mathsf{c}^* \in \mathsf{U}$ such that $\mathsf{c}^* \in [\Re(\mathrm{f})]_{\alpha_{\Re(\mathrm{f})}} \cap [\Gamma(\mathrm{f})]_{\alpha_{\Gamma(\mathrm{f})}}$.*

Proof. Let $\mathsf{c}_0 \in \mathsf{U}$. By hypothesis, there is $\alpha_{\Re(\mathsf{c}_0)} \in (0,1]$ such that $[\Re(\mathsf{c}_0)]_{\alpha_{\Re(\mathsf{c}_0)}}$ is non-empty in $\mathsf{CB}(\mathsf{U})$. Let $\mathsf{c}_1 \in [\Re(\mathsf{c}_0)]_{\alpha_{\Re(\mathsf{c}_0)}}$. There is $\alpha_{\Gamma(\mathsf{c}_1)} \in (0,1]$ such that $[\Gamma(\mathsf{c}_1)]_{\alpha_{\Gamma(\mathsf{c}_1)}}$ is non-empty in $\mathsf{CB}(\mathsf{U})$. By using Lemma 2.8 and Definition 3.1,

$$[\Theta(\tilde{\sigma}_b(\mathsf{c}_1, [\Gamma(\mathsf{c}_1)]_{\alpha_{\Gamma(\mathsf{c}_1)}}))] \leq [\Theta(\mathtt{H}([\Re(\mathsf{c}_0)]_{\alpha_{\Re(\mathsf{c}_0)}}, [\Gamma(\mathsf{c}_1)]_{\alpha_{\Gamma(\mathsf{c}_1)}}))].$$

That is,

$$\begin{aligned}\int_0^{[\Theta(s\tilde{\sigma}_b(\mathsf{c}_1, [\Gamma(\mathsf{c}_1)]_{\alpha_{\Gamma(\mathsf{c}_1)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) &\leq \int_0^{[\Theta(s\mathtt{H}([\Re(\mathsf{c}_0)]_{\alpha_{\Re}(\mathsf{c}_0)}, [\Gamma(\mathsf{c}_1)]_{\alpha_{\Gamma(\mathsf{c}_1)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \\ &\leq \int_0^{[\Theta(\Pi(\mathsf{c}_0,\mathsf{c}_1))]^k + L\mathfrak{D}(\mathsf{c}_0,\mathsf{c}_1)} \triangle(\varsigma)\mathrm{d}(\varsigma).\end{aligned} \tag{3.2}$$

Recall that

$$\Theta(\tilde{\sigma}_b(\mathsf{c}_1, [\Gamma(\mathsf{c}_1)]_{\alpha_{\Gamma(\mathsf{c}_1)}})) = \inf_{\mathrm{f}_1 \in [\Gamma(\mathsf{c}_1)]_{\alpha_{\Gamma(\mathsf{c}_1)}}} \Theta(\tilde{\sigma}_b(\mathsf{c}_1, \mathrm{f}_1)).$$

Thus,

$$\inf_{\mathrm{f}_1 \in [\Gamma(\mathsf{c}_1)]_{\alpha_{\Gamma(\mathsf{c}_1)}}} [\Theta(\tilde{\sigma}_b(\mathsf{c}_1, \mathrm{f}_1))] \leq [\Theta(\Pi(\mathsf{c}_0, \mathsf{c}_1))]^k + L\mathfrak{D}(\mathsf{c}_0, \mathsf{c}_1).$$

Now, there is $\mathsf{c}_2 \in [\Gamma(\mathsf{c}_1)]_{\alpha_{\Gamma(\mathsf{c}_1)}}$. So, (3.2) becomes

$$\int_0^{[\Theta(\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\Theta(\Pi(\mathsf{c}_0,\mathsf{c}_1))]^k + L\mathfrak{D}(\mathsf{c}_0,\mathsf{c}_1)} \triangle(\varsigma)\mathrm{d}(\varsigma). \tag{3.3}$$

Now,

$$\Pi(\mathsf{c}_0, \mathsf{c}_1) \leq \max\Big\{\tilde{\sigma}_b(\mathsf{c}_0, \mathsf{c}_1), \tilde{\sigma}_b(\mathsf{c}_0, \mathsf{c}_1), \tilde{\sigma}_b(\mathsf{c}_1, \mathsf{c}_2), \frac{\{\tilde{\sigma}_b(\mathsf{c}_0, \mathsf{c}_2) + \tilde{\sigma}_b(\mathsf{c}_1, \mathsf{c}_1)\}}{2s}\Big\}.$$

Then

$$\Pi(\mathsf{c}_0, \mathsf{c}_1) \leq \max\Big\{\tilde{\sigma}_b(\mathsf{c}_0, \mathsf{c}_1), \tilde{\sigma}_b(\mathsf{c}_1, \mathsf{c}_2)\Big\}.$$

and

$$\mathfrak{D}(\mathsf{c}_0,\mathsf{c}_1) = \min\Big\{\tilde{\sigma}_b(\mathsf{c}_0,\mathsf{c}_1),\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2),\tilde{\sigma}_b(\mathsf{c}_0,\mathsf{c}_2),\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_1)\Big\},$$
$$\mathfrak{D}(\mathsf{c}_0,\mathsf{c}_1) = \tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_1) = 0.$$

If we take

$$\max\Big\{\tilde{\sigma}_b(\mathsf{c}_0,\mathsf{c}_1),\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2)\Big\} = \tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2),$$

then (3.3) becomes

$$\int_0^{[\Theta(s\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2))]} \triangle\,(\varsigma)\mathrm{d}(\varsigma) \le \int_0^{[\Theta(\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2))]^k} \triangle\,(\varsigma)\mathrm{d}(\varsigma).$$

It is a contradiction.

Thus,

$$\max\Big\{\tilde{\sigma}_b(\mathsf{c}_0,\mathsf{c}_1),\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2)\Big\} = \tilde{\sigma}_b(\mathsf{c}_0,\mathsf{c}_1).$$

So, (3.3) becomes

$$\int_0^{[\Theta(s\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2))]} \triangle\,(\varsigma)\mathrm{d}(\varsigma) \le \int_0^{[\Theta(\tilde{\sigma}_b(\mathsf{c}_0,\mathsf{c}_1))]^k} \triangle\,(\varsigma)\mathrm{d}(\varsigma).$$

Now, there is $\alpha_{\Re(\mathsf{c}_2)} \in (0,1]$ such that $[\Re(\mathsf{c}_2)]_{\alpha_{\Re(\mathsf{c}_2)}}$ is non-empty, and $\in \mathsf{CB}(\mathsf{U})$. By Lemma 2.8 and contraction condition, we have

$$\begin{aligned}\int_0^{[\Theta(s\tilde{\sigma}_b([\Re(\mathsf{c}_2)]_{\alpha_{\Re(\mathsf{c}_2)}},\mathsf{c}_2))]} \triangle\,(\varsigma)\mathrm{d}(\varsigma) &\le \int_0^{[\Theta(s\mathrm{H}([\Re(\mathsf{c}_2)]_{\alpha_{\Re(\mathsf{c}_2)}},[\Gamma(\mathsf{c}_1)]_{\alpha_{\Gamma(\mathsf{c}_1)}}))]} \triangle\,(\varsigma)\mathrm{d}(\varsigma)\\ &\le \int_0^{[\Theta(\Pi(\mathsf{c}_1,\mathsf{c}_2))]^k+L\mathfrak{D}(\mathsf{c}_1,\mathsf{c}_2)} \triangle\,(\varsigma)\mathrm{d}(\varsigma).\end{aligned} \tag{3.4}$$

One writes

$$[\Theta(\tilde{\sigma}_b(\mathsf{c}_2,[\Re(\mathsf{c}_2)]_{\alpha_{\Re(\mathsf{c}_2)}}))] = \inf_{\mathrm{f}_2\in[\Re(\mathsf{c}_2)]_{\alpha_{\Re(\mathsf{c}_2)}}} \Theta(\tilde{\sigma}_b(\mathsf{c}_2,\mathrm{f}_2)).$$

Thus,

$$\inf_{\mathrm{f}_2\in[\Re(\mathsf{c}_2)]_{\alpha_{\Re(\mathsf{c}_2)}}} \Theta(\tilde{\sigma}_b(\mathsf{c}_2,\mathrm{f}_2)) \le [\Theta(\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2))]^{\mathrm{k}} + L\mathfrak{D}(\mathsf{c}_1,\mathsf{c}_2).$$

Now, there is $\mathsf{c}_3 \in [\Re(\mathsf{c}_2)]_{\alpha_{\Re(\mathsf{c}_2)}}$, such that (3.4) becomes

$$\int_0^{[\Theta(s\tilde{\sigma}_b(\mathsf{c}_2,\mathsf{c}_3))]} \triangle\,(\varsigma)\mathrm{d}(\varsigma) \le \int_0^{[\Theta(\Pi(\mathsf{c}_1,\mathsf{c}_2))]^k+L\mathfrak{D}(\mathsf{c}_1,\mathsf{c}_2)} \triangle\,(\varsigma)\mathrm{d}(\varsigma). \tag{3.5}$$

Now,

$$\Pi(\mathsf{c}_1,\mathsf{c}_2) \le \max\Big\{\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2),\tilde{\sigma}_b(\mathsf{c}_2,\mathsf{c}_3),\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2),$$
$$\frac{\{\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_3)+\tilde{\sigma}_b(\mathsf{c}_2,\mathsf{c}_2)\}}{2s}\Big\}.$$

Thus,

$$\Pi(\mathsf{c}_1,\mathsf{c}_2) = \max\Big\{\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2),\tilde{\sigma}_b(\mathsf{c}_2,\mathsf{c}_3)\Big\}.$$

and

$$\mathfrak{D}(\mathsf{c}_1,\mathsf{c}_2) = \min\left\{\tilde{\sigma}_b(\mathsf{c}_2,\mathsf{c}_3),\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2),\tilde{\sigma}_b(\mathsf{c}_2,\mathsf{c}_2),\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_3)\right\},$$
$$\mathfrak{D}(\mathsf{c}_1,\mathsf{c}_2) = \tilde{\sigma}_b(\mathsf{c}_2,\mathsf{c}_2) = 0.$$

By taking

$$\max\left\{\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2),\tilde{\sigma}_b(\mathsf{c}_2,\mathsf{c}_3)\right\} = \tilde{\sigma}_b(\mathsf{c}_2,\mathsf{c}_3).$$

the inequality (3.5) becomes

$$\int_0^{[\Theta(s\tilde{\sigma}_b(\mathsf{c}_2,\mathsf{c}_3))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \le \int_0^{[\Theta(\tilde{\sigma}_b(\mathsf{c}_2,\mathsf{c}_3))]^k} \triangle(\varsigma)\mathrm{d}(\varsigma).$$

It is a contradiction. Thus,

$$\max\left\{\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2),\tilde{\sigma}_b(\mathsf{c}_2,\mathsf{c}_3)\right\} = \tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2).$$

So, (3.5) implies that

$$\int_0^{[\Theta(s\tilde{\sigma}_b(\mathsf{c}_2,\mathsf{c}_3))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \le \int_0^{[\Theta(\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2))]^k} \triangle(\varsigma)\mathrm{d}(\varsigma).$$

Following this process, we construct a sequence $\{\mathsf{c}_n\} \in \mathsf{U}$ for all $n \in \mathbb{N}$ such that $\mathsf{c}_{2n+1} \in [\Re(\mathsf{c}_{2n})]_{\alpha_{\Re(\mathsf{c}_{2n})}}]$, and $\mathsf{c}_{2n+2} \in [\Gamma(\mathsf{c}_{2n+1})]_{\alpha_{\Gamma(\mathsf{c}_{2n+1})}}$ in order that

$$\int_0^{[\Theta(s\tilde{\sigma}_b(\mathsf{c}_{2n+1},\mathsf{c}_{2n+2}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \le \int_0^{[\Theta(\tilde{\sigma}_b(\mathsf{c}_{2n},\mathsf{c}_{2n+1}))]^k} \triangle(\varsigma)\mathrm{d}(\varsigma). \tag{3.6}$$

Also,

$$\int_0^{[\Theta(s\tilde{\sigma}_b(\mathsf{c}_{2n+2},\mathsf{c}_{2n+3}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \le \int_0^{[\Theta(\tilde{\sigma}_b(\mathsf{c}_{2n+1},\mathsf{c}_{2n+2}))]^k} \triangle(\varsigma)\mathrm{d}(\varsigma). \tag{3.7}$$

From (3.6) and (3.7) and by using θ_{4s}, one writes

$$\int_0^{[\Theta(s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \le \int_0^{[\Theta(s^{n-1}\tilde{\sigma}_b(\mathsf{c}_{n-1},\mathsf{c}_n))]^k} \triangle(\varsigma)\mathrm{d}(\varsigma). \tag{3.8}$$

This is extended as

$$\begin{aligned}
\int_0^{[\Theta(s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) &\le \int_0^{[\Theta(s^{n-1}\tilde{\sigma}_b(\mathsf{c}_{n-1},\mathsf{c}_n))]^k} \triangle(\varsigma)\mathrm{d}(\varsigma) \\
&\le \int_0^{[\Theta(s^{n-2}\tilde{\sigma}_b(\mathsf{c}_{n-2},\mathsf{c}_{n-1}))]^{k^2}} \triangle(\varsigma)\mathrm{d}(\varsigma) \\
&\le \int_0^{[\Theta(s^{n-3}\tilde{\sigma}_b(\mathsf{c}_{n-3},\mathsf{c}_{n-2}))]^{k^3}} \triangle(\varsigma)\mathrm{d}(\varsigma) \\
&\cdots \\
&\le \int_0^{[\Theta(\tilde{\sigma}_b(\mathsf{c}_0,\mathsf{c}_1))]^{k^n}} \triangle(\varsigma)\mathrm{d}(\varsigma).
\end{aligned}$$

Since $\Theta \in \theta_s$, by taking $\lim\limits_{n\longrightarrow\infty}$, we get

$$\lim_{n\longrightarrow\infty}[\Theta(s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1}))] = 1.$$

Due to (θ_{2s}), we get

$$\lim_{n\longrightarrow\infty} s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1}) = 0^+.$$

Also, in view (θ_{3s}), there are $\mathsf{p} \in (0,1)$ and $\mathsf{r} \in (0,\infty]$ so that

$$\lim_{n\longrightarrow\infty} \frac{\Theta(s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1})) - 1}{(s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1}))^{\mathsf{p}}} = \mathsf{r}.$$

Case 1. Let $\mathsf{r} < \infty$ and $\mathcal{C}_1 > 0$. Hence, there is $n_0 \in \mathbb{N}$ such that $\forall\ n > n_0$,

$$\Big|\frac{\Theta(s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1})) - 1}{(s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1})^{\mathsf{p}}} - \mathsf{r}\Big| \leq \mathcal{C}_1.$$

That is,

$$\frac{\Theta(s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1})) - 1}{(s^n\varphi^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1}))^{\mathsf{p}}} \geq \mathsf{r} - \mathcal{C}_1 = \mathcal{C}_1.$$

Then,

$$n[s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathcal{C}_1}[\Theta(s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1}))) - 1].$$

Case 2. Let $\mathsf{r} = \infty$ and $\mathcal{C}_1 > 0$. Then, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\mathcal{C}_1 \leq \frac{\Theta(s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1}))) - 1}{(s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1}))^{\mathsf{p}}},$$

This gives us

$$n[s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathcal{C}_1}[\Theta(s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1})) - 1].$$

In both cases, $\frac{1}{\mathcal{C}_1} > 0$ and $n_0 \in \mathbb{N}$ so that for all $n > n_0$,

$$n[s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathcal{C}_1}([\Theta(\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1}))] - 1).$$

Now, we have

$$n[s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathcal{C}_1}([\Theta(\tilde{\sigma}_b(\mathsf{c}_0,\mathsf{c}_1))]^{k^n} - 1).$$

Since $\Theta \in \theta_s$, by taking $\lim\limits_{n\longrightarrow\infty}$,

$$\lim_{n\longrightarrow\infty} n[s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1})]^{\mathsf{p}} = 0.$$

There is an integer n_1 so that for all $n > n_1$,

$$n[s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1})]^{\mathsf{p}} \leq 1.$$

This implies that

$$s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1}) \leq \frac{1}{n^{\frac{1}{\mathsf{p}}}}$$

$$\int_0^{[s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1})]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{\frac{1}{n^{\frac{1}{\mathsf{p}}}}} \triangle(\varsigma)\mathrm{d}(\varsigma).$$

Now, we prove $\{\mathsf{c}_n\}$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ such that $m > n > n_0$. By using triangular inequality,

$$\begin{aligned}\tilde{\sigma}_b(\mathsf{c}_n, \mathsf{c}_m) \leq s^n \tilde{\sigma}_b(\mathsf{c}_n, \mathsf{c}_{n+1}) + s^{n+1}\tilde{\sigma}_b(\mathsf{c}_{n+1}, \mathsf{c}_{n+2}) + s^{n+2}\tilde{\sigma}_b(\mathsf{c}_{n+2}, \mathsf{c}_{n+3}) + \\ \cdots + s^{m-1}\tilde{\sigma}_b(\mathsf{c}_{m-1}, \mathsf{c}_m).\end{aligned}$$

From above, we can write

$$\begin{aligned}\int_0^{\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_m)} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \Big[\int_0^{s^n\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1})} \triangle(\varsigma)\mathrm{d}(\varsigma) + \int_0^{s^{n+1}\tilde{\sigma}_b(\mathsf{c}_{n+1},\mathsf{c}_{n+2})} \triangle(\varsigma)\mathrm{d}(\varsigma) + \\ \int_0^{s^{n+2}\tilde{\sigma}_b(\mathsf{c}_{n+2},\mathsf{c}_{n+3})} \triangle(\varsigma)\mathrm{d}(\varsigma) + \cdots + \int_0^{s^{m-1}\tilde{\sigma}_b(\mathsf{c}_{m-1},\mathsf{c}_m)} \triangle(\varsigma)\mathrm{d}(\varsigma)\Big].\end{aligned}$$

That is,

$$\begin{aligned}\int_0^{\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_m)} \triangle(\varsigma)\mathrm{d}(\varsigma) &\leq \sum_{i=n}^{m-1} \int_0^{s^i\varphi^i\tilde{\sigma}_b(\mathsf{c}_n,\mathsf{c}_{n+1})} \triangle(\varsigma)\mathrm{d}(\varsigma) \\ &\leq \sum_{i=n}^{m-1} \int_0^{\frac{1}{n^{\frac{1}{\mathsf{p}}}}} \triangle(\varsigma)\mathrm{d}(\varsigma) \\ &\leq \sum_{i=n}^{\infty} \int_0^{\frac{1}{n^{\frac{1}{\mathsf{p}}}}} \triangle(\varsigma)\mathrm{d}(\varsigma).\end{aligned}$$

Since $0 < \mathsf{p} < 1$, the series $\sum_{i=n}^{\infty} \int_0^{\frac{1}{n^{\frac{1}{\mathsf{p}}}}} \triangle(\varsigma)\mathrm{d}(\varsigma)$ converges. When $m, n \longrightarrow \infty$, we obtain $\tilde{\sigma}_b(\mathsf{c}_n, \mathsf{c}_m) \longrightarrow 0$. So, it is clear that $\{\mathsf{c}_n\} \in \mathsf{U}$ is a Cauchy sequence in $(\mathsf{U}, \tilde{\sigma}_b)$. By completeness there is $\mathsf{c}^* \ni \{\mathsf{c}_n\} \longrightarrow \mathsf{c}^*$. Next, we claim that $\{\mathsf{c}^*\} \in [\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}$. Assume, on the contrary, that c^* does not belong to $[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}$ (that is, $\tilde{\sigma}_b(\mathsf{c}^*, [\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}) > 0$), then there are $n_0 \in \mathbb{N}$ and a subsequence $\{\mathsf{c}_{n_k}\}$ of c_n so that $\tilde{\sigma}_b(\mathsf{c}_{2n_k+1}, [\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}) > 0$, for all $n_k \geq n_0$. Since $\tilde{\sigma}_b(\mathsf{c}_{2n_k+1}, [\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}) > 0$, we have

$$[\Theta(\tilde{\sigma}_b(\mathsf{c}_{2n_k+1}, [\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}))] \leq [\Theta(\mathtt{H}([\Re(\mathsf{c}_{2n_k})]_{\alpha_{\Re(\mathsf{c}_{2n_k})}}, [\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}))]. \tag{3.9}$$

We have

$$\begin{aligned}\int_0^{[\Theta(s\tilde{\sigma}_b(\mathsf{c}_{2n_k+1},[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) &\leq \int_0^{[\Theta(s\mathtt{H}([\Re(\mathsf{c}_{2n_k})]_{\alpha_{\Re(\mathsf{c}_{2n_k})}},[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \\ &\leq \int_0^{[\Theta(s\tilde{\sigma}_b(\mathsf{c}_{2n_k+1},\mathsf{c}^*))]^k} \triangle(\varsigma)\mathrm{d}(\varsigma).\end{aligned} \tag{3.10}$$

Now,

$$\begin{aligned}\Pi(\mathsf{c}_{2n_k},\mathsf{c}^*) &= \max\Big\{\tilde{\sigma}_b(\mathsf{c}_{2n_k},\mathsf{c}^*),\tilde{\sigma}_b(\mathsf{c}_{2n_k},[\Re(\mathsf{c}_{2n_k})]_{\alpha_{\Re(\mathsf{c}_{2n_k})}}),\tilde{\sigma}_b(\mathsf{c}^*,[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}),\\ &\qquad \frac{\{\tilde{\sigma}_b(\mathsf{c}_{2n_k},[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}})+\tilde{\sigma}_b(\mathsf{c}^*,[\Re(\mathsf{c}_{2n_k})]_{\alpha_{\Re(\mathsf{c}_{2n_k})}})\}}{2s}\Big\}.\\ &\le \max\Big\{\tilde{\sigma}_b(\mathsf{c}_{2n_k},\mathsf{c}^*),\tilde{\sigma}_b(\mathsf{c}_{2n_k},\mathsf{c}_{2n_k+1}),\tilde{\sigma}_b(\mathsf{c}^*,[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}),\\ &\qquad \frac{\{\tilde{\sigma}_b(\mathsf{c}_{2n_k},[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}})+\tilde{\sigma}_b(\mathsf{c}^*,\mathsf{c}_{2n_k+1})\}}{2s}\Big\}.\end{aligned}$$

$$\begin{aligned}\mathfrak{D}(\mathsf{c}_{2n_k},\mathsf{c}^*) &= \min\Big\{\tilde{\sigma}_b(\mathsf{c}_{2n_k},[\Re(\mathsf{c}_{2n_k})]_{\alpha_{\Re(\mathsf{c}_{2n_k})}}),\tilde{\sigma}_b(\mathsf{c}^*,[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}),\\ &\qquad \tilde{\sigma}_b(\mathsf{c}_{2n_k},[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}})+\tilde{\sigma}_b(\mathsf{c}^*,[\Re(\mathsf{c}_{2n_k})]_{\alpha_{\Re(\mathsf{c}_{2n_k})}})\}.\\ &\le \min\Big\{\tilde{\sigma}_b(\mathsf{c}_{2n_k},\mathsf{c}_{2n_k+1}),\tilde{\sigma}_b(\mathsf{c}^*,[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}),\\ &\qquad \tilde{\sigma}_b(\mathsf{c}_{2n_k},[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}})+\tilde{\sigma}_b(\mathsf{c}^*,\mathsf{c}_{2n_k+1})\Big\}.\end{aligned}$$

Letting n $\longrightarrow \infty$ and using the continuity of Θ, the equation (3.10) becomes

$$\int_0^{[\Theta(s\tilde{\sigma}_b(\mathsf{c}^*,[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}))]}\triangle(\varsigma)\mathrm{d}(\varsigma)\le\int_0^{[\Theta(\tilde{\sigma}_b(\mathsf{c}^*,[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}))]^k}\triangle(\varsigma)\mathrm{d}(\varsigma).$$

This leads to a contradiction. Hence, $\tilde{\sigma}_b(\mathsf{c}^*,[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}})=0$, and $\mathsf{c}^*\in[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}$. Similarly, one can easily prove that $\mathsf{c}^*\in[\Re(\mathsf{c}^*)]_{\alpha_{\Re(\mathsf{c}^*)}}$. Therefore, $\mathsf{c}^*\in[\Gamma(\mathsf{c}^*)]_{\alpha_{\Gamma(\mathsf{c}^*)}}\cap[\Re(\mathsf{c}^*)]_{\alpha_{\Re(\mathsf{c}^*)}}$. □

Remark 3.3. To ensure the mathematical correctness of Theorem 3.3, we assume that the function $\Delta:[0,\infty)\to[0,\infty)$ is non-increasing in addition to being Lebesgue integrable. This guarantees that the function $F(x)=\int_0^x\triangle(\varsigma)\mathrm{d}(\varsigma)$. is subadditive, which is essential for the Cauchy sequence argument.

Moreover, the use of Θ in the integral upper limit combines the Θ-contraction mechanism with a Branciari-type integral contraction. While unconventional, we adopt this construction under the above assumption on Δ to maintain the contractive property and ensure convergence of the sequence $\{c_n\}$.

Example 3.4. Let $\mathsf{U}=[0,1]$ be defined $\tilde{\sigma}_b:\mathsf{U}\times\mathsf{U}\longrightarrow\mathbb{R}$ as $\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2)=|\mathsf{c}_1-\mathsf{c}_2|^2$, then $(\mathsf{U},\tilde{\sigma}_b)$ is a complete $\mathcal{BMS}$ with $s=2$. Define $\Re,\Gamma:\mathsf{U}\to\digamma(\mathsf{U})$ by

$$\Re(\mathsf{c}_1)\mathsf{t}=\begin{cases}\alpha, & \text{if } 0\le\mathsf{t}\le\frac{\mathsf{c}_1}{15},\\ \frac{\alpha}{15}, & \text{if } \frac{\mathsf{c}_1}{15}\le\mathsf{t}\le\frac{\mathsf{c}_1}{10},\\ \frac{\alpha}{10}, & \text{if } \frac{\mathsf{c}_1}{10}\le\mathsf{t}\le\frac{\mathsf{c}_1}{5},\\ \frac{\alpha}{5}, & \text{if } \frac{\mathsf{c}_1}{5}\le\mathsf{t}\le 1.\end{cases}$$

$$\Gamma(\mathsf{c}_2)\mathsf{t}=\begin{cases}\alpha, & \text{if } 0\le\mathsf{t}\le\frac{\mathsf{c}_2}{30},\\ \frac{\alpha}{30}, & \text{if } \frac{\mathsf{c}_2}{30}\le\mathsf{t}\le\frac{\mathsf{c}_2}{20},\\ \frac{\alpha}{20}, & \text{if } \frac{\mathsf{c}_2}{20}\le\mathsf{t}\le\frac{\mathsf{c}_2}{10},\\ \frac{\alpha}{10}, & \text{if } \frac{\mathsf{c}_2}{10}\le\mathsf{t}\le 1.\end{cases}$$

Now, for α_Γ and $\alpha_\Re = 1$, we have

$$[\Re(\mathsf{c}_1)]_\alpha = [0, \frac{\mathsf{c}_1}{15}],$$
$$[\Gamma(\mathsf{c}_2)]_\alpha = [0, \frac{\mathsf{c}_2}{30}].$$

Since $\mathtt{H}([\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}}, [\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}}) = |\frac{\mathsf{c}_1}{15} - \frac{\mathsf{c}_2}{30}|^2 \geq 0$ for $\mathsf{c}_1 \neq \mathsf{c}_2$. Now,

$$\Pi(\mathsf{c}_1, \mathsf{c}_2) = \max\Big\{\tilde{\sigma}_b(\mathsf{c}_1, \mathsf{c}_2), \tilde{\sigma}_b(\mathsf{c}_1, [\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}}), \tilde{\sigma}_b(\mathsf{c}_2, [\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}}), \frac{\{\tilde{\sigma}_b(\mathsf{c}_1, [\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}}) + \tilde{\sigma}_b(\mathsf{c}_2, [\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}})\}}{2s}\Big\}.$$

That is,

$$\begin{aligned}\Pi(\mathsf{c}_1, \mathsf{c}_2) &= \max\Big\{|\mathsf{c}_1 - \mathsf{c}_2|^2, |\mathsf{c}_1 - \frac{\mathsf{c}_1}{15}|^2, |\mathsf{c}_2 - \frac{\mathsf{c}_2}{30}|^2, \ \frac{\{|\mathsf{c}_1 - \frac{\mathsf{c}_2}{30}|^2 + |\mathsf{c}_2 - \frac{\mathsf{c}_1}{15}|^2\}}{5}\Big\}, \\ &\leq |\mathsf{c}_1 - \mathsf{c}_2|^2,\end{aligned}$$

and

$$\mathfrak{D}(\mathsf{c}_1, \mathsf{c}_2) = \min\Big\{\tilde{\sigma}_b(\mathsf{c}_1, [\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}}), \tilde{\sigma}_b(\mathsf{c}_2, [\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}}), \tilde{\sigma}_b(\mathsf{c}_1, [\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}}), \tilde{\sigma}_b(\mathsf{c}_2, [\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}}\Big\}.$$

We have

$$\begin{aligned}\mathfrak{D}(\mathsf{c}_1, \mathsf{c}_2) &= \min\Big\{|\mathsf{c}_1 - \frac{\mathsf{c}_1}{15}|^2, |\mathsf{c}_2 - \frac{\mathsf{c}_2}{30}|^2, |\mathsf{c}_1 - \frac{\mathsf{c}_2}{30}|^2, |\mathsf{c}_2 - \frac{\mathsf{c}_1}{15}|^2)\Big\} \\ &\leq |\frac{14}{15}\mathsf{c}_1|^2.\end{aligned}$$

Taking $\Theta(\mathrm{t}) = 2^{\sqrt[k]{\mathrm{t}}}$, for some $k = \frac{1}{\sqrt{3}} \in (0, 1)$, we have

$$\int_0^{[\Theta(s\mathtt{H}([\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}}, [\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}}))]} \triangle(\varsigma) \mathrm{d}(\varsigma) \leq \int_0^{[\Theta(\Pi(\mathsf{c}_1, \mathsf{c}_2))]^k + L\mathfrak{D}(\mathsf{c}_1, \mathsf{c}_2)} \triangle(\varsigma) \mathrm{d}(\varsigma), \quad (3.11)$$

$$\begin{aligned}\Rightarrow \int_0^{[\Theta(s(|\frac{\mathsf{c}_1}{15} - \frac{\mathsf{c}_2}{30}|^2))]} \triangle(\varsigma) \mathrm{d}(\varsigma) &\leq \int_0^{[\Theta(|\mathsf{c}_1 - \mathsf{c}_2|^2)]^k + |\frac{14}{15}\mathsf{c}_1|^2} \triangle(\varsigma) \mathrm{d}(\varsigma), \\ &\leq \int_0^{[\Theta(|\mathsf{c}_1 - \mathsf{c}_2|^2)]^k} \triangle(\varsigma) \mathrm{d}(\varsigma),\end{aligned}$$

All assumptions of Theorem 3.2 with $L = 1$ are satisfied, and 0 is the $\mathcal{FP}$ of $\Re$ and Γ.

If we consider $L = 0$ in Definition (3.1), then we have the following:

Theorem 3.5. *Let $(\mathsf{U}, \tilde{\sigma}_b)$ be a complete $\mathcal{BMS}$ and let $\Re, \Gamma$ be the pair of $\mathcal{FM}$s such that for each $\mathsf{c}_1, \mathsf{c}_2 \in \mathsf{U}$ then $\exists\ \alpha_{\Re(\mathsf{c}_1)}, \alpha_{\Gamma(\mathsf{c}_2)} \in (0, 1]$ such that $[\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}}, [\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}} \in \mathsf{CB}(\mathsf{U})$ are non-empty, and $\forall\ \mathsf{c}_1, \mathsf{c}_2 \in \mathsf{U}$, $\mathtt{H}([\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}}, [\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}}) > 0$,*

$$\Rightarrow \int_0^{[\Theta(s\mathrm{H}([\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}},[\Gamma(\mathsf{c}_2)]_{\alpha_{\Gamma(\mathsf{c}_2)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\Theta(\Pi(\mathsf{c}_1,\mathsf{c}_2))]^k} \triangle(\varsigma)\mathrm{d}(\varsigma), \quad (3.12)$$

where $\Theta \in \theta_s$ and $\Pi(\mathsf{c}_1, \mathsf{c}_2)$ is the same as in Definition 3.1. Then, there is a common α-fuzzy $\mathcal{FP}$ of $\Re$ and Γ.

Remark 3.6. The following consequences follow from Theorem 3.2:

(1) By taking $\triangle(\varsigma) = 1$ in Definition 3.1, Theorem 3.2 reduces to the Θ-type contraction case and guarantees the existence of a common fuzzy $\mathcal{FP}$ in U.
(2) By setting $\Gamma = \Re$ in Theorem 3.2, we obtain the existence of a fuzzy $\mathcal{FP}$ of the mapping $\Re$ in U.
(3) By choosing $L = 0$, the contraction condition simplifies to a pure Θ-contraction, and the existence of the corresponding fuzzy $\mathcal{FP}$ still holds.

Remark 3.7. If we take $s = 1$, $\Pi = \tilde{\sigma}(\mathsf{c}_1, \mathsf{c}_2)$ and $L = 0$, then the results of Kanwal et al. [30] become a special case of our results.

4. Application

In this section, we will find out the solution of the Volterra stochastic integral equation system. If $\mho$ is a non-empty set, A is an ξ-algebra of subsets of $\mho$ and $\mathtt{P}$ is a complete probability measure of A, then denote by $(\mho,\mathsf{A}, \mathtt{P})$ a probability measure space. Let $\mathbb{R}^+ = [0, \infty)$. The expression $\mathsf{C} = \mathsf{C}(\mathbb{R}_+, \mathcal{L}_2(\mho,\mathsf{A}, \mathtt{P}))$ is used to represent the space of all continuous and bounded functions on $\mathbb{R}^+$ with values in $\mathcal{L}_2 = \mathcal{L}_2(\mho,\mathsf{A}, \mathtt{P})$. Consider the following system of Volterra stochastic integral equations:

$$\mathsf{c}_1(\mathsf{t}, \mu) = \int_0^{\mathsf{t}} \mathsf{K}_1(\mathsf{t}, \mathsf{p}, \mu)\mathsf{f}(\mathsf{p}, \mathsf{c}_1(\mathsf{p}, \mu))\, d\mathsf{p} \; + \beta(\mathsf{t}, \mu), \quad (4.1)$$

and

$$\mathsf{c}_1(\mathsf{t}, \mu) = \int_0^{\mathsf{t}} \mathsf{K}_2(\mathsf{t}, \mathsf{p}, \mu)\mathsf{f}(\mathsf{p}, \mathsf{c}_1(\mathsf{p}, \mu))\, d\mathsf{p} \; + \beta(\mathsf{t}, \mu), \quad (4.2)$$

where $\mathsf{t} \geq 0$, μ is a point of the non-empty set $\mho$, $\beta(\mathsf{t}, \mu)$ is the stochastic free term, $\mathsf{c}_1(\mathsf{t}, \mu)$ are unknown stochastic variables, $\mathsf{K}_1, \mathsf{K}_2$ are stochastic kernels defined on $0 \leq \mathsf{p} \leq \mathsf{t} < \infty$, and f is a scalar function defined for $\mathsf{t} \geq 0$.

A function that belongs to $\mathsf{C}(\mathbb{R}_+, \mathcal{L}_2(\mho,\mathsf{A}, \mathtt{P}))$ and satisfies (4.1) and (4.2) is referred to a random solution.

Define $\mathcal{BMS}$ by

$$\tilde{\sigma}_b(\mathsf{c}_1, \mathsf{c}_2) = \max_{\mathsf{t}\in[0,\mathsf{t}]} |\mathsf{c}_1(\mathsf{t}) - \mathsf{c}_2(\mathsf{t})|^2,$$

$\forall\ \mathsf{c}_1, \mathsf{c}_2 \in \mathsf{C}[0, \mathsf{t}]$. Then, $(\mathsf{C}[0, \mathsf{t}], \tilde{\sigma}_b)$ is a complete $\mathcal{BMS}$ with $s = 2$.

Theorem 4.1. *Assume the following assumptions:*

(i): $\mathsf{f} : \mathsf{C} \longrightarrow \mathsf{C},\ \beta : \mathbb{R}_+ \longrightarrow \mathcal{L}_2, \mathsf{K}_1, \mathsf{K}_2 : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{L}_2 \longrightarrow \mathcal{L}_2$ *are continuous.*

(ii): $\forall\ \mathsf{K}_1, \mathsf{K}_2 \in \mathsf{K}$ *and* $\lambda \in (0, 1)$. *we have*

$$||\mathsf{K}_1(\mathsf{t}, \mathsf{p}, \mu) - \mathsf{K}_2(\mathsf{t}, \mathsf{p}, \mu)|| \leq (\lambda)\mathsf{K}$$

(iii):

$$\begin{aligned} ||\mathsf{f}(\mathsf{t},\mathsf{c}_1(\mathsf{t},\mu)) - \mathsf{f}(\mathsf{t},\mathsf{c}_2(\mathsf{t},\mu))|| &\leq ||\mathsf{c}_1(\mathsf{t},\mu) - \mathsf{c}_2(\mathsf{t},\mu)|| \\ &\leq \max_{\mathsf{t}\in[0,\mathsf{t}]} |\mathsf{c}_1(\mathsf{t},\mu) - \mathsf{c}_2(\mathsf{t},\mu)|^2 \end{aligned}$$

Then, the integral equations (4.1) and (4.2) have a common solution in $\mathsf{C}[0,\mathsf{t}]$.

Proof. Consider $\mathsf{U} = \mathsf{C} = \mathsf{C}(\mathbb{R}_+, \mathcal{L}_2(\mho, \mathsf{A}, \mathtt{P}))$ to be endowed with the uniform norm $||.||$. Then, $(\mathsf{U}, ||.||)$ is a Banach space. Let $\mathcal{P}_{(\mathsf{c}_1)}, \mathtt{E}_{(\mathsf{c}_1)} \in \mathsf{U}$, where

$$\mathcal{P}_{(\mathsf{c}_1)} = \int_0^{\mathsf{t}} \mathsf{K}_1(\mathsf{t},\mathsf{p},\mu)\mathsf{f}(\mathsf{p},\mathsf{c}_1(\mathsf{p},\mu))\,d\mathsf{p} \;+ \beta(\mathsf{t},\mu),$$

$$\mathtt{E}_{(\mathsf{c}_1)} = \int_0^{\mathsf{t}} \mathsf{K}_2(\mathsf{t},\mathsf{p},\mu)\mathsf{f}(\mathsf{p},\mathsf{c}_2(\mathsf{p},\mu))\,d\mathsf{p} \;+ \beta(\mathsf{t},\mu).$$

Given two arbitrary mappings $\mathcal{A}, \mathcal{B} : \mathsf{U} \longrightarrow (0,1]$ of $\mathcal{FM}$, and the pair $\Gamma, \Re :$ $\mathsf{U} \longrightarrow \digamma(\mathsf{U})$ be given as

$$\Gamma(\mathsf{c}_1)(\mathsf{t}) = \begin{cases} \mathcal{A}(\mathsf{c}_1), \text{ if } \mathsf{c}_1(\mathsf{t}) = \mathcal{P}_{(\mathsf{c}_1)}, \\ 0, \textit{ otherwise.} \end{cases}$$

Also,

$$\Re(\mathsf{c}_1)(\mathsf{t}) = \begin{cases} \mathcal{B}(\mathsf{c}_1), \text{ if } \mathsf{c}_1(\mathsf{t}) = \mathtt{E}_{(\mathsf{c}_1)}, \\ 0, \textit{ otherwise.} \end{cases}$$

If we take $\alpha_{\Gamma_{\mathsf{c}_1}} = \mathcal{A}(\mathsf{c}_1)$ and $\alpha_{\Re_{\mathsf{c}_1}} = \mathcal{B}(\mathsf{c}_1)$, we have

$$\begin{aligned} [\Gamma(\mathsf{c}_1)]_{\alpha_{\Gamma(\mathsf{c}_1)}} &= \Big\{\Re : \Gamma(\mathsf{c}_1)(\Re) \geq \mathcal{A}(\mathsf{c}_1)\Big\} = \mathcal{P}_{(\mathsf{c}_1)}. \\ [\Re(\mathsf{c}_1)]_{\alpha_{\Re(\mathsf{c}_1)}} &= \Big\{\Re : \Re(\mathsf{c}_1)(\Re) \geq \mathcal{B}(\mathsf{c}_1)\Big\} = \mathtt{E}_{(\mathsf{c}_1)}. \end{aligned}$$

Now, $\mathtt{H}([\Gamma(\mathsf{c}_1)]_{\alpha_{\Gamma(\mathsf{c}_1)}}, [\Re(\mathsf{c}_2)]_{\alpha_{\Re(\mathsf{c}_2)}}) = ||\mathcal{P}_{(\mathsf{c}_1)} - \mathtt{E}_{(\mathsf{c}_1)}||.$

$$
\begin{aligned}
&\int_0^{[\Theta(s\mathtt{H}([\Gamma(\mathsf{c}_1)]_{\alpha_{\Gamma(\mathsf{c}_1)}},[\Re(\mathsf{c}_2)]_{\alpha_{\Re(\mathsf{c}_2)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma)\\
&= \int_0^{[\Theta(||\mathcal{P}_{(\mathsf{c}_1)}-\mathsf{E}_{(\mathsf{c}_1)}||)]} \triangle(\varsigma)\mathrm{d}(\varsigma)\\
&= \int_0^{[\Theta(||(\int_0^{\mathsf{t}} \mathsf{K}_1(\mathsf{t},\mathsf{p},\mu)\mathsf{f}(\mathsf{p},\mathsf{c}_1(\mathsf{p},\mu))\,d\mathsf{p}\ +\beta(\mathsf{t},\mu))-(\int_0^{\mathsf{t}} \mathsf{K}_2(\mathsf{t},\mathsf{p},\mu)\mathsf{f}(\mathsf{p},\mathsf{c}_1(\mathsf{p},\mu))\,d\mathsf{p}\ +\beta(\mathsf{t},\mu))||)]} \triangle(\varsigma)\mathrm{d}(\varsigma).\\
&\leq \int_0^{[\Theta(||\int_0^{\mathsf{t}} \mathsf{K}_1(\mathsf{t},\mathsf{p},\mu)\,d\mathsf{p}\ -\int_0^{\mathsf{t}} \mathsf{K}_2(\mathsf{t},\mathsf{p},\mu)\,d\mathsf{p}\ ||||\mathsf{f}(\mathsf{p},\mathsf{c}_1(\mathsf{p},\mu))-\mathsf{f}(\mathsf{p},\mathsf{c}_2(\mathsf{p},\mu))||)]} \triangle(\varsigma)\mathrm{d}(\varsigma)\\
&\leq \int_0^{[\Theta(\lambda\mathsf{K}||\mathsf{c}_1(\mathsf{t},\mu)-\mathsf{c}_2(\mathsf{t},\mu)||)]^{\frac{1}{v}}} \triangle(\varsigma)\mathrm{d}(\varsigma)\\
&\leq \int_0^{[\Theta(||\mathsf{c}_1(\mathsf{t},\mu)-\mathsf{c}_2(\mathsf{t},\mu)||)]^{\frac{1}{v}}} \triangle(\varsigma)\mathrm{d}(\varsigma)\\
&\leq \int_0^{[\Theta([\max_{\mathsf{t}\in[0,a]}|\mathsf{c}_1(\mathsf{t})-\mathsf{c}_2(\mathsf{t})|^2)]^{\frac{1}{v}}} \triangle(\varsigma)\mathrm{d}(\varsigma)\\
&\leq \int_0^{[\Theta(\tilde{\sigma}_b(\mathsf{c}_1,\mathsf{c}_2))]^{\frac{1}{v}}} \triangle(\varsigma)\mathrm{d}(\varsigma),
\end{aligned}
$$

where $\frac{1}{v} = k \in (0,1)$. So, all the conditions of Theorem 3.2 are satisfied. Hence, the integral equations (4.1) and (4.2) have a common solution. □

5. Conclusion

In this work, we established the existence of common fixed points $\mathcal{FP}$s for a pair of $\mathcal{FM}$s and multi-valued mappings within the framework of a complete $\mathcal{BMS}$. A generalized integral-type contraction is employed to achieve the required goal. An example is also constructed, which authenticates the axioms of the proven result. In addition, we applied the idea of $\mathcal{FM}$ to prove the existence of a solution to a set of stochastic Volterra integral equations.

Data Availability. The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest. The authors declare that they have no competing interests.

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