Gulf Journal of Mathematics
Vol 22, Issue 2 (2026) 1-21

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v22i2.4157

THE SELF INVERSE ELEMENT GRAPH OVER A RING

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Abstract. In this paper, the self-inverse element graph $S_i(\mathcal{R})$ of a commutative ring $\mathcal{R}$ with unity is introduced as a simple undirected graph whose vertex set is $\mathcal{R}$, where two distinct vertices in $\mathcal{R}$ are adjacent iff their sum is a self-inverse element of $\mathcal{R}$. Various graph-theoretic properties of this graph, including size, regularity, connectedness, girth, completeness, bipartiteness, and planarity, are established for arbitrary commutative rings. Conditions under which the graph is a path or a cycle are also obtained. Furthermore, depending on the characteristic, graphical properties of the self-inverse element graph of the Cartesian product of the rings $\mathbb{Z}_n$ $(n \geq 2)$ are examined, and the structure of the graph for finite fields is completely determined.

Keywords. Self inverse element graph, Connectedness, Girth, Regular graph.
2020 Mathematics Subject Classification. Primary 05C25; Secondary 05C10, 05C07, 05C30.

1. Introduction

Algebraic graph theory is an vibrant research field in the present era and can be investigated at various mathematical skill levels having many applications in various disciplines. The understanding of the algebraic properties of rings is an inspiration for new ideas to investigate various graph theoretic properties and will help to settle many difficult problems. Moreover, by translating the algebraic attributes of rings into graph theoretic language, various complicated problems become easier to answer. The connection between ring theoretic and graph theoretic characteristics has enticed many researchers (see [4, 6, 14]).
An algebraic structure can be seen as a graph and there are numerous methods available to relate it with a graph. Firstly, this association was presented by Arthur Cayley by connecting a graph with a group and named it the Cayley graph of a group (see [11]). There is another significant category of such types of constructions which can be constructed over a ring. For example, Beck [6], introduced the concept of zero divisor graphs with the set of vertex set equal to a ring $\mathcal{R}$ and two distinct vertices are adjacent ot each other iff one is the zero divisor of the other. Later, Andrson and Livingston [4] extended this definition

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Date: Received: Jan 26, 2026; Revised: Feb 23, 2026; Accepted: March 7, 2026.
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by reducing the set of vertex $\mathcal{R}$ into a set of all nontrivial zero divisors in $\mathcal{R}$. In recent years, the researchers explored different definitions regarding algebraic graphs to analyze the properties of ring theory in terms of graph theory such as the total graph, square element graph, co-maximal graph, annihilator graphs of a ring etc. (see [3, 8, 13, 14]). Particularly, Ashrafi et. al. [5], defined the unit graph of R, in which vertex set is R, where two distinct vertices $x, y \in R$ are adjacent iff $x + y$ is a unit in R. Thereafter, Abdelkarim et. al [1], examined key structural and graph-theoretic properties of the unit graph over the ring $\mathbb{Z}_{p^k}$, including aspects such as adjacency, connectivity, and degree behavior. This motivated us to define and characterize the properties of the self inverse element graph $S_i(\mathcal{R})$ over a ring $\mathcal{R}$ for which we consider the set of all the self inverse elements in $\mathcal{R}$ given by $S(\mathcal{R}) = \{x \in \mathcal{R} | x^2 = 1\}$. In the unit graph, two vertices are adjacent whenever their sum is a unit, whereas in the self-inverse element graph, two vertices are adjacent whenever their sum is a self-inverse element. Since every self-inverse element is necessarily a unit, therefore the edge set of $S_i(\mathcal{R})$ is contained in the edge set of unit graph. Hence, $S_i(\mathcal{R})$ is a spanning subgraph of the unit graph. Moreover, the two graphs coincide precisely when every unit of $\mathcal{R}$ is self-inverse, whereas the presence of a unit of order greater than two yields a proper containment $S_i(\mathcal{R}) \subsetneq \mathcal{G}((R))$, where $\mathcal{G}((R))$ is the unit graph. This establishes a natural structural refinement of the unit graph determined by the involution structure of the unit group of the ring. For basic terminology and notations, we follow the references [15, 16].

Section 2 presents the notion of a self inverse element graph $S_i(\mathcal{R})$ over a finite commutative ring $\mathcal{R}$ with unity and analyze its various graph theoretic characteristics including size, completeness, regularity, etc. Furthermore, we completely determine the structure of self inverse element graph for finite fields. Section 3 focuses on the connectedness of self inverse element graph over the ring $\mathbb{Z}_n$ and their cartesian product, also examine all rings for which $S_i(\mathcal{R})$ is a path graph or cycle graph. In the last section, we completely characterize girth and bipartiteness of $S_i(\mathcal{R})$ for a ring $\mathcal{R}$. Along with this, we investigate the planarity of $S_i(\mathcal{R})$ for an arbitrary commutative ring $\mathcal{R}$ with unity.

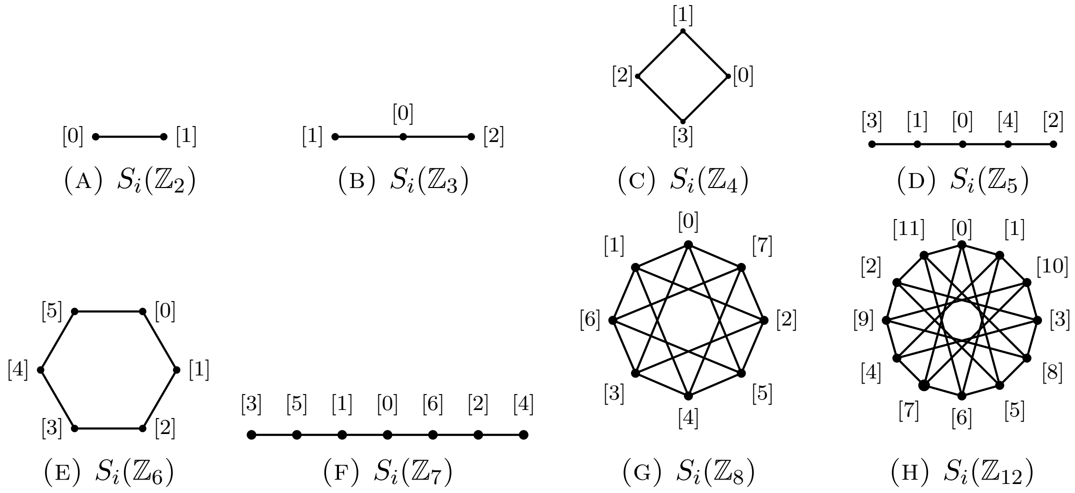
2. THE GRAPH $S_i(\mathcal{R})$ AND ITS CHARACTERIZATIONS

We begin this section with the definition of the self inverse element graph over a commutative ring with unity $\mathcal{R}$.

Definition 2.1. Let $S(\mathcal{R})$ be the set of self inverse elements of $\mathcal{R}$. The self inverse element graph $S_i(\mathcal{R})$ over $\mathcal{R}$ is a simple undirected graph $G = (V, E)$, where $V = \mathcal{R}$ and $ab \in E$ iff $a \neq b$ and $a + b = x$, for some $x \in S(\mathcal{R})$ i.e. $x^2 = 1$.

Some examples of self inverse element graphs are discussed below in Figure 1 and 2.

Observe that if $\mathcal{R} = 0$, then $S_i(\mathcal{R})$ is the trivial graph with one vertex, which is trivially a complete graph. Furthermore, it is observed that $S_i(\mathbb{Z}_2), S_i(\mathbb{Z}_3), S_i(\mathbb{Z}_5)$ and $S_i(\mathbb{Z}_7)$ are the path graphs P_2, P_3, P_5 and P_7 respectively, while $S_i(\mathbb{Z}_4)$ and $S_i(\mathbb{Z}_6)$ are the cycle graphs C_4 and C_6 respectively (see Figure 1a-1f). Also, it is significant to observe from Figure 1c and 2b that two non-isomorphic ring could



(A) $S_i(\mathbb{Z}_2)$ (B) $S_i(\mathbb{Z}_3)$ (C) $S_i(\mathbb{Z}_4)$ (D) $S_i(\mathbb{Z}_5)$

(E) $S_i(\mathbb{Z}_6)$ (F) $S_i(\mathbb{Z}_7)$ (G) $S_i(\mathbb{Z}_8)$ (H) $S_i(\mathbb{Z}_{12})$

FIGURE 1. Self inverse element graph of $\mathbb{Z}_n$, for small n.

(A) $S_i(\mathbb{Z}_2 \times \mathbb{F}_4)$ (B) $S_i(\mathbb{F}_2[x]/(x^2))$

FIGURE 2. Self inverse element graphs for $\mathbb{Z}_2 \times \mathbb{F}_4$ and $\mathbb{F}_2[x]/(x^2)$

have identical self inverse element graphs.

Recall that the set of self inverse elements in a ring $\mathcal{R}$ is $S(\mathcal{R}) = \{a \in \mathcal{R} | a^2 = 1\}$ and $S(\mathcal{R}) \neq \emptyset$ for every ring $\mathcal{R}$, as $1 \in S(\mathcal{R})$. Moreover, for $a \in S(\mathcal{R})$, we have $-a \in S(\mathcal{R})$. With this observation, we have

Lemma 2.2. *For a ring $\mathcal{R}$, $|S(\mathcal{R})| = 2^r$ for some non-negative integer r.*

Additionally, the degree of the vertex 0 in $S_i(\mathcal{R})$ is the number of self inverse element in $\mathcal{R}$ i.e., $deg(0) = |S(\mathcal{R})|$. In fact, it follows from the definition of $S_i(\mathcal{R})$ that the self inverse elements are the only elements which are adjacent to 0. With these observations in mind, the remaining section is devolved to study the basic properties of $S_i(\mathcal{R})$ like completeness, regularity, size and number of connected components in $S_i(\mathcal{R})$ etc.

Theorem 2.3. *If $\mathcal{R}$ is a ring, then $S_i(\mathcal{R})$ is complete iff $\mathcal{R} = \mathbb{Z}_2$.*

Proof. The graph $S_i(\mathcal{R})$ is complete iff each pair of distinct vertices in $S_i(\mathcal{R})$ are adjacent to each other. In particular, every non-zero vertex of $S_i(\mathcal{R})$ is adjacent to the zero vertex of $S_i(\mathcal{R})$. This infers that each non-zero vertex of $S_i(\mathcal{R})$ is a self inverse element in $\mathcal{R}$. Consequently, every nonzero element of $\mathcal{R}$ is invertible, so $\mathcal{R}$ is a field. Since a field has at most two self-inverse elements, namely 1 and -1, we obtain $|\mathcal{R}| \leq 3$. The only commutative rings with unity of order at most 3 are $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Hence, the result follows from Figures 1a and 1b. $\square$

Next, to determine the size of $S_i(\mathcal{R})$, we examine some algebraic properties of the set $T(\mathcal{R}) = \{a \in \mathcal{R} |\, \exists\, b \in \mathcal{R}$ satisfying $a = b + b\}$. Observe that

Lemma 2.4. *If $\mathcal{R}$ is a ring with unity, then $T(\mathcal{R})$ is an ideal of $\mathcal{R}$.*

Before proceeding further recall Theorem 15.5 from [12], given below

Theorem 2.5. *Let $\mathcal{R}$ be a ring with unity 1. Then mapping $\phi : \mathbb{Z} \to \mathcal{R}$ described by $n \mapsto n.1$ is a ring homomorphism. Moreover, $\mathcal{R}$ contains a subring isomorphic to $\mathbb{Z}_n$ if its characteristic is $n > 0$.*

The next proposition establishes a relationship between an ideal $T(\mathcal{R})$ in $\mathcal{R}$ and the set of self inverse elements in $\mathcal{R}$.

Proposition 2.6. *If $\mathcal{R}$ is a finite ring with unity and $char(\mathcal{R}) = n$, then $T(\mathcal{R}) \cap S(\mathcal{R})$ is either $\emptyset$ or $S(\mathcal{R})$ based on whether n is odd or even, respectively.*

Proof. Let $\mathcal{R}$ be a ring with unity and $char(\mathcal{R}) = n$. Then by Theorem 2.5, there exists a subring $X \subseteq \mathcal{R}$ isomorphic to $\mathbb{Z}_n$ and we have the following two cases:
Case 1: When n is an even integer. Suppose $a \in T(\mathcal{R}) \cap S(\mathcal{R})$, then $a = b + b$, where $b \in \mathcal{R}$ and $a^2 = 1$ or $1 = a^2 = (b + b)^2 = (2b)^2 = 2(2b^2)$. Multiplying $1 = 2(2b^2)$ by $n/2$ yields $n/2 = n(2b^2) = 0$, which is a contradiction. Hence, there is no element in $T(\mathcal{R}) \cap S(\mathcal{R})$, thus $T(\mathcal{R}) \cap S(\mathcal{R}) = \emptyset$.
Case 2: When n is an odd integer. Notice that 1 can be written as $1 = [\frac{n+1}{2}] + [\frac{n+1}{2}]$, where $[\frac{n+1}{2}] \in \mathbb{Z}_n$. This infers that $1 \in T(\mathcal{R})$. Further, let a be any element in $\mathcal{R}$. Then $a = a.1 = a\left([\frac{n+1}{2}] + [\frac{n+1}{2}]\right) = a\left([\frac{n+1}{2}]\right) + a\left([\frac{n+1}{2}]\right)$ this implies that $a = b + b$, where $b = a\left([\frac{n+1}{2}]\right) \in \mathcal{R}$. Thus, every element in $\mathcal{R}$ lies in $T(\mathcal{R})$ and so, $T(\mathcal{R}) = \mathcal{R}$. Hence, $T(\mathcal{R}) \cap S(\mathcal{R}) = S(\mathcal{R})$.

□

As a consequence of the above result, the following two theorems providing the size of $S_i(\mathcal{R})$ for any $\mathcal{R}$.

Theorem 2.7. *If $\mathcal{R}$ is a finite ring and $char(\mathcal{R}) = n$, where n is an even integer, then $S_i(\mathcal{R})$ is k-regular, where $k = |S(\mathcal{R})|$. Consequently, the size of $S_i(\mathcal{R}) = \frac{k|\mathcal{R}|}{2}$.*

Proof. Assume that $\mathcal{R}$ is a finite ring and $char(\mathcal{R}) = n$ (an even integer). Let $S(\mathcal{R}) = \{x_1, x_2, x_3, ..., x_k\}$ and $|S(\mathcal{R})| = k$. For $a \in \mathcal{R}$, consider the set $N(a) = \{x \in \mathcal{R} | x + a \in S(\mathcal{R})\}$. Notice that for a given $x_i \in S(\mathcal{R})$, the equation $x + a = x_i$, has a unique solution for x, given by $x = x_i - a$. Thus $|N(a)| = |S(\mathcal{R})| = k$. Further, $a \notin N(a)$. For this, if $a \in N(a)$, then there exists a self inverse element x_i such that $x_i = a + a$. This implies that $x_i \in T(\mathcal{R})$ and so $x_i \in T(\mathcal{R}) \cap S(\mathcal{R}) \neq \emptyset$, which is a contradiction to Proposition 2.6. Thus, we conclude that for $a \in \mathcal{R}$, the set $N(a)$ is the deleted neighborhood of a and therefore, its order becomes the degree of the vertex a in $S_i(\mathcal{R})$. Since $|N(a)| = k \;\forall\; a \in \mathcal{R}$, thus each vertex in $S_i(\mathcal{R})$ is adjacent to exactly k vertices in it. Hence, $S_i(\mathcal{R})$ is k-regular and the size of $S_i(\mathcal{R}) = \frac{k|\mathcal{R}|}{2}$.

□

Theorem 2.8. *If $char(\mathcal{R})$ is an odd number, then the size of $S_i(\mathcal{R})$ is $\frac{(|\mathcal{R}|-1)|S(\mathcal{R})|}{2}$.*

Proof. Let $\mathcal{R}$ be a ring with odd characteristic. Consider the set $N(a) = \{x \in \mathcal{R} | x + a \in S(\mathcal{R})\}$. Again, notice that for each $x_i \in S(\mathcal{R})$, the equation $x + a = x_i$ has a unique solution for x, given by $x = x_i - a$. So, for all $a \in \mathcal{R}$, $N(a) = |S(\mathcal{R})|$. Now, it is important to see that form Proposition 2.6, we have $\mathcal{R} = T(\mathcal{R})$, in particular, each $x_i \in S(\mathcal{R})$ can be uniquely written as $x_i = b + b$ for some $b \in \mathcal{R}$. This implies that $b \in N(b)$ and therefore $N(b) \setminus \{b\}$ will becomes the deleted neighborhood of such an element $b \in \mathcal{R}$, as the self inverse element graph under consideration is simple, i.e., self loops are not allowed. Thus, we conclude that $|S(\mathcal{R})|$ number of elements in $\mathcal{R}$ has $|S(\mathcal{R})| - 1$ elements in their deleted neighborhood, however the remaining $|\mathcal{R}| \setminus |S(\mathcal{R})|$ elements in $\mathcal{R}$ has $|S(\mathcal{R})|$ elements as their corresponding neighborhoods. Hence, only $|S(\mathcal{R})|$ number of elements have their degree $|S(\mathcal{R})| - 1$, but all the remaining vertices in $S_i(\mathcal{R})$ have their degrees equal to $|S(\mathcal{R})|$. Finally, the total number of edges in $S_i(\mathcal{R})$ is $\frac{(|\mathcal{R}|-|S(\mathcal{R})|)|S(\mathcal{R})|+|S(\mathcal{R})|(|S(\mathcal{R})|-1)}{2} = \frac{|\mathcal{R}||S(\mathcal{R})|-|S(\mathcal{R})|^2+|S(\mathcal{R})|^2-|S(\mathcal{R})|}{2} = \frac{(|\mathcal{R}|-1)|S(\mathcal{R})|}{2}$. ☐

It follows from Theorem 2.7 and 2.8 that for any commutative ring with unity $\mathcal{R}$, the degree of a vertex in $S_i(\mathcal{R})$ is either $|S(\mathcal{R})|$ or $|S(\mathcal{R})| - 1$ i.e., $S_i(\mathcal{R})$ is either $|S(\mathcal{R})|$-regular or almost $|S(\mathcal{R})|$-regular graph, where $|S(\mathcal{R})|$ is the number of self inverse elements in $\mathcal{R}$.

By the above note it is easy to see that the difference between the degrees of a pair of vertices in a self inverse element graph cannot exceed 1.

Remark 2.9. If a_1 and a_2 are two vertices in $S_i(\mathcal{R})$, then $|deg\,(a_1) - deg\,(a_2)\,| \leq 1$.

As an application of the above, we get

Theorem 2.10. *$S_i(\mathcal{R})$ is a star graph iff $\mathcal{R} = \mathbb{Z}_2$ or $\mathbb{Z}_3$.*

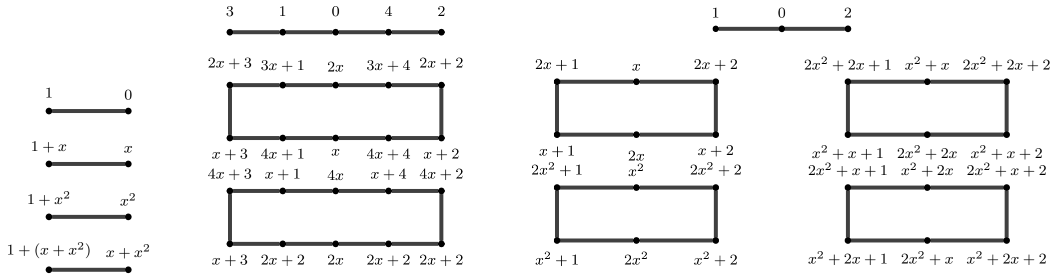
Proof. Suppose $\mathcal{R}$ is a ring with n elements and $S_i(\mathcal{R})$ is a star graph. Therefore, the degree of exactly one vertex in $S_i(\mathcal{R})$ is $n-1$, while the degree of all the other vertices must be 1. Let $deg(a_1) = n - 1$ and a_2 be any other vertex. Then by Remark 2.9, $|deg\,(a_1) - deg\,(a_2)\,| \leq 1$ implies that $|(n-1)-1| \leq 1$ which deduces that $n \leq 3$. Thus, $\mathcal{R} \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. Also, it is clear from Figure 1a and 1b, that both $S_i(\mathbb{Z}_2)$ and $S_i(\mathbb{Z}_3)$ are star graphs. Hence, the result follows. ☐

It is noteworthy to mention that if $char(\mathcal{R}) \neq 2$, then the number of self inverse elements in $\mathcal{R}$ is even. Since, in this case, an element a and its additive inverse $-a$ are distinct and either both or none of the two are self inverse elements simultaneously. Also, recall from Theorem 6.2 in [16] that a connected graph Γ is Eulerian iff the degree of each vertex in Γ is even. Also, if $char(\mathcal{R})$ is even (other than 2), then the number of self inverse elements are also even. Thus, the degree of each vertex in $\mathcal{R}$ is again even. Moreover, by Theorem 2.8, if $char(\mathcal{R})$ is odd, then there exist vertices with odd degree in $S_i(\mathcal{R})$. In view of this, we have

Theorem 2.11. *A connected graph $S_i(\mathcal{R})$ is Eulerian iff $char(\mathcal{R})$ is even.*

Next, we examine the structure of the self inverse element graph of finite fields. For this, first we investigate the self inverse element graphs of some finite fields, given in Figure 3. From this, we try to analyze the information concerning the number of connected components and their graphical structure with the help the following example:

Example 2.12. A finite field $\mathbb{F}$ with p elements is isomorphic to $\mathbb{Z}_p$ and $S_i(\mathbb{Z}_p)$ is a connected graph isomorphic to path P_p, for $p = 2, 3, 5, 7$, as depicted in Figure 1. Whereas, the graph $S_i(\mathbb{F}_{2^3})$ has four connected components each isomorphic to K_2, while $S_i(\mathbb{F}_{3^3})$ has five connected components, out of which one is isomorphic to P_3 and the other four are isomorphic to C_6. Furthermore, $S_i(\mathbb{F}_{5^2})$ has three components, out of which one is isomorphic to P_5 and the remaining two are isomorphic to C_{10}, as shown in Figure 3.



(A) $S_i(\mathbb{F}_{2^3})$ (B) $S_i(\mathbb{F}_{5^2})$ (C) $S_i(\mathbb{F}_{3^3})$

FIGURE 3. Self inverse element graph of finite fields $\mathbb{F}_q$ for $q = 2^3, 5^2, 3^3$

The subsequent theorem offers the possibilities for the degrees of the vertices in $S_i(\mathbb{F})$.

Theorem 2.13. *Let $\mathbb{F}$ be a finite field. If char$(\mathbb{F}) = 2$, then $S_i(\mathbb{F})$ is 1-regular i.e., the degree of each vertex in $S_i(\mathbb{F})$ is one. On the other hand, if char$(\mathbb{F}) = p$, a prime > 2, then the degree of all the vertices is two, except the vertices corresponding to the elements namely; $[\frac{p-1}{2}]$ and $[\frac{p+1}{2}]$.*

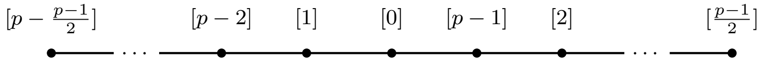
Proof. Suppose that $\mathbb{F}$ be a finite field. If *char*$(\mathbb{F})$ is even, then *char*$(\mathbb{F}) = 2$. However, in this case, the number of self inverse elements in $\mathbb{F}$ is only one, namely; $1(= -1)$, as the field can have at most two self inverse elements 1 and -1. Therefore, Theorem 2.7 infers that $S_i(\mathbb{F})$ is 1-regular with $\frac{|\mathbb{F}|}{2}$ edges. On the other hand, if *char*$(\mathbb{F}) = p > 2$, then the only two self inverse elements in $\mathbb{F}$ are 1 and -1. Now, by Theorem 2.8, $S_i(\mathbb{F})$ is almost 2-regular graph with all the vertices have degree 2, except the two vertices corresponding to the elements $[\frac{p-1}{2}]$ and $[\frac{p+1}{2}]$ in $\mathbb{F}$. Also, the degree of these two vertices is 1. $\square$

In the upcoming result, we discuss the connectedness of $S_i(\mathbb{F})$.

Theorem 2.14. *Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^n$, then $S_i(\mathbb{F})$ is disconnected iff $n > 1$. Moreover, $S_i(\mathbb{F}_p)$ is the path graph of length p.*

Proof. Suppose that $\mathbb{F}$ is a finite field with $|\mathbb{F}| = p^n$ and $n > 1$. Then by Theorem 2.5, there exists a subfield $\mathbb{F}_1 \subset \mathbb{F}$ such that $\mathbb{F}_1 \cong \mathbb{Z}_p$. Now, we claim that the subgraph of $S_i(\mathbb{F})$ induced by the set of vertices $\mathbb{F}_1$ and $\mathbb{F} \setminus \mathbb{F}_1$ are disconnected. For this, let a be an arbitrary element in $\mathbb{F} \setminus \mathbb{F}_1$. Then, we claim that no element in $\mathbb{F}_1$ is adjacent to a. If possible, let a be adjacent to an element

$x \in \mathbb{F}_1$. Therefore, by definition of the self inverse element graph $a + x = \pm 1$ and so $a = \pm 1 - x$. This infers that $a \in \mathbb{F}_1$, a contradiction. Thus, we conclude that each element of $\mathbb{F}_1$ is non-adjacent to every element in $\mathbb{F} \setminus \mathbb{F}_1$ i.e., $S_i(\mathbb{F})$ has at least two connected components. Hence $S_i(\mathbb{F})$ is disconnected.
Conversely, let $\mathbb{F}$ be a finite field such that $S_i(\mathbb{F})$ is disconnected. We shall show that $n > 1$. On contrary, if $n = 1$ then $\mathbb{F} \cong \mathbb{Z}_p$ and it is simple to observe that $S_i(\mathbb{Z}_p)$ is the path graph P_p with p vertices gives by



which is a connected graph. Hence, it follows that for a finite field $\mathbb{F}$ of order p^n, $S_i(\mathbb{F})$ is disconnected then $n > 1$. ☐

Now, we characterize the structure of the self inverse element graphs of finite fields.

Theorem 2.15. *Let $\mathbb{F}$ be a finite field of order 2^n. Then $S_i(\mathbb{F})$ has 2^{n-1} connected components each isomorphic to K_2.*

Proof. Let $\mathbb{F}$ be a finite field of order 2^n. Next, we examine the algebraic structure of $\mathbb{F}$ as: $\mathbb{F} = \left\{ \sum_{i=0}^{n-1} a_i \alpha^i | a_i \in \mathbb{Z}_2, \alpha^n = \alpha + 1 \right\}$. Since, $char(\mathbb{F}) = 2$, this implies that $\mathbb{F}$ has only one self inverse element namely; 1, the multiplicative identity of $\mathbb{F}$. Now, in $S_i(\mathbb{F})$, an element $x \in \mathbb{F}$ is adjacent to an element y iff $y = x + 1$. Now, take $x = a_1\alpha + a_2\alpha^2 + ... + a_{n-1}\alpha^{n-1}$. Then there are exactly 2^{n-1} choices for the element x and correspondingly the remaining 2^{n-1} elements in $\mathbb{F}$ are of the form $x + 1$. Thus, 2^n elements in $\mathbb{F}$ are pairwise adjacent to each other, specifically the elements x and $x + 1$ as discussed above. Hence, $S_i(\mathbb{F})$ is isomorphic to 2^{n-1} copies of K_2. ☐

Notice that a finite field and a Boolean ring of the same order (say 2^n) need not be isomorphic, in general. But, in the next theorem, we see that their self inverse element graphs are isomorphic.

Theorem 2.16. *If $\mathcal{R}$ is a finite Boolean ring and $|\mathcal{R}| = 2^n$, then $S_i(\mathcal{R})$ has 2^{n-1} copies of K_2.*

Proof. Let $\mathcal{R}$ be a finite Boolean ring and $x \in \mathcal{R}$. If x is a self inverse element, then $x^2 (= x) = 1$, i.e., the only self inverse element in $\mathcal{R}$ is 1. Now, for any $x \in \mathcal{R}$ we have $x + x + 1 = 1$, this implies that for any $x \in \mathcal{R}$; $x, x + 1$ are adjacent in $S_i(\mathcal{R})$. Further, by cancellation laws in $\mathcal{R}$, x cannot be adjacent to an element other than $x + 1$. Thus, we conclude that every vertex has exactly one adjacent vertex in $S_i(\mathcal{R})$. Hence, $S_i(\mathcal{R})$ is a union of 2^{n-1} copies of the complete graph with 2 vertices. ☐

Next, we explore the structure of the self inverse element graph of finite fields having characteristic other than 2. For this, we state the following lemma and omit its proof, as it is straightforward.

Lemma 2.17. *For $\alpha, \beta, \gamma \in \mathbb{Z}_p$, we have*
(i) *If $\alpha + \beta \equiv 0 \, (mod\, p)$, then $\beta = p - \alpha$.*
(ii) *If $\alpha + \beta \equiv 0 \, (mod\, p)$ and $\beta + \gamma \equiv 0 \, (mod\, p)$, then $\alpha = \gamma$.*

Proposition 2.18. *Let* $|\mathbb{F}| = p^n$, $p > 2$ *and* $a = \sum_{i=0}^{n-1} a_i\alpha^i$, $b = \sum_{i=0}^{n-1} b_i\alpha^i \in \mathbb{F}$. *If* a *and* b *are connected in* $S_i(\mathbb{F})$ *and* $d = d(a, b)$, *then*

$$a_i - (-1)^d b_i = \begin{cases} \pm d; & i = 0, \\ 0; & \forall i = 1, ..., n-1. \end{cases}$$

Proof. Let a finite field of order p^n be represented by: $\mathbb{F} = \left\{\sum_{i=0}^{n-1} a_i\alpha^i | a_i \in \mathbb{Z}_p\right\}$, where α is an algebraic element of degree n over $\mathbb{Z}_p$. Let $a = \sum_{i=0}^{n-1} a_i\alpha^i$ and $b = \sum_{i=0}^{n-1} b_i\alpha^i \in \mathbb{F}$, where $a_i, b_i \in \mathbb{Z}_p$, for each i. Assume that there is a path between a and b in $S_i(\mathbb{F})$ and $d = d(a, b)$. Let $a = a_0 \leftrightarrow a_1 \leftrightarrow a_2 \leftrightarrow a_3 \leftrightarrow ... \leftrightarrow a_{d-1} \leftrightarrow a_d = b$ be the shortest path of length d connecting a and b. Consequently, there exist n sequences in $\mathbb{Z}_p$ corresponding to the coefficients of $\alpha^0, \alpha^1, \alpha^2, ..., \alpha^{n-1}$ in the presentation of these vertices as elements in $\mathbb{F}$. Suppose that the sequence corresponding to the coefficients of α^j is ${a_j}^0, {a_j}^1, {a_j}^2, ..., {a_j}^{n-1}$. Since, the vertices a_j and a_{j+1} are adjacent for $j = 0, 1, 2, ..., n-1$, therefore by definition of $S_i(\mathcal{R})$, for each $j = 1, 2, 3, ..., n-1$, we have

$$\begin{aligned} {a_j}^0 + {a_j}^1 &\equiv 0 \ (\mathrm{mod} p) \\ {a_j}^1 + {a_j}^2 &\equiv 0 \ (\mathrm{mod} p) \\ &\vdots \\ {a_j}^{d-1} + {a_j}^d &\equiv 0 \ (\mathrm{mod} p). \end{aligned} \tag{2.1}$$

While, for $j = 0$;

$$\begin{matrix} a_0^0 + a_0^1 \\ a_0^1 + a_0^2 \\ \vdots \\ a_0^{d-1} + a_0^d \end{matrix} \equiv \text{ either } \begin{cases} (-1)^1 \\ (-1)^2 \\ \vdots \\ (-1)^d \end{cases} \text{ or } \begin{cases} (-1)^0 \\ (-1)^1 \\ \vdots \\ (-1)^{d-1} \end{cases} \pmod p. \tag{2.2}$$

Next, multiplying the i^{th} equation of system (2.2) by $(-1)^{i+1}$and then adding, we get $a_0 + (-1)^{d+1}b_0 = \pm d$. Applying the same technique for the equations in system (2.1) corresponding to $j = 1, 2, 3, ..., n-1$, we obtain $a_j + (-1)^{d+1}b_j = 0$ $\forall j = 1, 2, 3, ..., d-1$. Hence, $a_i - (-1)^d b_i = \begin{cases} \pm d; & i = 0, \\ 0; & \forall i = 1, 2, ..., n-1. \end{cases}$ □

As an immediate consequence, we have

Theorem 2.19. *Let* $\mathbb{F}_{p^n}$ *be a finite field with* $p \neq 2$. *Then the distance between any two connected vertices in* $S_i(\mathbb{F}_{p^n})$ *cannot exceed* p.

Proof. Suppose that $\mathbb{F}_{p^n}$ is a finite field with $p > 2$. Let a, b be a pair of connected vertices in $S_i(\mathbb{F}_{p^n})$ with $d(a, b) = d$. Then by Proposition 2.18, we have $a_0 \pm b_0 = \pm d$. Since $a_0, b_0 \in \mathbb{Z}_p$, therefore $d \in \mathbb{Z}_p$. This concludes that the distance between a and b in $S_i(\mathbb{F}_{p^n})$ cannot exceed p, which proves the result. □

Remark 2.20. In view of Lemma 2.17, the system of equations (2.1) infers that for $j \in \{1, 2, ..., n-1\}$, ${a_j}^i = {a_j}^{i+2}$ and ${a_j}^{i+1} = p - {a_j}^i$, for all $i = 0, 1, 2, ..., d(a, b)$.

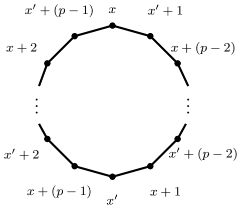
This means that if a pair of vertices $a = \sum_{i=0}^{n-1} a_i\alpha^i$ and $b = \sum_{i=0}^{n-1} b_i\alpha^i \in S_i(\mathbb{F}_{p^n})$, where $a_i, b_i \in \mathbb{Z}_p$ for each i, are connected and $d(a,b) = d$, then $a - a_0 = b - b_0$ if d is even, while $a - a_0 = p - (b - b_0)$ if d is odd, i.e., all the coefficient in the presentation of a and b as an element of $\mathbb{F}_{p^n}$ either coincides or negative of each other, except for the coefficients corresponding to α^0.

This naturally induces a relation on $S_i(\mathbb{F}_{p^n})$ as: Let $a = \sum_{i=0}^{n-1} a_i\alpha^i$ and $b = \sum_{i=0}^{n-1} b_i\alpha^i$, where $a_i, b_i \in \mathbb{Z}_p$. Then the two vertices $a, b \in S_i(\mathbb{F}_{p^n})$ are connected iff either $a_i = b_i$ or $a_i = -b_i \; \forall \; i = 1, 2, ..., n-1$. Clearly, this is an equivalence relation on vertices of $S_i(\mathbb{F}_{p^n})$ and the residue classes for this relation correspond to the connected components in $S_i(\mathbb{F}_{p^n})$. Further an residue class containing an element $a \in \mathbb{Z}_{p^n} \setminus \mathbb{F}$ has order $2p$ and is given by $[a] = \{x, x+1, x+2, , ..., x+(p-1), x', x'+1, x'+2, ..., x'+(p-1)\}$, where $x = a - a_0$ and $x' = (p-1)x$. Whereas, an residue class containing an element $a \in \mathbb{Z}_p$ is $[a] = \{0, 1, 2, , ..., p-1\} \, (= \mathbb{Z}_p)$ and it has p elements.

By virtue of the above discussion, we have

Theorem 2.21. *If $\mathbb{F}$ is a field with $|\mathbb{F}| = p^n, n > 1, p > 2$, then $S_i(\mathbb{F})$ has one connected component isomorphic to path P_p, while all the remaining $\frac{p^{n-1}-1}{2}$ connected components are isomorphic to cycle graph C_{2p}.*

Proof. It is clear that the connected components in $S_i(\mathbb{F}_{p^n})$ are the subgraphs of $S_i(\mathbb{F}_{p^n})$ induced by the residue classes under the aforesaid equivalence relation. Now, as discussed in the proof of Theorem 2.14, the subgraph formed by the residue class $\mathbb{Z}_p$ is isomorphic to the path P_p. On the other hand, if $a \in \mathbb{Z}_{p^n} \setminus \mathbb{Z}_p$, then the subgraph induced by the residue class of a i.e., $[a] = \{x, x+1, x+2, , ..., x+(p-1), x', x'+1, x'+2, ..., x'+(p-1)\}$, where $x = a - a_0$ and $x' = (p-1)x$ is the cycle graph C_{2p} represented as below:



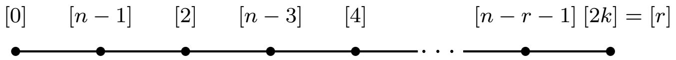
Moreover, observe that $\mathbb{Z}_p$ is the only residue class with p elements, while all the remaining residue classes have $2p$ elements. Also, the number of elements in all distinct residue classes of order $2p$ is $p^n - p$, as they together constitute $\mathbb{F}_{p^n} \setminus \mathbb{Z}_p$. Thus, the number of residue classes with $2p$ elements is $\frac{p^n - p}{2p} = \frac{p^{(n-1)}-1}{2}$. i Hence, we conclude that $S_i(\mathbb{F}_{p^n})$ has a single connected component isomorphic to P_p and the remaining $\frac{p^{(n-1)}-1}{2}$ connected components are isomorphic to C_{2p}. $\square$

3. THE STRUCTURE OF SELF INVERSE ELEMENT GRAPH OF $\mathbb{Z}_n$ AND THEIR CARTESIAN PRODUCT

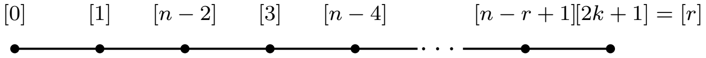
Here, we analyze the structure of the self inverse element graph of the ring of integers modulo a positive integer n. We also study the self inverse element graph of the cartesian product of these rings. First, in the forthcoming two theorems, we establish the connectedness of $S_i(\mathbb{Z}_n)$ and $S_i(\mathbb{Z})$.

Theorem 3.1. *$S_i(\mathbb{Z}_n)$ is a connected graph, for every $n \in \mathbb{N}$.*

Proof. Let $r \in \mathbb{Z}_n$. We claim that the vertex r is connected to the vertex corresponding to the element $0 \in \mathbb{Z}_n$. If the element r is of the form $2k$, then a path connecting the vertex r and vertex 0 in $S_i(\mathbb{Z}_n)$ is given below. Since, for any two consecutive vertices in this graph, their sum is either $[1]$ or $[n-1]$, which are precisely the self-inverse elements of $\mathbb{Z}_n$.



Otherwise, if $r = 2k+1$ is odd, then a connection between the vertices r and 0 is given by



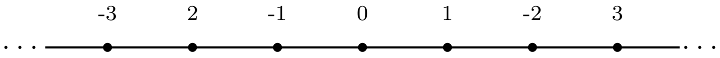
Thus, we see that for each $r \in \mathbb{Z}_n$ the vertex r is connected to the vertex corresponding to vertex $0 \in \mathbb{Z}_n$. Hence, $S_i(\mathbb{Z}_n)$ is a connected graph, for every n.

□

We extend the same idea to see the connectedness of $S_i(\mathbb{Z})$.

Theorem 3.2. *The graph $S_i(\mathbb{Z})$ is a connected graph. In particular, it is a path of infinite length.*

Proof. Since, $S(\mathbb{Z}) = \{1, -1\}$ and the cancellation laws hold under addition in $\mathbb{Z}$, therefore we conclude that the degree of any vertex in $S_i(\mathbb{Z})$ is exactly 2. In other words, a vertex $n \in \mathbb{Z}$ has precisely two neighbors $-n+1$ and $-n-1$ only. Consequently, the self inverse element graph of $\mathbb{Z}$ is given by:



Clearly, each integer appears as a vertex having degree 2 in the above graph. Hence $S_i(\mathbb{Z})$ is a connected path of infinite length. □

As a minor changes in Theorem 5.5 in [2], we have the next result providing the number of self inverse elements in $\mathbb{Z}_n$.

Theorem 3.3. *If $n = 2^{e_0}\prod_{i=1}^{r} p_i^{e_i}$, is the prime factorization of n, where each p_i is a distinct odd prime and $e_i \geq 0$ for $0 \leq i \leq r$, then $x^2 \equiv 1 \pmod{n}$ has exactly 2^{r+e} solutions in $\mathbb{Z}_n$, as a consequence, $|S(\mathbb{Z}_n)| = 2^{r+e}$, where* $e = \begin{cases} 0, & e_0 = 0 \text{ or } 1 \\ 1, & e_0 = 2 \\ 2, & e_0 \geq 3. \end{cases}$

The forthcoming corollaries are direct applications of Theorem 2.7 and 3.3.

Corollary 3.4. *For a positive integer n with $n = 2^{e_0}p_1^{e_1}p_2^{e_2}...p_r^{e_r}$, where $p_1, p_2, ...p_r$ are distinct odd primes. Let e be the integer given by* $e = \begin{cases} 0; e_0 = 1 \\ 1; e_0 = 2. \\ 2; e_0 \geq 3 \end{cases}$ *If $e_0 \neq 0$, then the self inverse element graph of the ring $\mathbb{Z}_n$ i.e., $S_i(\mathbb{Z}_n)$ is 2^{r+e}-regular. While, if $e_0 = 0$, then $S_i(\mathbb{Z}_n)$ is almost 2^r-regular (specifically, 2^r vertices in $S_i(\mathbb{Z}_n)$ are of degree $2^r - 1$ and the degrees of the remaining $(n - 2^r)$ vertices is 2^r).*

Corollary 3.5. *The size of $S_i(\mathbb{Z}_n)$ is* $\begin{cases} n2^{r+e-1}; & \text{if } n \text{ even} \\ (n-1)2^{r-1}; & \text{if } n \text{ odd} \end{cases}$.

The next two theorems characterize all commutative rings with unity, whose self inverse element graph is a path graph or a cycle graph, respectively.

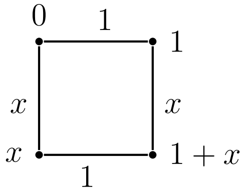
Theorem 3.6. *The graph $S_i(\mathcal{R})$ is a path iff $\mathcal{R} = \mathbb{Z}_2$ or $\mathbb{Z}_{p^r}$, where p is an odd prime.*

Proof. Let $\mathcal{R} = \mathbb{Z}_2$. From Figure 1a, $S_i(\mathbb{Z}_2)$ is a path. Now let $\mathcal{R} = \mathbb{Z}_{p^r}$. By Theorem 3.3, $\mathbb{Z}_{p^r}$ has exactly two self-inverse elements, $[1]$ and $[n-1]$, where $n = p^r$. By Theorem 2.8, precisely two vertices, $[\frac{n-1}{2}]$ and $[\frac{n+1}{2}]$, have degree 1, while all others have degree 2. Since $S_i(\mathbb{Z}_{p^r})$ is connected (Theorem 3.1), it is a path of length $p^r - 1$.
Conversely, suppose $S_i(\mathcal{R})$ is a path. Then it is connected with exactly two vertices of degree 1 and all others of degree 2, implying that $\mathcal{R}$ has at most two self-inverse elements. If $char(\mathcal{R}) = k$, then by Theorem 2.5, $\mathbb{Z}_k$ is a subring of $\mathcal{R}$. Hence, $\mathbb{Z}_k$ also has at most two self-inverse elements, so by Theorem 3.3, $k \in 2, 4, p^r, 2p^r$. Thus, following two cases arise:
Case 1: If $|S(\mathcal{R})| = 1$, then $S_i(\mathcal{R})$ is a path only when $|\mathcal{R}| = 2$, giving $\mathcal{R} = \mathbb{Z}_2$.
Case 2: If $|S(\mathcal{R})| = 2$ and $char(\mathcal{R})$ is even $(2, 4, 2p^r)$, then by Theorem 2.7, $S_i(\mathcal{R})$ is 2-regular, hence not a path. Thus $char(\mathcal{R}) = p^r$. In this case, $\mathbb{Z}_{p^r}$ is a subring and induces a path of length $p^r - 1$. Because, if $\mathcal{R} \neq \mathbb{Z}_{p^r}$, then some $a \in \mathcal{R} \setminus \mathbb{Z}_{p^r}$ must be adjacent to an end vertex of this path. By Definition 2.1, this forces $a \in \mathbb{Z}_{p^r}$, which is a contradiction. Hence $\mathcal{R} = \mathbb{Z}_{p^r}$. Therefore, $S_i(\mathcal{R})$ is a path iff $\mathcal{R} = \mathbb{Z}_2$ or $\mathbb{Z}_{p^r}$. □

Theorem 3.7. *The graph $S_i(\mathcal{R})$ is a cycle iff $\mathcal{R} = \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ or $\mathbb{Z}_{2p^r}$, where p is an odd prime.*

Proof. It is clear, from Figures 1c and 2b, $S_i(\mathbb{Z}_4)$ and $S_i(\mathbb{Z}_2[x]/(x^2))$ are cycles. If $\mathcal{R} = \mathbb{Z}_{2p^r}$, then by Theorems 2.7 and 3.1, $S_i(\mathbb{Z}_{2p^r})$ is connected and 2-regular. Hence it is a cycle.
Conversely, suppose $S_i(\mathcal{R})$ is a cycle. Then it is connected and 2-regular, so $\mathcal{R}$ has exactly two self-inverse elements. Let $char(\mathcal{R}) = k$. As we discussed in the proof of above theorem, $\mathcal{R}$ contains a subring isomorphic to $\mathbb{Z}_k$, where $k \in 2, 4, p^r, 2p^r$. Since $S_i(\mathcal{R})$ is 2-regular, k cannot be odd, thus $k \in 2, 4, 2p^r$.

Case 1: $char(\mathcal{R}) = 2$. Let 1 and $x(\neq -1)$ be the two self-inverse elements. Then $0, 1, x, 1 + x$ induces a 4-cycle. If $\mathcal{R}$ had any additional element, it would be non-adjacent to these vertices, making the graph disconnected. Hence $|\mathcal{R}| = 4$ and $\mathcal{R} \cong \mathbb{Z}_2[x]/(x^2)$.



Case 2: $char(\mathcal{R}) = 4$ or $2p^r$. Then $\mathcal{R}$ contains a subring $\mathbb{Z}_k$ with $k = 4$ or $2p^r$. The induced subgraph on $\mathbb{Z}_k$ is a cycle (C_4 or C_{2p^r}). Since $S_i(\mathcal{R})$ is 2-regular, no additional adjacencies are possible; otherwise the graph would be disconnected. Thus $\mathcal{R} = \mathbb{Z}_4$ or $\mathbb{Z}_{2p^r}$. □

PROPERTIES OF $S_i(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times ... \times \mathbb{Z}_{m_k})$

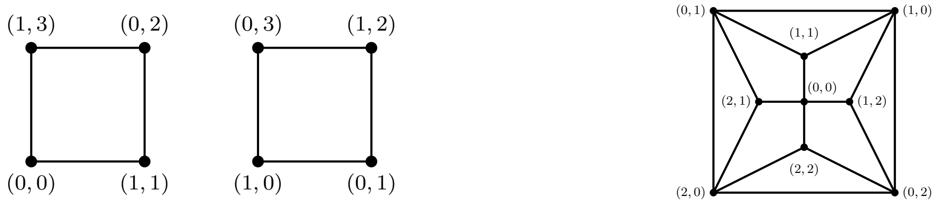
In this subsection, we explore the self inverse element graph of the cartesian product of the rings of integer modulo n, $n \geq 2$. Suppose that $\mathcal{R} = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times ... \times \mathbb{Z}_{m_k} = \prod_{i=1}^{k} \mathbb{Z}_{m_i}$. Observe that if $S(\mathbb{Z}_{m_i})$ is the set of self inverse elements in $\mathbb{Z}_{m_i}$ then the self inverse element of $\mathcal{R}$ are the members of the set $\prod_{i=1}^{k} S(\mathbb{Z}_{m_i})$. Moreover, an element $a \in \mathcal{R}$ is of the form $a = (a_1, a_2, ..., a_k)$, where $a_i \in \mathbb{Z}_{m_i}$ for $i = 1, 2, ..., k$. Therefore, two elements $a = (a_1, a_2, ..., a_k)$ and $b = (b_1, b_2, ..., b_k)$ are adjacent in $S_i(\mathcal{R})$ iff $a_i + b_i \in S(\mathbb{Z}_{m_i})$ (or $b_i = -a_i \pm s_i$, for some $s_i \in S(\mathbb{Z}_{m_i})$) for all i, $1 \leq i \leq k$. Further, depending upon the parity of the coordinates of an element in $\mathcal{R}$, we define:

Definition 3.8. Two elements $a, b \in \mathcal{R}$ are said to have same parity patterns, if the corresponding coordinates of both a and b are of same parity i.e., both are either even or odd simultaneously.
In this case, we say these elements are of same *parity type* and the collection of all elements in $\mathcal{R} = \prod_{i=1}^{k} \mathbb{Z}_{m_i}$ having the same parity type is a *parity class* in $\mathcal{R}$ and is written by using any element in it, when its coordinates are replaced with e or o accordingly as the corresponding coordinate is even or odd respectively.

For instance, the possible parity classes are the sets of elements having the same parity type in $\mathbb{Z}_2 \times \mathbb{Z}_4$ are $(e, e) = \{(0, 0), (0, 2)\}, (e, o) = \{(0, 1), (0, 3)\}, (o, e) = \{(1, 0), (1, 2)\}, (o, o) = \{(1, 1), (1, 3)\}$, while for $\mathbb{Z}_3 \times \mathbb{Z}_3$, this is give by $(e, e) = \{(0, 0), (0, 2), (2, 0), (2, 2)\}, (e, o) = \{(0, 1), (2, 1)\}, (o, e) = \{(1, 0), (1, 2)\}$ and $(o, o) = \{(1, 1)\}$. Furthermore, their self inverse element graphs are presented in Figure 4.

Clearly, $S_i(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is connected, whereas $S_i(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is a disconnected graph with 2 connected components each isomorphic to $S_i(\mathbb{Z}_4)$. Thus, in the rest of this section, we investigate the connectedness and connected components of $S_i(\mathcal{R})$, where $\mathcal{R}$ is a cartesian product $\prod_{i=1}^{k} \mathbb{Z}_{m_i}$. In this, direction, we have

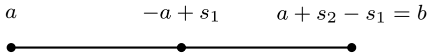
Lemma 3.9. *Let $\mathcal{R} = \prod_{i=1}^{k} \mathbb{Z}_{m_i}$ and $a, b \in \mathcal{R}$. If coordinates of the elements a and b differ from each other exactly at one place by a difference of 2, then a and b are connected in $S_i(\mathcal{R})$.*



(A) $S_i(\mathbb{Z}_2 \times \mathbb{Z}_4)$ (B) $S_i(\mathbb{Z}_3 \times \mathbb{Z}_3)$

FIGURE 4. Self inverse element graph of cartesian product of rings (a) $\mathbb{Z}_2 \times \mathbb{Z}_4$ (b) $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Proof. Without loss of generality, assume that a and b differ only at first coordinate by 2. Therefore, $b_1 = a_1+2$ and $b_j = a_j \forall j = 2, 3, ..., k$. So, $a = (a_1, a_2, ..., a_k)$ and $b = (a_1 + 2, b_2, ..., b_k)$. Take $s_1 = (1, 1, ..., 1)$ and $s_2 = (-1, 1, ..., 1)$, clearly $s_1, s_2 \in S(\mathcal{R}) \subsetneq \mathcal{R}$. Since, $\mathcal{R}$ is closed under addition. Therefore, $-a + s_1$ and $a + s_2 - s_1$ are also members of $\mathcal{R}$. This gives a connection between a and b as:



Consequently, a connected path connects a and b. □

On applying the above result in succession to a fixed coordinate, we conclude that

Proposition 3.10. *Let $a, b \in \mathcal{R} = \prod_{i=1}^{k} \mathbb{Z}_{m_i}$. If the coordinates of a and b differ exactly at one place by an even integer, then the vertices a and b are connected in $S_i(\mathcal{R})$.*

The next result provides the connectedness between the elements having the same parity type.

Theorem 3.11. *Let $\mathcal{R} = \prod_{i=1}^{k} \mathbb{Z}_{m_i}$. If $a, b \in \mathcal{R}$ are of the same parity type, then there exists a path in $S_i(\mathcal{R})$ from a to b.*

Proof. Let $a = (a_1, a_2, ..., a_k), b = (b_1, b_2, ..., b_k) \in \mathcal{R}$ be two elements having the same parity type. Therefore, the corresponding coordinates of a and b are either both even or odd. This infers that the corresponding difference between the coordinates must be even i.e., $d_i = a_i - b_i$ is even for every i, $1 \leq i \leq k$. Now, since, d_1 is even (say) $2r$. Therefore, in view of the above proposition, there exists a path P_1 in $S_i(\mathcal{R})$ connecting the vertices a and u_1, where $u_1 = (a_1 + d_1, a_2, ..., a_k) \in \mathcal{R}$. Next, we switch to the next coordinate. Clearly, the second coordinate of u_1 coincide with that of a, while the first coordinate of u_1 is b_1. Further, $a_2 - b_2 = d_2$ is again even, so again using Proposition 3.10, there exists a path P_2 in $S_i(\mathcal{R})$ connecting the vertices u_1 and $u_2 = (b_1, b_2, a_3, ..., a_k) \in \mathcal{R}$. Continuing in this manner, at the k^{th} stage we have a path P_k connecting $u_{k-1} = (b_1, b_2, ..., b_{k-1}, a_k)$ and b. Finally, a path connecting the vertices a and b in $S_i(\mathcal{R})$ can be obtained as a union of the paths $P_1, P_2, ..., P_k$. Thus, we conclude that a and b are connected in $S_i(\mathcal{R})$. This concludes the proof. □

Consequently, we have

Corollary 3.12. *All the vertices corresponding to the elements having the same parity type belong to a single connected component in* $S_i(\mathcal{R})$.

Next, in view of Theorem 3.11, in order to establish the connectedness of the self inverse element graph of the cartesian product $\prod_{i=1}^{k}\mathbb{Z}_{m_i}$, it is enough to prove that at least one element from each possible parity class is connected to each other, as all elements lying within the same parity class are connected to each other. Further, observe that a self inverse element in $\mathbb{Z}_m$ is always an odd integer if m is even, but for an odd m, $\mathbb{Z}_m$ has at least one even as well as an odd integer as self inverse elements. With these observation in mind, we have

Theorem 3.13. *Let* $\mathcal{R} = \prod_{i=1}^{k}\mathbb{Z}_{m_i}$, $m_i \geq 2$ *for* $1 \leq i \leq k$. *Then* $S_i(\mathcal{R})$ *is a connected graph iff* m_i *is even, for at most one* i. *Moreover, if* $r = |\{i \mid m_i \text{ is even}, 1 \leq i \leq k\}| \geq 2$, *then* $S_i(\mathcal{R})$ *is disconnected and the number of connected components in* $S_i(\mathcal{R})$ *is* 2^{r-1}.

Proof. Let $\mathcal{R} = \prod_{i=1}^{k}\mathbb{Z}_{m_i}$. First assume that each m_i is odd. Let $a \in \mathcal{R}$ have all coordinates even. Since m_i is odd, the set of self-inverse elements of $\mathbb{Z}_{m_i}$ contains both an odd and an even representative (notably 1 and $-1 = m_i - 1$). Hence, for a suitable $s \in S(\mathcal{R})$ whose coordinates are each 1 or -1, the elements $-a \pm s$ realize all remaining parity classes. Because a is adjacent to $-a \pm s$ (their sum equals $\pm s \in S(\mathcal{R})$), Theorem 3.11 implies that $S_i(\mathcal{R})$ is connected.

Next suppose that exactly one m_i is even; without loss of generality take m_1 even. Then every self-inverse element of $\mathbb{Z}_{m_1}$ is odd, while for $i \neq 1$ both parities may occur. As before, an appropriate $s \in S(\mathcal{R})$ connects a to representatives of all parity classes, so $S_i(\mathcal{R})$ remains connected. Therefore, if at most one m_i is even, the graph $S_i(\mathcal{R})$ is connected.

Conversely, assume that at least two of the m_i are even. Without loss of generality, let $m_1, \ldots, m_r$ be even with $2 \leq r \leq k$. Then every self-inverse element in the first r coordinates is odd, so adding any self-inverse element simultaneously switches the parity of these coordinates. In particular, each $a = (a_1, \ldots, a_k)$ is adjacent to $b = (-a_1 + 1, \ldots, -a_k + 1)$, since $a + b = (1, \ldots, 1) \in S(\mathcal{R})$. Thus, elements having exactly opposite parity in the first r coordinates lie in the same connected component, and no edges connect different such parity pairs. The number of parity classes in the first r coordinates is 2^r; pairing opposite classes yields 2^{r-1} connected components, which is at least two for $r \geq 2$. Hence, $S_i(\mathcal{R})$ is disconnected.

□

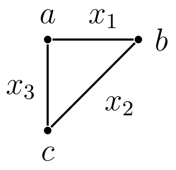
4. GIRTH AND PLANARITY OF $S_i(\mathcal{R})$

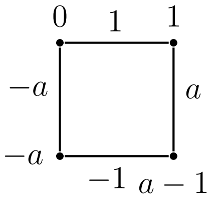
Recall that the girth of a graph is a structural characteristic, defined as the length of a shortest cycle existing in the graph. In case, if a graph does not contain any cycle, then it is said to have infinite girth. Moreover, a planar representation of a graph refers to a presentation of the graph on a plane, in which the edges do not cross each other except at their endpoints. A graph with such representation is called planar, otherwise, non-planar.

In this section, we study girth, bipartiteness and planarity of the self inverse element graph of rings. In fact, we completely characterize the girth of the self inverse element graph of $\mathbb{Z}_n$, for every n.

Theorem 4.1. *If $\mathcal{R}$ is a ring with even characteristic having at least four self inverse elements, then girth of $S_i(\mathcal{R})$ is 4.*

Proof. Let $char(\mathcal{R}) = 2n$. Since, $S_i(\mathcal{R})$ is a simple graph, therefore the girth of $S_i(\mathcal{R})$ must be greater than 2. Next, we claim that there is no cycle of length 3 in $S_i(\mathcal{R})$. If possible, suppose that there exists a cycle of length 3, then by definition of self inverse element graph there exist three distinct vertices a, b and c in $S_i(\mathcal{R})$ such that $a + b = x_1, b + c = x_2$ and $c + a = x_3,$ for some $x_1, x_2, x_3 \in S(\mathcal{R})$.

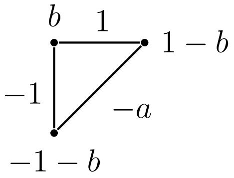




Notice that the elements x_1, x_2 and x_3 are pairwise distinct, as are a, b and c. Thus, we have $b = x_1 - a$ and $c = x_3 - a$ which implies that $x_2 = b + c = x_1 - a + x_3 - a = x_1 + x_3 - 2a$. On squaring both sides, we get $1 = (x_1 + x_3 - 2a)^2 = x_1^2 + x_3^2 + 4a^2 + 2x_1x_3 - 4x_3a - 4x_1a$ which gives $1 = 2 + 4a^2 + 2x_1x_3 - 4x_3a - 4x_1a$. Now, multiplying this equation with n, we have $1.n = 2n + 4na^2 + 2nx_1x_3 - 4nx_3a - 4nx_1a$. Further, by hypothesis $char(\mathcal{R}) = 2n$, so this equation reduces to $1.n = 0$, which is a contradiction. Thus, we conclude that $S_i(\mathcal{R})$ cannot have a cycle of length 3. Consequently, girth of $S_i(\mathcal{R}) \geq 4$. Next, consider 4 distinct vertices $0, 1, -a, a - 1 \in S_i(\mathcal{R})$, for some self inverse element a other than ± 1. Now, the subgraph induced by these four vertices in $S_i(\mathcal{R})$ is the cycle graph C_4. Thus, there exists a cycle of length 4 in $S_i(\mathcal{R})$. Hence, the girth of $S_i(\mathcal{R})$ is 4. ☐

Theorem 4.2. *If $\mathcal{R}$ is a ring with odd characteristic having at least four self inverse elements, then the girth of $S_i(\mathcal{R})$ is 3.*

Proof. Assume that a ring $\mathcal{R}$ has at least four self inverse elements. Let $a \in S(\mathcal{R})$ be a self inverse element in $\mathcal{R}$ other than ± 1. Since, $char(\mathcal{R})$ is odd, therefore by Lemma 2.6, $a \in T(\mathcal{R})$. This implies that $a = b + b$ for some $b \in \mathcal{R}$. Notice that there exists a cycle of length 3 with vertices $b, 1 - b$ and $-1 - b$ in $S_i(\mathcal{R})$, because $b + (1 - b) = 1, b + (-1 - b) = -1$ and $(1 - b) + (-1 - b) = -(b + b) = -a$, given by



Further, observe that the vertices $b, 1 - b, -1 - b$ are pairwise distinct, otherwise, if any two of them coincide i.e., $b = 1 - b$ or $b = -1 - b$ or $1 - b = -1 - b$, then it will lead to $a = 1$ or $a = -1$ or $1 = -1$, respectively. This is absurd, as the elements $1, -1$ and a are distinct. Hence, the girth of $S_i(\mathcal{R})$ is 3. □

Further, Theorem 3.6 infers that $S_i(\mathbb{Z}_2)$ and $S_i(\mathbb{Z}_{p^r})$ are path graphs of length 2 and p^r respectively, whereas, from Theorem 3.7, we see that $S_i(\mathbb{Z}_4)$ and $S_i(\mathbb{Z}_{2p^r})$ are cycles of length 4 and $2p^r$. Therefore, as an immediate application of Theorem 4.1 and 4.2, we provide a complete characterization of the girth of $S_i(\mathbb{Z}_n)$, for every n.

Theorem 4.3. *Let $\mathbb{Z}_n$ denote the ring of integers modulo n, where p is an odd prime and $r \in \mathbb{N}$. Then* $\text{Girth of } S_i(\mathbb{Z}_n) = \begin{cases} \infty; & n = 2, p^r, \\ 3; & n \text{ is odd except } p^r, \\ 4; & n \text{ is even except } 2, 4, 2p^r, \\ n; & n = 4, 2p^r. \end{cases}$

By virtue of Theorem 2.21, the girth of self inverse element graph of finite fields can be established as

Theorem 4.4. *If $\mathbb{F}$ is a field with $|\mathbb{F}| = p^n$, then the girth of* $S_i(\mathbb{F}) = \begin{cases} \infty; & p = 2 \text{ or } n = 1 \\ 2p; & \text{otherwise}. \end{cases}$

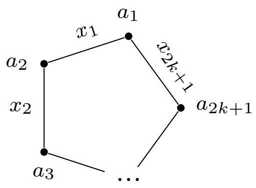
Next, we provide a characterization for $S_i(\mathcal{R})$ to be a bipartite graph. For this, first, we prove the following proposition.

Proposition 4.5. *Let $\mathcal{R}$ is a commutative ring with unity having odd characteristic. If $|S(\mathcal{R})| \leq 2^r$ then $char(\mathcal{R})$ cannot have more than r distinct odd prime divisors. In fact, if $char(\mathcal{R})$ has an odd prime divisor, then $|S(\mathcal{R})| \neq 1$.*

Proof. Assume that $|S(\mathcal{R})| = 2^r$ and $char(\mathcal{R}) = k$. Suppose that $k = {p_1}^{\alpha_1}{p_2}^{\alpha_2}...{p_m}^{\alpha_m}$, where $p_1, p_2, ..., p_m$ are distinct odd primes divisors of k. If possible, let $m > r$. Then, in view of Theorem 2.5, $\mathbb{Z}_k$ is a subring of the ring $\mathcal{R}$. Further, Theorem 3.3 infers that $|S(\mathbb{Z}_k)| = 2^m$. Now, since $m > r$, therefore $|S(\mathcal{R})| \geq |S(\mathbb{Z}_k)| = 2^m > 2^r$, which is a contradiction. Hence, the number of distinct primes dividing $char(\mathcal{R})$ cannot exceed r. Further, it follows that if $m > 1$, there exists an odd prime which divides k, then $|S(\mathcal{R})| \geq 2$. □

Theorem 4.6. *The graph $S_i(\mathcal{R})$ is bipartite iff either $char(\mathcal{R})$ is even or $char(\mathcal{R}) = p^r$ along with $|S(\mathcal{R})| = 2$, where p is an odd prime and $r \geq 1$.*

Proof. Assume that $char(\mathcal{R})$ is even (say $2n$). Now, if the graph $S_i(\mathcal{R})$ is acyclic i.e., without any cycle, then it is trivially bipartite. Otherwise, we claim that $S_i(\mathcal{R})$ has no cycles of odd length. On contrary, let $S_i(\mathcal{R})$ contains a cycle of odd length $2k + 1$, where $k \geq 1$. This implies that there exist a sequence $a_1, a_2, a_3, ..., a_{2k+1}$ of $2k+1$ elements in $\mathcal{R}$ such that $a_i + a_{i+1} = x_i$, $i = 1, 2, 3, ..., 2k-1$ and $a_{2k+1} + a_1 = x_{2k+1}$, where $x_i \in S(\mathcal{R})$ for $1 \leq i \leq 2k + 1$.



Adding all these equations, we get $\Sigma_{i=1}^{2k+1}2a_i = \Sigma_{i=1}^{2k+1}x_i$. On squaring both sides, we obtain $\left(\Sigma_{i=1}^{2k+1}2a_i\right)^2 = \left(\Sigma_{i=1}^{2k+1}x_i\right)^2$ i.e., $\Sigma_{i=1}^{2k+1}4a_i^2 + \Sigma_{\substack{i,j=1\\i<j}}^{2k+1}8a_ia_j = \Sigma_{i=1}^{2k+1}1 + \Sigma_{\substack{i,j=1\\i<j}}^{2k+1}2x_ix_j$, $1 \leq i,j \leq 2k+1$. This simplifies to $\Sigma_{i=1}^{2k+1}4a_i^2 + \Sigma_{\substack{i,j=1\\i<j}}^{2k+1}8a_ia_j = 2k+1+\Sigma_{\substack{i,j=1\\i<j}}^{2k+1}2x_ix_j$. Thereafter, multiplying the above equation by n and using the fact that $char(\mathcal{R}) = 2n$, we left with $n.1 = 0$ which is a contradiction. Thus, $S_i(\mathcal{R})$ has no cycle of odd length.

On the other hand, suppose that $|S(\mathcal{R})| = 2$ and $char(\mathcal{R}) = p^r$ for odd prime p, $r > 0$. Again, we show that $S_i(\mathcal{R})$ cannot have a cycle of odd length. If not, then by the Pigeonhole principle, one can find two adjacent edges in that cycle of odd length such that their end vertices add up to a single self inverse element $x \in S(\mathcal{R})$, because $|S(\mathcal{R})| = 2$. So, there exist distinct elements $a, b, c \in \mathcal{R}$ satisfying $a + b = x = b + c$. But, this leads to $a = c$, which is a contradiction. Thus, $S_i(\mathcal{R})$ again has no cycle of odd length. Finally, form Theorem 1.2.18 [16], we conclude that $S_i(\mathcal{R})$ is bipartite.

Conversely, suppose that $char(\mathcal{R}) > 1$ is neither even nor it is a power of a single odd prime. Therefore, $char(\mathcal{R})$ has at least 2 prime divisor and in view of Proposition 4.5, $\mathcal{R}$ must have at least 4 self inverse elements i.e., $|S(\mathcal{R})| \geq 4$. Thus, it follows from Theorem 4.2, that girth of $S_i(\mathcal{R})$ is 3 i.e., $S_i(\mathcal{R})$ contains a cycle of length 3. Hence $S_i(\mathcal{R})$ cannot be bipartite.

□

Remark 4.7. Notice that even if $char(\mathcal{R}) = p^s$, $s \geq 1$, it is not necessary that $|S(\mathcal{R})| = 2$. For example, $\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s}$. Thus, the condition that $|S(\mathcal{R})| = 2$ in the above theorem cannot be ignored.

As an application of the above result, we have

Corollary 4.8. *For a finite field $\mathbb{F}$, $S_i(\mathbb{F})$ is always bipartite.*

From Lemma 2.2 it follows that the number of self inverse elements in a commutative ring is always a non-negative power of 2. Based on this, in the rest of this section, we discuss the planarity of $S_i(\mathcal{R})$ and begin with analyzing this for finite fields.

Theorem 4.9. *If $\mathbb{F}$ is a finite field, then $S_i(\mathbb{F})$ is a planar graph.*

Proof. Let $\mathbb{F}$ be a finite field with $|\mathbb{F}| = p^n$. Then by Theorem 2.21, any connected component of $S_i(\mathbb{F})$ is either a path or a cycle of finite lengths, however both are planar. Since, each connected component of $S_i(\mathbb{F})$ is planar, therefore we conclude that $S_i(\mathbb{F})$ is a planar graph. □

Thereafter, we see that the planarity of $S_i(\mathcal{R})$, for a finite commutative ring $\mathcal{R}$ can be completely established, whenever $|S(\mathcal{R})| \neq 4$ (i.e., the number of self inverse elements in $\mathcal{R}$ is other than 4).

Theorem 4.10. *If $\mathcal{R}$ is a ring with $|S(\mathcal{R})| \leq 2$, then $S_i(\mathcal{R})$ is planar. Whereas, $S_i(\mathcal{R})$ is non-planar for $|S(\mathcal{R})| \geq 8$.*

Proof. Let $|S(\mathcal{R})| = 1$. Then each vertex in $S_i(\mathcal{R})$ can have exactly one adjacent vertex, which implies that $S_i(\mathcal{R})$ is a finite union of paths of length 2 i.e., K_2. Therefore, $S_i(\mathcal{R})$ is planar. Again, if $|S(\mathcal{R})| = 2$, then degree of a vertex in $S_i(\mathcal{R})$ is either one or two. So, each connected component of $S_i(\mathcal{R})$ is either a path or a cycle. Hence, $S_i(\mathcal{R})$ is again planar.
On the other hand, assume that $\mathcal{R}$ is a ring such that $|S(\mathcal{R})| = k \geq 8$. First, recall that a simple planar graph must have at least one vertex of degree at most 5 (see Theorem 13.6 [16]). But it follows from Remark 2.9 that the degree of a vertex in $S_i(\mathcal{R})$ is either $(k-1)$ or k, where $k = |S(\mathcal{R})| \geq 8$. Hence, we conclude that $S_i(\mathcal{R})$ is non-planar, whenever $|S(\mathcal{R})| \geq 8$. $\square$

As, it remains to check the planarity of $S_i(\mathcal{R})$ when $|S(\mathcal{R})| = 4$, so we provide the following result for the case of even characteristic and this is a sound application of Euler's formula.

Theorem 4.11. *If $\mathcal{R}$ is a commutative ring with unity such that $char(\mathcal{R})$ is even and $|S(\mathcal{R})| = 4$, then $S_i(\mathcal{R})$ is non-planar.*

Proof. Let $char(\mathcal{R})$ be even and $|S(\mathcal{R})| = 4$. Since, characteristic of ring is a divisor of its order, therefore the number of elements in $\mathcal{R}$ is even (say) $2m$. However, Theorem 2.7 infers that the graph $S_i(\mathcal{R})$ is a 4-regular graph having $\frac{4\times 2m}{2} = 4m$ edges. Further, in view of Theorem 4.1, $S_i(\mathcal{R})$ is triangle free i.e., it cannot have a cycle of length 3. On contrary, suppose that $S_i(\mathcal{R})$ is planar. Now, if $S_i(\mathcal{R})$ is connected, then as a consequence of Euler's formula, the number of edges in $S_i(\mathcal{R})$ cannot exceed $2n-4$, where n is the order of $S_i(\mathcal{R})$ (see Theorem 6.1.23 [15]). Although, the order and size of $S_i(\mathcal{R})$ is $2m$ and $4m$, respectively and therefore, we have $4m \leq 4m-4$, which is a contradiction. Thus, $S_i(\mathcal{R})$ cannot be a planar graph. Observe that even if $S_i(\mathcal{R})$ is disconnected, then on applying the same argument to any connected component G' of $S_i(\mathcal{R})$, non-planarity of $S_i(\mathcal{R})$ can be established, because G' is also triangle free and order of G' is at least 5. Hence, the graph $S_i(\mathcal{R})$ is non-planar. $\square$

Next, in the case of odd characteristic, we prove

Theorem 4.12. *If $|S(\mathbb{Z}_n)| = 4$ and n is an odd number, then $S_i(\mathbb{Z}_n)$ is a planar graph.*

Before proving Theorem 4.12, we shall discuss an example for better understanding of the planar representation of $S_i(\mathbb{Z}_n)$, where n is an integer as required in the aforesaid theorem.

Example 4.13. Consider $\mathcal{R} = \mathbb{Z}_{15} \cong \mathbb{Z}_3 \times \mathbb{Z}_5$ and contains four self-inverse elements. To examine the planarity of the self-inverse element graph of finite rings $\mathbb{Z}_n$, the graph is visualized as embedded on a right circular cylinder of finite height. The cylinder is divided into two equal parts by a plane through its diameter in such a way that adjacent vertices do not lie on opposite sides of the resulting intersection lines; consequently, some vertices lie on these lines, and all edges remain entirely within one of the two subfaces. Edges on one half of the cylinder are represented by solid lines, while those on the other half are shown by dotted lines. Accordingly, the self-inverse element graph $\mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{15}$ can be visualized on the cylindrical surface as illustrated in Figure 5, and therefore it is a planar graph.

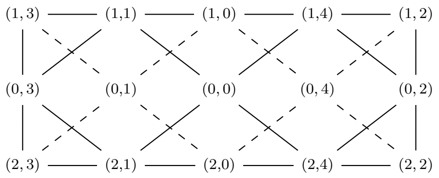
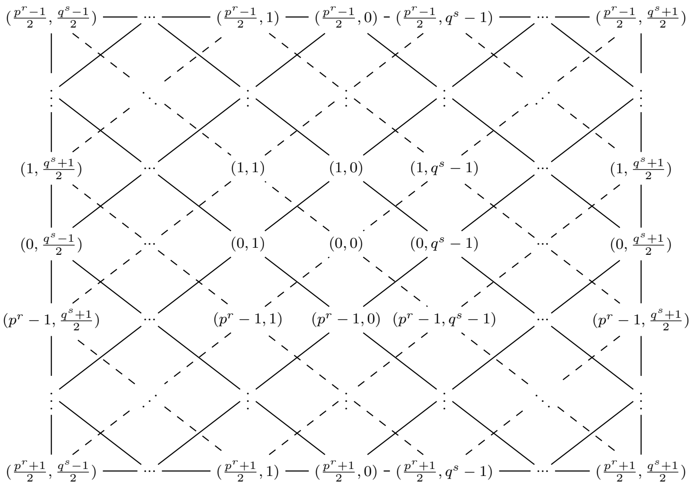


FIGURE 5. $S_i(\mathbb{Z}_3 \times \mathbb{Z}_5)$

Now, we will give the proof of Theorem 4.12.

Proof. Let n be an odd integer and $|S(\mathbb{Z}_n)| = 4$. Then, in view of Theorem 3.3, this holds iff $n = p^r q^s$, where p and q are distinct odd primes and $r, s \geq 1$. Clearly, elements of $\mathbb{Z}_{p^r q^s}$ can be identified as an element in the ring $\mathbb{Z}_{p^r} \times \mathbb{Z}_{q^s}$, as $\mathbb{Z}_{p^r q^s} \cong \mathbb{Z}_{p^r} \times \mathbb{Z}_{q^s}$. Also, $S(\mathbb{Z}_p^r \times \mathbb{Z}_q^s) = \{(1,1), (p^r-1,1), (1,q^s-1), (p^r-1,q^s-1)\}$. Further, in view of Theorem 2.8, the degree of each vertex in $\mathbb{Z}_p^r \times \mathbb{Z}_q^s$ is 4, as $S(\mathbb{Z}_p^r \times \mathbb{Z}_q^s) = 4$, except the four vertices namely; $\left(\frac{p^r-1}{2}, \frac{q^s-1}{2}\right), \left(\frac{p^r-1}{2}, \frac{q^s+1}{2}\right), \left(\frac{p^r+1}{2}, \frac{q^s-1}{2}\right)$ and $\left(\frac{p^r+1}{2}, \frac{q^s+1}{2}\right)$ have their degree 3. Thus, the planarity of the graph $S_i(\mathbb{Z}_{p^r q^s})$ can be visualized by wrapped its vertices around a right circular cylinder as given below:



Hence, the graph $S_i(\mathbb{Z}_n)$ is planar, when $n = p^r q^s$, where p, q are odd primes and $r, s > 0$. □

As a consequence of Proposition 4.5, we observe that if $char(\mathcal{R}) = k$ is odd and $|S(\mathcal{R})| = 4$, then $char(\mathcal{R}) = k$ can not have more than two prime divisors. Thus, we conclude that $char(\mathcal{R})$ is either p^r or $p^r q^s$, where p, q are odd primes and $r, s > 0$. Now, in case $char(\mathcal{R}) = p^r q^s$, we have

Theorem 4.14. *If $\mathcal{R}$ is a commutative ring with unity such that $|S(\mathcal{R})| = 4$ and $char(\mathcal{R}) = p^r q^s$, where p and q are odd primes, $r, s > 0$. Then $S_i(\mathcal{R})$ is planar if $\mathcal{R} = \mathbb{Z}_{p^r q^s}$, otherwise, non-planar.*

Proof. Let $char(\mathcal{R}) = p^r q^s$ and $|S(\mathcal{R})| = 4$, then $\mathcal{R}$ has a subring isomorphic to $\mathbb{Z}_{p^r q^s}$.Thus, if $\mathcal{R} = \mathbb{Z}_{p^r q^s}$, then by Theorem 4.12, $S_i(\mathcal{R})$ is a planar graph. Otherwise, if the subring $\mathbb{Z}_{p^r q^s}$ is a proper subring of $\mathcal{R}$, then there exists an element $a \in \mathcal{R} \setminus \mathbb{Z}_{p^r q^s}$. Again, since $\mathbb{Z}_{p^r q^s}$ is closed under addition. Therefore, $S_i(\mathcal{R})$ has a connected component isomorphic to $S_i(\mathbb{Z}_{p^r q^s})$ and $S_i(\mathcal{R})$ cannot be connected. Now, in view of Theorem 3.3, $|S(\mathbb{Z}_{p^r q^s})| = 4$, so $S(\mathcal{R}) = S(\mathbb{Z}_{p^r q^s})$, as $|S(\mathbb{Z}_{p^r q^s})| = 4$. Therefore Theorem 2.8, infers that $S_i(\mathcal{R})$ have exactly 4 vertices of degree 3 and the degree of the remaining vertices is 4. However, all the vertices of degree 3 lies in the connected component of $S_i(\mathcal{R})$ induced by its subring $\mathbb{Z}_{p^r q^s}$. Thus, all the other components in $S_i(\mathcal{R})$ are 4-regular graphs with at least 5 vertices in them. Now, as observed in the proof of Theorem 4.11, the size of a planar graph cannot exceed $2n - 4$, where n is the order of the graph. Now, applying this to any connected component X of $S_i(\mathcal{R})$ of order m, containing the element $a \in \mathcal{R} \setminus \mathbb{Z}_{p^r q^s}$. We have $2m \leq 2m - 4$, as X is 4-regular and it cannot contain any cycle of length three i.e., X is triangle free. This is impossible. Thus, we conclude that any connected component of $S_i(\mathcal{R})$ other than $S_i(\mathbb{Z}_{p^r q^s})$ (if exists) is non-planar.

□

Till now, in this section we have completely characterized the planarity of $S_i(\mathcal{R})$, for an arbitrary commutative ring $\mathcal{R}$ with unity, except for the case when $\mathcal{R}$ has exactly 4 self inverse elements and $char(\mathcal{R}) = p^r$, for some odd prime p, $r > 0$. Consequently, we are posing the following open problem:

Problem: If $|S(\mathcal{R})| = 4$ and $char(\mathcal{R}) = p^r$, then determine the planarity of $S_i(\mathcal{R})$.

CONCLUSIONS

Algebraic graphs have attracted significant attention in recent years. In this paper, we introduced the self-inverse element graph $S_i(\mathcal{R})$ over a finite commutative ring $\mathcal{R}$ with unity and investigated its graph-theoretic properties, including size, completeness, bipartiteness, girth, and regularity. We completely characterized the structure of this graph for finite fields and determined the connectedness of $S_i(\mathcal{R})$ over the ring $\mathbb{Z}_n$ and its Cartesian products. We also identified all rings for which $S_i(\mathcal{R})$ is a path or a cycle. Finally, we conclude with an open problem concerning the remaining case in the planarity characterization of $S_i(\mathcal{R})$ for an arbitrary commutative ring with unity.

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