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HIGHER ORDER MELNIKOV FUNCTIONS FOR PIECEWISE SMOOTH DIFFERENTIAL SYSTEMS WITH FOUR ZONES

KRISHNAT MASALKAR[1]* and SUBHASH KENDRE[2]

ABSTRACT. In this work, we develop a systematic algorithm for computing Melnikov functions of arbitrary order for planar piecewise smooth Hamiltonian systems separated by the coordinate axes and perturbed within the class of polynomial differential systems. Two equivalent formulations of the Melnikov functions are obtained. The first formulation involves the flight times and divergence integrals of certain vector fields along the trajectories of the underlying Hamiltonian system, while the second formulation avoids the explicit use of flight times and trajectory expressions. By applying the resulting Melnikov functions, we determine the number of limit cycles bifurcating from planar piecewise Hamiltonian systems with four regions under polynomial perturbations.

1. INTRODUCTION

Smooth dynamical systems play a fundamental role in modeling and analyzing a wide range of physical phenomena, including fluid motion, elastic deformations, nonlinear optical effects, and biological processes. Nevertheless, many important real-world systems fall outside the scope of smooth theory. Typical examples include electrical circuits with switching elements, mechanical systems involving impacts or free-play, frictional and sliding processes, and various control systems. Such models naturally lead to nonsmooth or piecewise smooth dynamical systems; see, for instance, [1].

The study of limit cycles is of central importance in the qualitative theory of dynamical systems. One of the most famous open problems in this area is Hilbert's sixteenth problem, which asks for the maximum number and relative positions of limit cycles of planar polynomial differential systems. When a planar smooth differential system possesses a period annulus and is subjected to perturbations within a class of analytic or polynomial systems, a natural question is how many limit cycles bifurcate from the period annulus.

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* Corresponding author.
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To address this problem, one commonly introduces a Poincaré map on a suitable transversal section and defines the associated displacement map. The number of isolated zeros of the displacement map corresponds exactly to the number of limit cycles bifurcating from the period annulus. Expanding the displacement map as a power series in the perturbation parameter ε, the coefficient $M_k(h)$ of ε^k is called the Melnikov function of order k. If $M_1(h) \equiv M_2(h) \equiv \cdots \equiv M_{k-1}(h) \equiv 0$, then the number of zeros of $M_k(h)$ determines the number of bifurcating limit cycles. In [2], explicit expressions for the first- and second-order Melnikov functions were derived for polynomial perturbations, and these results were applied to classical models such as the harmonic oscillator and the Duffing oscillator.

For nonsmooth or piecewise smooth differential systems, the computation of Melnikov functions is considerably more challenging, and the complexity increases rapidly for higher-order terms. In the smooth setting, an algorithm for computing higher-order Melnikov functions was developed by Iliev and Françoise; see [2]. However, no analogous general algorithm is currently available for piecewise smooth differential systems.

The Melnikov function approach has nevertheless been extended to certain classes of piecewise smooth systems. In [10], Liu and Han derived the first-order Melnikov function for planar piecewise smooth Hamiltonian systems and applied it to the study of limit cycle bifurcations. Subsequently, Liang, Han, and Ramanovski [9] investigated limit cycles arising from perturbations of a piecewise smooth linear Hamiltonian system with a generalized homoclinic loop involving a linear saddle and a linear center. Furthermore, Wang, Han, and Constantinescu studied piecewise Hamiltonian systems with four switching lines under first-order analytic perturbations in [12, 11]. More recently, Yang [13] obtained the first-order Melnikov function for piecewise smooth near-Hamiltonian systems separated by two straight lines.

In parallel, averaging theory and Melnikov methods have been extensively developed for piecewise differential systems in recent years; see, for example, [15], [16], and [17]. Both approaches are classical tools for estimating the number of limit cycles bifurcating from families of periodic orbits by analyzing the zeros of the corresponding averaging or Melnikov functions. However, averaging theory typically relies on an appropriate polar coordinate transformation. Recently, it was shown in [14] that the averaging method and the Melnikov method are not equivalent in the context of piecewise differential systems. In [6], the authors studied the dynamics of solutions of semilinear partial differential equations which is useful in biological applications. Dynamical systems originate from modelling problems in biology. In [7], the authors studied the dynamics of solutions of special kinds of delay differential equations. This motivates the study of piecewise smooth dynamical systems.

2. Melnikov functions in terms of time of flights

Let

$$H_i, f_{ij}, g_{ij} \in C^{\infty}(\mathbb{R}^2) \text{ and } |\varepsilon| << 1$$
$$\Omega_1 = \{(x,y)\colon x>0, y<0\}, \Omega_2 = \{(x,y)\colon x>0, y>0\}$$
$$\Omega_3 = \{(x,y)\colon x<0, y>0\}, \Omega_4 = \{(x,y)\colon x<0, y<0\}$$

We aim to find the number of crossing limit cycles of a discontinuous piece-wise linear near Hamiltonian differential system on $\mathbb{R}^2$ given by

$$(\dot{x},\dot{y}) = (H_{iy}(x,y), -H_{ix}(x,y)) + \sum_{j=1}^{n} \varepsilon^j (f_{ij}(x,y),\ g_{ij}(x,y)),\ (x,y) \in \Omega_i. \quad (2.1)$$

For $\varepsilon = 0$, we get unperturbed system

$$(\dot{x},\dot{y}) = (H_{iy}(x,y), -H_{ix}(x,y)), \qquad (x,y) \in \Omega_i, \qquad i \in \{1,2,3,4\} \quad (2.2)$$

and the four zones Ω_i $i \in \{1,2,3,4\}$ separated by rays

$$\Gamma_1 = \{(x,y)\colon x=0, y<0\}, \Gamma_2 = \{(x,y)\colon x>0, y=0\}$$
$$\Gamma_3 = \{(x,y)\colon x=0, y>0\}, \Gamma_4 = \{(x,y)\colon x<0, y=0\}.$$

For the unperturbed system (2.2) to have a period annulus with crossing periodic orbits, the following assumptions must be made.

(A_1) There is an open interval $J = (a,b)$ and a period annulus consisting of family of crossing periodic orbits of type 1, L_h with $h \in J$ such that each orbit crosses the rays Γ_1, Γ_2, Γ_3 and Γ_4 in points

$$A_1(h) = (0, a_1(h)),\ A_2(h) = (a_2(h), 0),$$
$$A_3(h) = (0, a_3(h)) \text{ and } A_4(h) = (a_4(h), 0)$$

respectively, satisfying the following equations

$$H_i(A_i(h)) = H_i(A_{i+1}(h)),\ i \in \{1,2,3,4\} \text{ where } A_5(h) = A_1(h) \text{ for all } h \in J.$$

and

(A_2) crossing conditions(transversal conditions): For all $h \in J$.

$$H_{ix}(A_2(h)) \neq 0,\ (i=1,2) \qquad H_{ix}(A_4(h)) \neq 0\ (i=3,4)$$
$$H_{iy}(A_1(h)) \neq 0\ (i=1,4) \qquad H_{iy}(A_3(h)) \neq 0\ (i=2,3).$$

Moreover for each $h \in J$, the crossing periodic orbit

$$L_h : \widehat{A_1A_2} \cup \widehat{A_2A_3} \cup \widehat{A_3A_4} \cup \widehat{A_4A_1}$$

passes through the four points $A_1(h), A_2(h), A_3(h), A_4(h)$.

Let the solution of the perturbed system (2.1) in Ω_1 starts at $A_1(h) = (a_1(h), 0)$ and meets at $A_{2\varepsilon}(h) = (a_{2\varepsilon}(h), 0)$ on Γ_2. Let the solution in Ω_2 starts at $A_{2\varepsilon}(h) = (a_{3\varepsilon}(h), 0)$ and meets the point $A_{3\varepsilon}(h) = (0, a_3(\varepsilon, h))$. Let the solution in Ω_3 starts at $A_{3\varepsilon}(h)$ and meets at $A_{4\varepsilon} = (a_{4\varepsilon}(h), 0)$. Finally the solution in Ω_4 starts at $A_{4\epsilon}(h)$ and meets at $A_{1\varepsilon}(h) = (0, a_1(\varepsilon, h))$ on Γ_1.

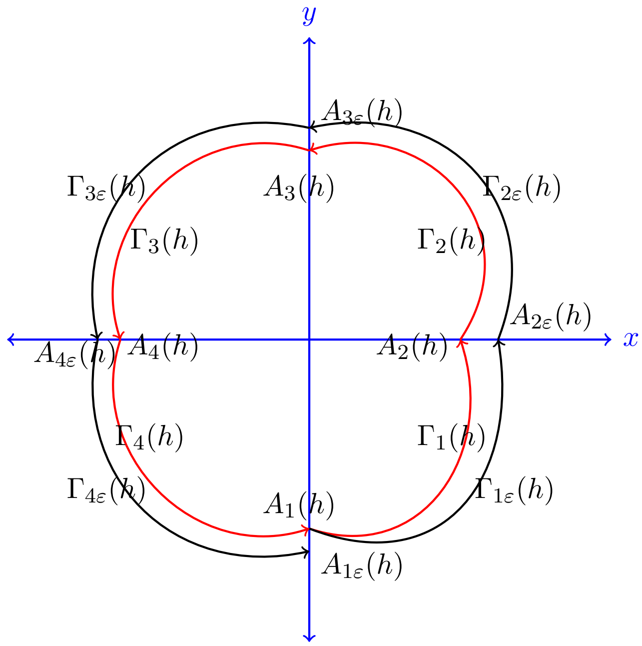

FIGURE 1. Periodic trajectory of (2.2) and trajectory of the system (2.1) .

The displacement map whose zeros give the number of limit cycles of the system, has expansion in the powers of ε as below:

$$H_1(A_{1\varepsilon}(h)) - H_1(A(h)) = \sum_{i=1}^{k-1} \varepsilon^i M_i(h) + \mathcal{O}(\varepsilon^n), \text{ for all } h \in J, \tag{2.3}$$

where M_i is i^{th} order Melnikov function. If M_t is first non-vanishing Melnikov function on J then number of zeros of M_t is exactly equal to number of zeros of the bifurcation function for sufficiently small $\varepsilon \neq 0$.

The displacement map can be written as

$$\begin{aligned} d(\varepsilon, h) =& H_1(A_{1\varepsilon}(h)) - H_1(A_1) \\ =& H_1(A_{1\varepsilon}(h)) - H_4(A_{1\varepsilon}(h)) + H_4(A_{4\varepsilon}(h)) - H_3(A_{4\varepsilon}(h)) \\ & + H_3(A_{3\varepsilon}(h)) - H_2(A_{3\varepsilon}(h)) + H_2(A_{2\varepsilon}(h)) - H_1(A_{2\varepsilon}(h)) \\ & + H_1(A_{2\varepsilon}(h)) - H_1(A_1(h)) + H_4(A_{1\varepsilon}(h)) - H_4(A_{4\varepsilon}(h)) \\ & + H_3(A_{4\varepsilon}(h)) - H_3(A_{3\varepsilon}(h)) + H_2(A_{3\varepsilon}(h)) - H_2(A_{2\varepsilon}(h)). \end{aligned}$$

Let us define for $i \in \{1, 2, 3, 4\}$, $h \in J$ and $|\varepsilon| << 1$

$$d_{i+1,i}(\varepsilon, h) = H_{i+1}(A_{i+1,\varepsilon}(h)) - H_i(A_{i+1,\varepsilon}(h)) \tag{2.4}$$

$$d_{i,i}(\varepsilon, h) = H_i(A_{i+1,\varepsilon}(h)) - H_i(A_{i\varepsilon}(h) \tag{2.5}$$

where $H_5(x,y) = H_1(x,y),\ A_{5,\varepsilon}(h) = A_{1,\varepsilon}(h)$.
Therefore, the displacement map becomes

$$d(\varepsilon,h) = \sum_{i=1}^{4} d_{i+1,i}(\varepsilon,h) + \sum_{i=1}^{4} d_{ii}(\varepsilon,h). \tag{2.6}$$

In [4], the authors obtained the first and second order Melnikov functions for the system (2.1), which contains the time of flights of each subsystem as limits of some line integrals involved. Motivated by this, we can develop an algorithm to find higher Melnikov functions for the system (2.1).

The perturbed system is equivalent to the 1-form

$$dH_i + \sum_{j=1}^{\infty} \varepsilon^j w_{ij} = 0, \text{where } dH_i = H_{ix}dx + H_{iy}dy,\ w_{ij} = f_{ij}dy - g_{ij}dx,$$

$i \in \{1,2,3,4\}$ $\qquad w_{ij} \equiv 0\ j > n$.
Inductively, for $i \in \{1,2,3,4\}$ and $j \in \mathbb{N}$, We define the vector fields, associated 1–forms, and functions as follows.

$$\text{For } j = 1 : \psi_{i0} = -1, \Psi_{i1} = -\psi_{i0}\omega_{i1} = f_{i1}\,dy - g_{i1}\,dx = F_{i1}\,dy - G_{i1}\,dx,$$
$$\chi_{i1} = (F_{i1},\, G_{i1}), \psi_{i1} = \int_0^{T_i(x,y)} \operatorname{div}(\chi_{i1}) \circ \phi_t^i(x_0^i, y_0^i)\,dt,$$
$$(x,y) = \phi^i\big(T_i(x,y), x_0^i, y_0^i\big).$$
$$\text{For } j \geq 2 : \Psi_{ij} = -\sum_{k=1}^{j} \psi_{i,j-k}\omega_{ik} = F_{ij}\,dy - G_{ij}\,dx,$$
$$\chi_{ij} = (F_{ij},\, G_{ij}), \psi_{ij} = \int_0^{T_i(x,y)} \operatorname{div}(\chi_{ij}) \circ \phi_t^i(x_0^i, y_0^i)\,dt,$$
$$(x,y) = \phi^i\big(T_i(x,y), x_0^i, y_0^i\big).$$

Here $T_i(x,y)$ denotes the time of flight of the trajectory starting at (x_0^i, y_0^i) and ending at (x,y), and $\phi^i(t,x,y)$ is the flow of the unperturbed Hamiltonian system (2.2) in the region Ω_i.

To develop an algorithm for computing all higher-order Melnikov functions, we first establish two auxiliary lemmas.

Lemma 2.1. *For the system (2.1) and for each $i = 1,2,3,4$, we have*

$$dH_i(x,y) = -\sum_{j=0}^{k-1} dR_{ij}(x,y)\varepsilon^j - \Psi_{ik}(x,y)\varepsilon^k + \mathcal{O}(\varepsilon^{k+1}),\ k \in \mathbb{N}, \tag{2.7}$$

where $R_{ij}(x,y)$ satisfies the following decomposition

$$\Psi_{ij}(x,y) = \psi_{ij}(x,y)dH_i(x,y) + dR_{ij}(x,y),\ j \in \mathbb{N}. \tag{2.8}$$

and in particular, $R_{i0}(x,y) = 1$.

Proof. Since $\chi_{i0} = \dfrac{\partial H_{i0}}{\partial y}\dfrac{\partial}{\partial x} - \dfrac{\partial H_{i0}}{\partial x}\dfrac{\partial}{\partial y}$, and since ψ_{ij} is defined as the integral of the scalar function $\operatorname{div}(\chi_{ij})$ along the trajectories of χ_{i0}, we have

$$\chi_{i0}(\psi_{ij}) = \frac{d\psi_{ij}}{dt} = \operatorname{div}(\chi_{ij}).$$

Moreover,

$$\begin{aligned} dH_i \wedge d\psi_{ij} &= (H_{ix}\psi_{ijy} - H_{iy}\psi_{ijx})\, dx \wedge dy = -\chi_{i0}(\psi_{ij})\, dx \wedge dy \\ &= -\operatorname{div}(\chi_{ij})\, dx \wedge dy = -d\Psi_{ij}. \end{aligned}$$

Therefore, $d(\psi_{ij} dH_i) = d\psi_{ij} \wedge dH_i = d\Psi_{ij}$, and hence $d\big(\Psi_{ij} - \psi_{ij} dH_i\big) = 0$. That is, the 1–form $\Psi_{ij} - \psi_{ij} dH_i$ is closed.
Since the domain Ω_i is connected and 1-form $\Psi_{ij} - \psi_{ij} dH_i$ is closed, there exist a unique function R_{ij} such that $\Psi_{ij} = \psi_{ij} dH_i + dR_{ij}$.
Now using the above decomposition, we can prove (2.7) inductively.
For $k = 1$, we note that $R_{i0} = 1$ and $dH_i + \sum_{j=1}^{n} \varepsilon^j \omega_{ij} = 0$.
Hence, it is clear that $dH_i = -\varepsilon\omega_{i1} + \mathcal{O}(\varepsilon^2) = -\varepsilon\Psi_{i1} + \mathcal{O}(\varepsilon^2)$.
Further, for $k \geq 2$, consider the equation

$$\left(-\sum_{l=0}^{k-1} \psi_{il}\varepsilon^l\right)\left(dH_i + \sum_{j=1}^{n} \omega_{ij}\varepsilon^j\right) = 0.$$

Expanding sums, we get,

$$-\psi_{i0} dH_i - \left(\sum_{j=1}^{k-1} \psi_{ij} dH_i \varepsilon^j\right) - \sum_{l=0}^{k-1}\sum_{j=1}^{k} \psi_{il}\omega_{ij}\varepsilon^{l+j} = 0.$$

Re-indexing last double sum, we have

$$\begin{aligned} \sum_{l=0}^{k-1}\sum_{j=1}^{k} \psi_{il}\omega_{ij}\varepsilon^{l+j} &= \sum_{m\geq 1}\left(\sum_{\substack{l+j=m \\ 0\leq l\leq k-1,\ 1\leq j\leq k}} \psi_{il}\omega_{ij}\right)\varepsilon^m \\ &= \sum_{m=1}^{k-1}\left(\sum_{j=1}^{m} \psi_{i,m-j}\omega_{ij}\right)\varepsilon^m + \left(\sum_{j=1}^{k} \psi_{i,k-j}\omega_{ij}\right)\varepsilon^k + \mathcal{O}(\varepsilon^{k+1}). \end{aligned}$$

Therefore, we get

$$dH_i - \sum_{j=1}^{k-1} (\psi_{ij} dH_i)\,\varepsilon^j - \sum_{l=1}^{k-1}\left(\sum_{j=1}^{l} \psi_{i,l-j}\omega_{ij}\right)\varepsilon^l - \left(\sum_{j=1}^{k} \psi_{i,k-j}\omega_{ij}\right)\varepsilon^k + \mathcal{O}(\varepsilon^{k+1}) = 0.$$

Therefore, $$dH_i - \sum_{j=1}^{k-1} \psi_{ij} dH_i \varepsilon^j + \sum_{j=1}^{k-1} \Psi_{ij}\varepsilon^j = -\Psi_{ik}\varepsilon^k + \mathcal{O}(\varepsilon^{k+1})$$

$$ie., \quad dH_i + \sum_{j=1}^{k-1} (\Psi_{ij} - \psi_{ij} dH_i)\varepsilon^j = -\Psi_{ik}\varepsilon^k + \mathcal{O}(\varepsilon^{k+1}),$$

$$ie., \quad dH_i + \sum_{j=1}^{k-1} R_{ij}\varepsilon^j = -\Psi_{ik}\varepsilon^k + \mathcal{O}(\varepsilon^{k+1}).$$

Hence, we get the equation (2.8). □

For integers $n \geq k \geq 1$, the *partial Bell polynomial* $B_{n,k} = B_{n,k}(x_1, \dots, x_{n-k+1})$ is defined by

$$B_{n,k} = \sum \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!} \prod_{m=1}^{n-k+1} \left(\frac{x_m}{m!}\right)^{j_m}, \tag{2.9}$$

where the sum runs over all nonnegative integers $j_1, j_2, \dots, j_{n-k+1}$ satisfying

$$\sum_{m=1}^{n-k+1} j_m = k, \qquad \sum_{m=1}^{n-k+1} m j_m = n.$$

The *complete Bell polynomial* $B_n = B_n(x_1, x_2, \dots, x_n)$ is defined by

$$B_n = \sum_{k=1}^{n} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}). \tag{2.10}$$

Theorem 2.2 (Faà di Bruno)**.** *Let f and g be sufficiently smooth functions. Then for any integer $n \geq 1$, the n-th derivative of the composite function $f \circ g$ is given by*

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x))\, B_{n,k}\left(g'(x), g''(x), \dots, g^{(n-k+1)}(x)\right). \tag{2.11}$$

Let us write each of the coordinate function $a_i(\varepsilon, h)$ of the point $A_{i\varepsilon}$ as power series of ε. Suppose

$$a_i(\varepsilon, h) = \sum_{j=0}^{\infty} \frac{1}{j!} a_{ij}(h)\varepsilon^j, \text{ for } i = 1, 2, 3, 4$$

be Taylor's expansion of functions $a_i(\varepsilon, h)$, and $A_1(h) = (0, a_1(h)) = (0, a_{10}(h))$, $A_2(h) = (a_2(h), 0) = (a_{20}(h), 0)$, $A_3(h) = (0, a_3(h)) = (0, a_{30}(h))$ and $A_4(h) = (a_4(h), 0) = (a_{40}(h), 0)$ be the points of intersection of the trajectory of the unperturbed system (2.2).
Using Faá di Bruno's formula and Taylor's expansion, we have the following lemma. Let us define Bell's polynomial functions as $B^{(j)}_{i,m} = B_{i,m}(a_{j1}, \dots, a_{j-m+1,1})$

Lemma 2.3. *For system (2.1), the following expansions hold:*

$$H_i(A_{i\varepsilon}) - H_i(A_i) = \sum_{l=1}^{k} \left(\sum_{r=1}^{l} \frac{1}{r!} \frac{\partial^r H_i}{\partial z^r}(A_i)\, B_{l,r}^{(i)} \right) \varepsilon^l + \mathcal{O}(\varepsilon^{k+1}),$$

$$R_{ij}(A_{i\varepsilon}) - R_{ij}(A_i) = \sum_{l=1}^{k} \left(\sum_{r=1}^{l} \frac{1}{r!} \frac{\partial^r R_{ij}}{\partial z^r}(A_i)\, B_{l,r}^{(i)} \right) \varepsilon^l + \mathcal{O}(\varepsilon^{k+1}),$$

where $z = y$ for $i = 1, 3$ and $z = x$ for $i = 2, 4$, and $B_{l,r}$ denotes the partial Bell polynomials.

Proof. We expand $H_i(A_{i\varepsilon})$ and $R_{ij}(A_{i\varepsilon})$ in powers of ε about the point

$$A_i = A_i(h) = A_i(0, h).$$

Since $A_{i\varepsilon}$ admits an expansion of the form

$$A_{i\varepsilon} = A_i + \sum_{m \geq 1} a_{im} \varepsilon^m,$$

the compositions $H_i(A_{i\varepsilon})$ and $R_{ij}(A_{i\varepsilon})$ can be written as Taylor expansions in the single variable z. Applying Faà di Bruno's formula yields the stated expressions, with coefficients given by the partial Bell polynomials. □

Now we will construct an algorithm to find Melnikov functions of higher order.

Theorem 2.4. *The k^{th} order Melnikov function M_k for the system (2.1) is given by*

$$M_k(h) = a_{1,k}.$$

provided $M_i(h) = a_{1,i}(h) \equiv 0$ for all $i \in \{1, 2, \ldots, k-1\}$ where $a_{1,i}'s$ are given by (2.16), (2.17), (2.18) and (2.19).

Proof. We have the displacement function

$$d(\varepsilon, h) = \sum_{i=1}^{4} d_{i+1,i}(\varepsilon, h) + \sum_{i=1}^{4} d_{ii}(\varepsilon, h).$$

Using lemma 2.3, we can write

$$\begin{aligned} d_{i+1,i}(\varepsilon, h) &= H_{i+1}(A_{i+1,\varepsilon}) - H_i(A_{i+1,\varepsilon}) = H_{i+1}(A_{i+1}) - H_i(A_{i+1}) \\ &+ \sum_{l=1}^{k} \left(\sum_{r=1}^{l} \frac{\partial H_{i+1}^r}{\partial z^r}(A_{i+1}) \frac{B_{l,r}^{(i+1)}}{l!} \right) \varepsilon^l + \mathcal{O}(\varepsilon^{k+1}) \\ &- \sum_{l=1}^{k} \left(\sum_{r=1}^{l} \frac{\partial H_i^r}{\partial z^r}(A_{i+1}) \frac{B_{l,r}^{(i+1)}}{l!} \right) \varepsilon^l + \mathcal{O}(\varepsilon^{k+1}), \end{aligned}$$

where $z=x$ if $i=1,3$ and $z=y$ if $i=2,4$ and note that $A_5=A_1$.
Similarly,

$$d_{i,i}(\varepsilon,h)=H_i(A_{i+1,\varepsilon})-H_i(A_{i,\varepsilon})=H_i(A_{i+1})-H_i(A_i)$$
$$+\sum_{l=1}^{k}\left(\sum_{r=1}^{l}\frac{\partial H_i^r}{\partial z^r}(A_{i+1})\frac{B_{l,r}^{(i+1)}}{l!}\right)\varepsilon^l+\mathcal{O}(\varepsilon^{k+1})-\sum_{l=1}^{k}\left(\sum_{r=1}^{l}\frac{\partial H_i^r}{\partial z^r}(A_i)\frac{B_{l,r}^{(i)}}{l!}\right)\varepsilon^l$$
$$+\mathcal{O}(\varepsilon^{k+1}).$$

Therefore the displacement function $d(\varepsilon,h)$ becomes

$$d(\varepsilon,h)=\sum_{i=1}^{4}\sum_{l=1}^{k}\left(\sum_{r=1}^{l}\frac{\partial H_{i+1}^r}{\partial z^r}(A_{i+1})\frac{B_{l,r}^{(i+1)}}{l!}\right)\varepsilon^l-\sum_{i=1}^{4}\sum_{l=1}^{k}\left(\sum_{r=1}^{l}\frac{\partial H_i^r}{\partial z^r}(A_i)\frac{B_{l,r}^{(i)}}{l!}\right)\varepsilon^l$$
$$+\mathcal{O}(\varepsilon^{k+1}).$$

Since

$$d(\varepsilon,h)=\sum_{i=1}^{k-1}\varepsilon^iM_i(h)+\mathcal{O}(\varepsilon^n)\text{ for all }h\in J,$$

the k^{th} order Melnikov function M_k is given by

$$M_k(h)=\sum_{i=1}^{4}\sum_{r=1}^{k}\left(\frac{\partial H_{i+1}^r}{\partial z^r}(A_{i+1})\frac{B_{l,r}^{(i+1)}}{l!}-\frac{\partial H_i^r}{\partial z^r}(A_i)\frac{B_{l,r}^{(i)}}{l!}\right)+\mathcal{O}(\varepsilon^{k+1}). \tag{2.12}$$

From (2.7), for $i=2,3,4$, we have

$$dH_i=-\sum_{j=0}^{k-1}\varepsilon^jdR_{ij}-\varepsilon^k\Psi_{ik}+\mathcal{O}(\varepsilon^{k+1}).$$

Integrating along the arc of the trajectory of the perturbed system in Ω_i, we get

$$H_i(A_{i+1,\varepsilon})-H_i(A_{i,\varepsilon})=-\sum_{j=0}^{k-1}(R_{ij}(A_{i+1,\varepsilon})-R_{ij}(A_{i,\varepsilon}))\varepsilon^j-\left(\int_{\widehat{A_{i,\varepsilon}A_{i+1,\varepsilon}}}\Psi_{ik}\right)\varepsilon^k$$
$$+\mathcal{O}(\varepsilon^{k+1}).$$

This implies that,

$$H_i(A_{i+1,\varepsilon})-H_i(A_{i,\varepsilon})=-\sum_{j=0}^{k-1}(R_{ij}(A_{i+1,\varepsilon})-R_{ij}(A_{i,\varepsilon}))\varepsilon^j-\left(\int_{\widehat{A_iA_{i+1}}}\Psi_{ik}\right)\varepsilon^k$$
$$+\mathcal{O}(\varepsilon^{k+1}).$$

Using lemma 2.3, we get

$$\sum_{l=1}^{k}\left(\sum_{r=1}^{l}\frac{\partial H_i^r}{\partial z^r}(A_{i+1})\frac{B_{l,r}^{(i+1)}}{l!}-\frac{\partial H_i^r}{\partial z^r}(A_i)\frac{B_{l,r}^{(i)}}{l!}\right)\varepsilon^l$$
$$=-\sum_{j=0}^{k-1}(R_{ij}(A_{i+1})-R_{ij}(A_i))\varepsilon^j+\sum_{l+j=1}^{k}\left(\sum_{r=1}^{l}\frac{\partial R_{i,j}^r}{\partial z^r}(A_{i+1})\frac{B_{l,r}^{(i+1)}}{l!}\right)\varepsilon^{l+j}$$
$$-\sum_{l+j=1}^{k}\left(\sum_{r=1}^{l}\frac{\partial R_{i,j}^r}{\partial z^r}(A_i)\frac{B_{l,r}^{(i)}}{l!}\right)\varepsilon^{l+j}-\left(\int_{\widehat{A_iA_{i+1}}}\Psi_{ik}\right)\varepsilon^k+\mathcal{O}(\varepsilon^{k+1}).$$

Comparing the coefficient of ε^k on both sides of the above equation we get

$$\frac{1}{k!}\left(\frac{\partial H_i^r}{\partial z^r}(A_{i+1})B_{k,r}^{(i+1)}-\frac{\partial H_i^r}{\partial z^r}(A_i)B_{k,r}^{(i)}\right)$$
$$=\sum_{r=1}^{l}\left(\frac{\partial R_{i,k-l}^r}{\partial z^r}(A_{i+1})\frac{B_{l,r}^{(i+1)}}{l!}-\frac{\partial R_{i,k-l}^r}{\partial z^r}(A_i)\frac{B_{l,r}^{(i)}}{l!}\right)-\int_{\widehat{A_iA_{i+1}}}\Psi_{ik}. \tag{2.13}$$

Let

$$\mathcal{L}_{i,j,k}=\sum_{r=2}^{k}\frac{\partial H_i^r}{\partial z^r}(A_j)\frac{B_{k,r}^{(j)}}{k!},\qquad \mathcal{R}_{i,j,k}=\sum_{l=1}^{k}\left(\sum_{r=1}^{l}\frac{\partial R_{i,k-l}^r}{\partial z^r}(A_j)\frac{B_{l,r}^{(j)}}{l!}\right).$$

Hence, the equation (2.13) becomes,

$$\frac{1}{k!}\frac{\partial H_i}{\partial z}(A_{i+1})a_{i+1,k}-\frac{1}{k!}\frac{\partial H_i}{\partial z}(A_i)a_{i,k}+\mathcal{L}_{i,i+1,k}-\mathcal{L}_{i,i,k}$$
$$=\mathcal{R}_{i,i+1,k}-\mathcal{R}_{i,i,k}-\int_{\widehat{A_iA_{i+1}}}\Psi_{ik}$$

Therefore,

$$a_{i+1,k}=\frac{\frac{\partial H_i}{\partial z}(A_i)}{\frac{\partial H_i}{\partial z}(A_{i+1})}a_{i,k}-\frac{k!}{\frac{\partial H_i}{\partial z}(A_{i+1})}$$
$$\left((\mathcal{L}_{i,i+1,k}-\mathcal{L}_{i,i,k})-(\mathcal{R}_{i,i+1,k}-\mathcal{R}_{i,i,k})+\int_{\widehat{A_iA_{i+1}}}\Psi_{ik}\right) \tag{2.14}$$

Note that $z=y$ when $A_i=A_1$ or A_3 and $z=x$ when $A_i=A_2$ or A_4 and last equation holds for $i=2,3,4$ with $A_5=A_1$.
Similarly for $i=1$, we have

$$\frac{1}{k!}\frac{\partial H_1}{\partial z}(A_2)a_{2,k}+\mathcal{L}_{1,2,k}=\mathcal{R}_{1,2,k}-\int_{\widehat{A_1A_2}}\Psi_{1,k}.$$

$$\text{Therefore, }\ a_{2k}=-\frac{k!}{\frac{\partial H_1}{\partial z}(A_2)}\left(\mathcal{L}_{1,2,k}-\mathcal{R}_{1,2,k}+\int_{\widehat{A_1A_2}}\Psi_{1,k}\right). \tag{2.15}$$

Observe that $\mathcal{L}_{i,j,k}$ is a function of $a_{j,1},\ldots,a_{j,k-1}$ only, and since $\mathcal{R}_{i,j,k}=0$ if $l=k$ due to $R_{i,0}=1$. Hence $\mathcal{R}_{i,j,k}$ becomes

$$\mathcal{R}_{i,j,k}=\sum_{l=1}^{k-1}\left(\sum_{r=1}^{l}\frac{\partial R^r_{i,k-l}}{\partial z^r}(A_j)\frac{B^{(j)}_{l,r}}{l!}\right).$$

and hence is a function of $a_{j,1},\ldots a_{j,k-1}$ only. Hence the equations, (2.14) and (2.15) can be solved explicitly for $a'_{i,j}s$. We write the equations and can find $a_{1,1},a_{1,2},a_{1,3},\ldots$.

$$a_{1,k}=\frac{\frac{\partial H_4}{\partial x}(A_4)}{\frac{\partial H_4}{\partial y}(A_1)}a_{4,k}-\frac{k!}{\frac{\partial H_4}{\partial y}(A_1)}\left((\mathcal{L}_{4,1,k}-\mathcal{L}_{4,4,k})-(\mathcal{R}_{4,1,k}-\mathcal{R}_{4,4,k})+\int_{\widehat{A_4A_1}}\Psi_{4k}\right), \tag{2.16}$$

$$a_{4,k}=\frac{\frac{\partial H_3}{\partial y}(A_3)}{\frac{\partial H_3}{\partial x}(A_4)}a_{3,k}-\frac{k!}{\frac{\partial H_3}{\partial x}(A_4)}\left((\mathcal{L}_{3,4,k}-\mathcal{L}_{3,3,k})-(\mathcal{R}_{3,4,k}-\mathcal{R}_{3,3,k})+\int_{\widehat{A_3A_4}}\Psi_{3k}\right), \tag{2.17}$$

$$a_{3,k}=\frac{\frac{\partial H_2}{\partial x}(A_2)}{\frac{\partial H_2}{\partial y}(A_3)}a_{2,k}-\frac{k!}{\frac{\partial H_2}{\partial y}(A_3)}\left((\mathcal{L}_{2,3,k}-\mathcal{L}_{2,2,k})-(\mathcal{R}_{2,3,k}-\mathcal{R}_{2,2,k})+\int_{\widehat{A_2A_3}}\Psi_{2k}\right), \tag{2.18}$$

$$a_{2,k}=-\frac{k!}{\frac{\partial H_1}{\partial x}(A_2)}\left(\mathcal{L}_{1,2,k}-\mathcal{R}_{1,2,k}+\int_{\widehat{A_1A_2}}\Psi_{1,k}\right). \tag{2.19}$$

We can solve the above equations recursively to find $M_k=a_{1,k}$. ☐

3. LIMIT CYCLE BIFURCATION

In this section, we find the number of limit cycle bifurcated from the period annulus of some piecewise smooth systems with four zones using Melnikov functions.

Example 3.1. Consider the piecewise smooth system on $\mathbb{R}^2$ with

$$(\sigma_1,\sigma_2,\sigma_3,\sigma_4)=(1,-1,-1,1),$$

$$(\dot{x},\dot{y})=(\sigma_i,\,2x)+\sum_{j=1}^{n}\varepsilon^j\big(f_{ij}(x,y),\,g_{ij}(x,y)\big),\qquad (x,y)\in\Omega_i, \tag{3.1}$$

where

$$f_{ij}(x,y)=\sum_{r+s=j}a_{i,rs}x^ry^s,\qquad g_{ij}(x,y)=\sum_{r+s=j}b_{i,rs}x^ry^s,$$

and

$$\Omega_1=\{x>0,y<0\},\quad \Omega_2=\{x>0,y>0\},$$
$$\Omega_3=\{x<0,y>0\},\quad \Omega_4=\{x<0,y<0\}.$$

For $(\sigma_1,\sigma_2,\sigma_3,\sigma_4)=(1,-1,-1,1)$, the unperturbed system

$$(\dot{x},\dot{y})=(\sigma_i,2x)$$

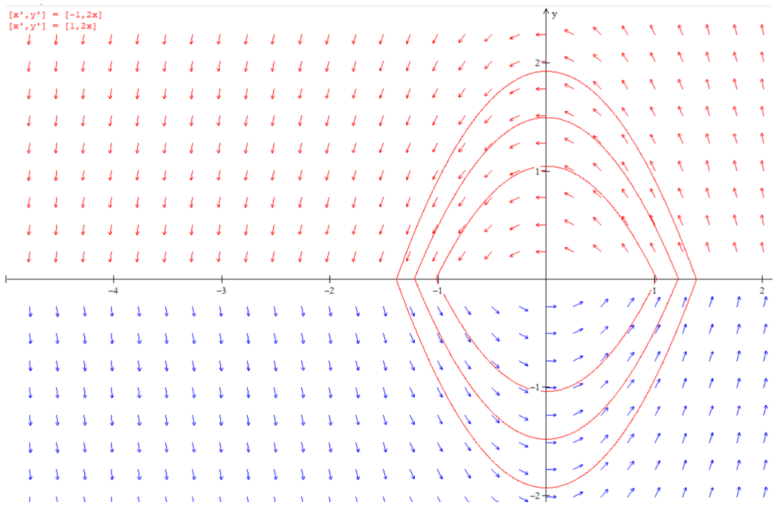

FIGURE 2. Period annulus of the system 3.1 at $\varepsilon = 0$

is Hamiltonian with $H_i(x,y) = \sigma_i y - x^2$. The periodic orbit γ_h is given by

$$\gamma_h : \begin{cases} y = x^2 - h, & (x,y) \in \Omega_1, \\ y = h - x^2, & (x,y) \in \Omega_2, \\ y = h - x^2, & (x,y) \in \Omega_3, \\ y = x^2 - h, & (x,y) \in \Omega_4, \end{cases} \quad 0 < h < \infty,$$

with intersection points $A_1 = (\sqrt{h}, 0)$, $A_2 = (0, h)$, $A_3 = (-\sqrt{h}, 0)$, $A_4 = (0, -h)$. For $j = 1$, $\omega_{i1} = (a_{i10}x + a_{i01}y)\,dy - (b_{i10}x + b_{i01}y)\,dx$. Hence, $M_1(h) = \sum_{i=1}^{4} \int_{\widehat{A_i A_{i+1}}} \omega_{i1}$. A direct computation yields $M_1(h) = Ah + Bh^{3/2} + Ch^{5/2}$, where

$$\begin{aligned} A &= -\frac{1}{2}\big(b_{110} + b_{210} + b_{310} + b_{410}\big), \\ B &= \frac{2}{3}\big(a_{110} - a_{210} - a_{310} + a_{410}\big) - \frac{1}{3}\big(b_{101} - b_{201} - b_{301} + b_{401}\big), \\ C &= \frac{2}{5}\big(a_{101} + a_{201} + a_{301} + a_{401}\big). \end{aligned}$$

Therefore, $M_1(h)$ has at most two positive zeros, and this bound is sharp. Consequently, at most two limit cycles bifurcate from the period annulus under first-order perturbations.

Assume that $M_1(h) \equiv 0$. By the higher-order Melnikov lemma, there exist functions ψ_{i1} such that $\omega_{i1} = dR_{i1} + \psi_{i1} dH_i$, and the second-order Melnikov function is

$$M_2(h) = \sum_{i=1}^{4} \int_{\widehat{A_i A_{i+1}}} \left(\omega_{i2} + \psi_{i1}\omega_{i1}\right).$$

Since $\omega_{i1} = (a_{i10}x + a_{i01}y)\,dy - (b_{i10}x + b_{i01}y)\,dx,\ dH_i = \sigma_i dy - 2x\,dx,$
we can choose

$$\psi_{i1}(x) = (a_{i10} + b_{i01})x, \qquad i \in \{1, 2, 3, 4\}.$$

Moreover,

$$\omega_{i2} = (a_{i20}x^2 + a_{i11}xy + a_{i02}y^2)\,dy - (b_{i20}x^2 + b_{i11}xy + b_{i02}y^2)\,dx.$$

Therefore, $\Psi_{i2} = \omega_{i2} + \psi_{i1}\omega_{i1}$. Restricting Ψ_{i2} to $\Gamma_i(h)$ and integrating over each arc yields

$$M_2(h) = \alpha_1 h + \alpha_2 h^{3/2} + \alpha_3 h^2 + \alpha_4 h^{5/2} + \alpha_5 h^3,$$

where α_k are explicit linear combinations of a_{irs}, b_{irs} $(r+s=2)$ and quadratic combinations of $a_{i10}, a_{i01}, b_{i10}, b_{i01}$. Consequently, $M_2(h)$ has at most four positive zeros.

Assume that the first- and second-order Melnikov functions vanish identically, $M_1(h) \equiv 0,\ M_2(h) \equiv 0$. Then the third-order Melnikov function for system (3.1) is given by $M_3(h) = a_{1,3}(h) = \sum_{k=1}^{7} \eta_k h^{k/2}$, where η_k are explicit linear combinations of the third-order perturbation coefficients $\{a_{i,rs}, b_{i,rs}\}$ and quadratic expressions in lower-order coefficients.

That is, $M_3(h) = s^2\Big(\beta_0 + \beta_1 s + \beta_2 s^2 + \beta_3 s^3 + \beta_4 s^4 + \beta_5 s^5 + \beta_6 s^6\Big),\ s = \sqrt{h}$. The function $M_3(h)/h$ is a polynomial in $\sqrt{h}$ of degree 6. Therefore, $M_3(h)$ can have at most six isolated positive zeros in $(0,1)$ (counting multiplicity). By choosing the free third-order parameters so that $\beta_0, \beta_1, \ldots, \beta_6$ are independent and alternate in sign, one can construct a parameter set for which the equation $M_3(h) = 0$ has six simple roots $0 < h_1 < h_2 < \cdots < h_6 < 1$. Each simple zero corresponds to a hyperbolic limit cycle bifurcating from the period annulus of the unperturbed system. Hence, the perturbed piecewise smooth system admits at least six limit cycles near the origin. This shows that the upper bound six is sharp.

In [5], authors studied the general piecewise linear 4-star-symmetric system of the form

$$x' = a^{\pm}x - y + b^{\pm}, \qquad y' = x - a^{\pm}y + c^{\pm}, \qquad (x,y) \in \mathbb{R}^2, \tag{3.2}$$

In first and third quadrants we take a^+, b^+c^+, while in third and fourth quadrants we take a^-, b^-, c^-. They have given the conditions under which the system (3.2) has center at infinity also they have shown that the existence of parameter values such that the system (3.2) has five limit cycles.

Using Melnikov functions, we will give the number of limit cycle bifurcated from the period annulus of the system (3.2) due to perturbation in the class of systems having linear centers, as below.

Example 3.2. Consider the perturbed system obtained as a perturbation of the linear center $\dot{x} = -y,\ \dot{y} = x$, within the class of systems (3.2). More precisely, consider

$$(\dot{x}, \dot{y}) = (-y, x) + \begin{cases} \displaystyle\sum_{j=1}^{n} \varepsilon^j (f_{1j}, g_{1j}), & (x, y) \in \Omega_1 \cup \Omega_3, \\ \displaystyle\sum_{j=1}^{n} \varepsilon^j (f_{2j}, g_{2j}), & (x, y) \in \Omega_2 \cup \Omega_4, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} & f_{ij} = a_{ij}x - y + b_{ij}, \qquad g_{ij} = x + a_{ij}y + c_{ij}, \qquad i \in \{1, 2\},\ j \in \mathbb{N}, \\ & \Omega_1 = \{(x, y) \colon x > 0,\ y < 0\}, \qquad \Omega_2 = \{(x, y) \colon x > 0,\ y > 0\}, \\ & \Omega_3 = \{(x, y) \colon x < 0,\ y > 0\}, \qquad \Omega_4 = \{(x, y) \colon x < 0,\ y < 0\}. \end{aligned}$$

Observe that, in this configuration,

$$\Sigma^+ = \Omega_2 \cup \Omega_3, \qquad \Sigma^- = \Omega_1 \cup \Omega_4.$$

From equations (2.16)–(2.19), we obtain the recursive relations for the coefficients $\alpha_{1,k}, \alpha_{2,k}, \alpha_{3,k}, \alpha_{4,k}$.
For $i \in \{1, 2, 3, 4\}$ we have $\psi_{i0} = -1$. For instance,

$$\begin{aligned} \int_{\widehat{A_1A_2}} \Psi_{11} &= \int_{\widehat{A_1A_2}} (a_{21}x - y + b_{21})\, dy - (x + a_{21}y + c_{21})\, dx \\ &= \int_{3\pi/2}^{2\pi} \Big[(ha_{21}\cos\theta - h\sin\theta + b_{21})(h\cos\theta) \\ &\qquad - (h\cos\theta + a_{21}h\sin\theta + c_{21})(-h\sin\theta) \Big] d\theta. \end{aligned}$$

Consequently,

$$\begin{aligned} & \int_{\widehat{A_1A_2}} \Psi_{11} = -h^2 - (b_{21} + c_{21})h, \ \int_{\widehat{A_2A_3}} \Psi_{21} = h^2 - (b_{11} - c_{11})\, h, \\ & \int_{\widehat{A_3A_4}} \Psi_{31} = -h^2 + (b_{21} + c_{21})\, h, \ \int_{\widehat{A_4A_1}} \Psi_{41} = h^2 + (b_{11} - c_{11})\, h. \end{aligned}$$

Thus,

$$\begin{aligned} & \alpha_{21} = \frac{h}{2} + \frac{b_{21} + c_{21}}{2}, \ \alpha_{31} = \alpha_{21} - \frac{h}{2} + \frac{b_{11} - c_{11}}{2}, \\ & \alpha_{41} = -\alpha_{31} - \frac{h}{2} + \frac{b_{21} + c_{21}}{2}, \ \alpha_{11} = \alpha_{41} + \frac{h}{2} + \frac{b_{11} - c_{11}}{2} = 0. \end{aligned}$$

Since

$$\mathcal{L}_{i,i+1,1} = \mathcal{L}_{i,i,1} = \mathcal{R}_{i,i+1,1} = \mathcal{R}_{i,i,1} = 0,$$

we conclude that

$$M_1(h) = \frac{1}{2h} \sum_{i=1}^{4} \int_{\widehat{A_iA_{i+1}}} \Psi_{i1} \equiv 0, \qquad h > 0.$$

Proceeding to second order, one obtains

$$M_2(h) = h\alpha_{12} = C_2h^2 + C_1h + C_0,$$

where C_0, C_1, C_2 are explicit functions of the system parameters (see Appendix). Hence, $M_2(h)$ has at most two positive zeros. Since $\{1, h, h^2\}$ is a Chebyshev system on $(0, \infty)$ and the coefficients are functionally independent, this upper bound is sharp.

If $M_2(h) \equiv 0$, then the third-order Melnikov function $M_3(h) = h\alpha_{13}$ is a polynomial of degree four. Therefore, the system (3.3) admits at most four limit cycles generated at third order. Since $\{1, h, h^2, h^3, h^4\}$ is a Chebyshev system on $(0, \infty)$ and the coefficients are free, this bound is also attainable.

Consequently, under third-order perturbations, system (3.3) can exhibit at most six limit cycles bifurcating from the period annulus.

4. Melnikov functions without time of flights

We can find all Melnikov functions $M_i(h)$, which do not depend upon the time of flights. The expressions of the Melnikov function, which do not contain the time of flight explicitly, are useful when the solutions of the unperturbed system(2.2) are difficult to find, and explicitly getting the time of flight is difficult. First, we will find the expressions of d_{ii} and $d_{i+1,i}$ in the following series of lemmas. Then they are used to find the expression of the displacement map $d(\varepsilon, h)$. Finally, the coefficient of ε^k in Taylor's expansion of $d(\varepsilon, h)$ is the k^{th} order Melnikov function $M_k(h)$. Consider the periodic orbit and trajectory of the unperturbed system ((2.1)) shown in following fig., which is useful to prove the series of lammas.

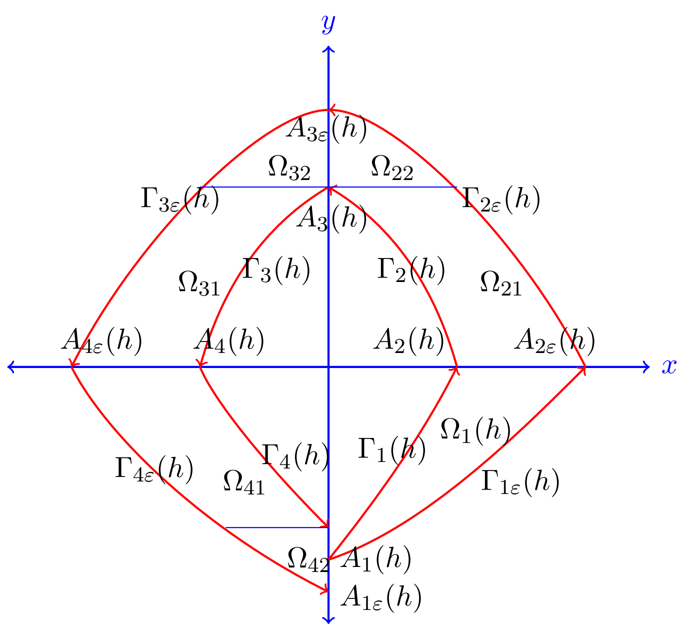

Figure 3. Unperturbed and perturbed trajectories and switching regions in the four quadrants.

Define the auxiliary coefficients c_{ti} as follows

$$c_{t,0} = 1, \qquad c_{ti} = \frac{f_{ti}}{H_{ty}} + \frac{1}{H_{tx}} \sum_{j=1}^{i-1} c_{t,j}\, g_{t,i-j} + \frac{g_{t,i}}{H_{tx}}, \quad i \in \mathbb{N},\ t \in \{1, 2, 3, 4\}.$$

For $i \in \mathbb{N}\cup\{0\}$, $t \in \{1,2,3,4\}$ define integrals I_1, I_2, I_3, I_4, 1-forms ω_{ti}, and Bell's polynomial $B^{(k)}_{i,m}$ and functions Δ_{tr} as below

$$I_{1i}(y) = \int_y^{a_1} \frac{H_{1y}}{H_{1x}}(x_0(y),u)\, c_{1i}(x_0(y),u)\, du,$$
$$I_{2i}(y) = \int_y^{a_3} \frac{H_{2y}}{H_{2x}}(x_0(y),u)\, c_{2i}(x_0(y),u)\, du,$$
$$I_{3i}(y) = \int_y^{a_3} \frac{H_{3y}}{H_{3x}}(x_0(y),u)\, c_{3i}(x_0(y),u)\, du,$$
$$I_{4i}(y) = \int_y^{a_1} \frac{H_{4y}}{H_{4x}}(x_0(y),u)\, c_{4i}(x_0(y),u)\, du,$$
$$B^{(k)}_{i,m} = B_{i,m}(a_{k,1},\dots a_{k,i-m+1,1}),\ \omega_{ti} = f_{ti}(x,y)dx + g_{ti}(x,y)dy,\ \Delta_{tr} = \frac{\partial f_{3r}}{\partial y} - \frac{\partial g_{3r}}{\partial x}$$

Lemma 4.1. *Assume that system (2.1) satisfies (A_1) and (A_2). Then $d_{11}(\varepsilon,h)$ defined in (2.5) admits the expansion*

$$d_{11}(\varepsilon,h) = \sum_{i=0}^{k-1}\left(\sum_{m=1}^{i} \frac{\partial^m H_1}{\partial x^m}(a_2,0)\, B^{(2)}_{im}\right)\varepsilon^i + \mathcal{O}(\varepsilon^k).$$

The coefficients a_{2j}, $j\in\mathbb{N}$, are determined recursively by

$$\sum_{m=1}^{i} \frac{\partial^m H_1}{\partial x^m}(a_2,0)\, B^{(2)}_{i,m} - \sum_{r+t=i}\sum_{m=1}^{t} \frac{1}{t!}\frac{\partial^{m-1} f_{1r}}{\partial x^{m-1}}(a_2,0)\, B^{(2)}_{t,m}$$
$$= \int_{\widehat{A_1A_2}} \omega_{1i} + \sum_{r+t=i}\int_0^{a_1} \Delta_{1r}(x,y) I_{1t}(y) dy.$$

Note that in following lemma take $a_{s,0} = a_1$ and for d_{22} and d_{33} take upper limit of integral as $a_{3,s}$ at appropriate place. For $l \in \{2,3,4\}$, we have

Lemma 4.2. *Assume that system (2.1) satisfies (A_1) and (A_2). Then $d_{ll}(\varepsilon,h)$ in (2.5) expands as*

$$d_{ll}(\varepsilon,h) = \sum_{i=0}^{k-1}\left(\sum_{m=1}^{i} \frac{\partial^m H_l}{\partial y^m}(A_{l+1}) B^{(l+1)}_{i,m} - \sum_{m=1}^{i} \frac{\partial^m H_l}{\partial x^m}(A_l) B^{(l)}_{i,m}\right)\varepsilon^i + \mathcal{O}(\varepsilon^k).$$

The coefficients a_{lj} and a_{lj} satisfy

$$\sum_{m=1}^{i}\Big(\frac{\partial^m H_l}{\partial y^m}(A_l) B^{(l+1)}_{i,m} - \frac{\partial^m H_l}{\partial x^m}(A_l) B^{(l)}_{i,m}\Big) + \sum_{r+t=i}\frac{1}{t!}\sum_{m=1}^{t}\frac{\partial^{m-1} f_{lr}}{\partial x^{m-1}}(A_l) B^{(l)}_{t,m}$$
$$- \sum_{r+t=i}\frac{1}{t!}\sum_{m=1}^{t}\frac{\partial^{m-1} g_{lr}}{\partial y^{m-1}}(A_{l+1}) B^{(l+1)}_{t,m}$$
$$= \int_{\widehat{A_lA_{l+1}}} \omega_{li} - \sum_{r+t=i}\int_0^{a_{l+1}} \Delta_{lr}(x,y) I_{lt}(y)\, dy + \sum_{s+r+t=i}\frac{1}{l!}\int_0^{a_{l+1,s}} \Delta_{2r}(x,y) I_{lt}(y)\, dy.$$

Proof of Lemma 4.1. : Consider the perturbed system (2.1) and the displacement

$$d_{11}(\varepsilon,h)=H_1(A_{2,\varepsilon})-H_1(A_1)=\int_{\widehat{A_1A_{2,\varepsilon}}}dH_1=\int_{\widehat{A_1A_{2,\varepsilon}}}(H_{1x}dx+H_{1y}dy).$$

Along the perturbed trajectory, we have

$$\dot{x}=H_{1y}+\sum_{i=1}^{k-1}\varepsilon^i f_{1i}+\mathcal{O}(\varepsilon^k),\quad \dot{y}=-H_{1x}+\sum_{i=1}^{k-1}\varepsilon^i g_{1i}+\mathcal{O}(\varepsilon^k),$$

which gives

$$H_{1x}\dot{x}+H_{1y}\dot{y}=\sum_{i=1}^{k-1}\varepsilon^i(H_{1x}f_{1i}+H_{1y}g_{1i})+\mathcal{O}(\varepsilon^k)=\sum_{i=1}^{k-1}\varepsilon^i\omega_{1i}+\mathcal{O}(\varepsilon^k),$$

where $\omega_{1i}=f_{1i}dx-g_{1i}dy$.

Let Ω_1 be the region bounded by the unperturbed orbit arc $\Gamma_1:=\widehat{A_1A_2}$, the perturbed arc $\Gamma_{1,\varepsilon}:=\widehat{A_1A_{2,\varepsilon}}$, and the segment $\overrightarrow{A_2A_{2,\varepsilon}}$. By Green's theorem,

$$\int_{\widehat{A_1A_{2,\varepsilon}}}\omega_{1i}=\int_{\widehat{A_1A_2}}\omega_{1i}+\int_{\overrightarrow{A_2A_{2,\varepsilon}}}f_{1i}dx+\iint_{\Omega_1}\left(\frac{\partial f_{1i}}{\partial y}-\frac{\partial g_{1i}}{\partial x}\right)dxdy.$$

Expressing $a_2(\varepsilon,h)=\sum_{j=0}^{\infty}\frac{1}{j!}a_{2j}\varepsilon^j$, the line integral along $\overrightarrow{A_2A_{2,\varepsilon}}$ can be expanded using Faà di Bruno's formula and Taylor series as

$$\int_{\overrightarrow{A_2A_{2,\varepsilon}}}f_{1i}(x,0)dx=\sum_{j=1}^{k-1}\varepsilon^j\sum_{m=1}^{j}\frac{\partial^{m-1}f_{1i}}{\partial x^{m-1}}(a_2)B_{j,m}^{(2)}+\mathcal{O}(\varepsilon^k).$$

To compute $\iint_{\Omega_1}(\partial f_{1i}/\partial y-\partial g_{1i}/\partial x)dxdy$, consider the homotopy

$$x(y,s)=x_0(y)+s(x_\varepsilon(y)-x_0(y)),\quad 0\le s\le 1.$$

Then

$$\iint_{\Omega_1}\left(\frac{\partial f_{1i}}{\partial y}-\frac{\partial g_{1i}}{\partial x}\right)dxdy=\int_{a_1}^{0}\left(\frac{\partial f_{1i}}{\partial y}-\frac{\partial g_{1i}}{\partial x}\right)\big(x_\varepsilon(y)-x_0(y)\big)dy.$$

Along $\Gamma_{1,\varepsilon}$,

$$\frac{\partial x_\varepsilon}{\partial y}=\frac{\dot{x}_\varepsilon}{\dot{y}_\varepsilon}=-\frac{H_{1y}}{H_{1x}}\Big(1+\sum_{i=1}^{k-1}c_{1i}\varepsilon^i+\mathcal{O}(\varepsilon^k)\Big),$$

where the coefficients c_{1i} satisfy the recursive relation

$$c_{1,0}=1,\qquad c_{1i}=\frac{f_{1i}}{H_{1y}}+\frac{1}{H_{1x}}\sum_{j=1}^{i-1}c_{1j}g_{1,i-j},\quad i\in\mathbb{N}.$$

Integrating gives

$$x_\varepsilon(y)-x_0(y)=-\sum_{j=1}^{k-1}\varepsilon^j\int_{a_1}^{y}\frac{H_{1y}}{H_{1x}}(x_0(u),u)c_{1j}(x_0(u),u)du+\mathcal{O}(\varepsilon^k).$$

Finally, combining all contributions, we obtain the compact formula for $d_{11}(\varepsilon,h)$ and determines the coefficients a_{2j} recursively. □

Proof of lemma 4.2 for l=2. : The proof is analogous to that of Lemma 4.1. By definition,

$$d_{22}(\varepsilon,h) = H_2(A_{3,\varepsilon}) - H_2(A_{2,\varepsilon}) = \int_{\widehat{A_{2,\varepsilon}A_{3,\varepsilon}}} (H_{2x}dx + H_{2y}dy)$$
$$= \int_{\widehat{A_{2,\varepsilon}A_{3,\varepsilon}}} (H_{2x}\dot{x} + H_{2y}\dot{y})dt.$$

Along the perturbed trajectory in the first quadrant,

$$\dot{x} = H_{2y} + \sum_{i=1}^{k-1} \varepsilon^i f_{2i} + \mathcal{O}(\varepsilon^k), \quad \dot{y} = -H_{2x} + \sum_{i=1}^{k-1} \varepsilon^i g_{2i} + \mathcal{O}(\varepsilon^k),$$

So that, $d_{22}(\varepsilon,h) = \sum_{i=1}^{k-1} \varepsilon^i \int_{\widehat{A_{2,\varepsilon}A_{3,\varepsilon}}} \omega_{2i} + \mathcal{O}(\varepsilon^k).$
Using Green's theorem and the regions Ω_{21}, Ω_{22} as in Figure (3),

$$\int_{\widehat{A_{2,\varepsilon}A_{3,\varepsilon}}} \omega_{2i} = \int_{\widehat{A_2A_3}} \omega_{2i} + \int_{\overrightarrow{A_3A_{3,\varepsilon}}} g_{2i}dy - \int_{\overrightarrow{A_2A_{2,\varepsilon}}} f_{2i}dx + \iint_{\Omega_{21}\cup\Omega_{22}} d\omega_{2i}.$$

The line integrals along $\overrightarrow{A_2A_{2,\varepsilon}}$ and $\overrightarrow{A_3A_{3,\varepsilon}}$ are expanded via Faà di Bruno and Taylor series:

$$\int_{\overrightarrow{A_2A_{2,\varepsilon}}} f_{2i}dx = \sum_{j=1}^{k-1} \varepsilon^j \sum_{m=1}^{j} \frac{\partial^{m-1} f_{2i}}{\partial x^{m-1}}(a_2,0) B^{(2)}_{j,m} + \mathcal{O}(\varepsilon^k),$$

$$\int_{\overrightarrow{A_3A_{3,\varepsilon}}} g_{2i}dy = \sum_{j=1}^{k-1} \varepsilon^j \sum_{m=1}^{j} \frac{\partial^{m-1} g_{2i}}{\partial y^{m-1}}(0,a_3) B^{(3)}_{j,m} + \mathcal{O}(\varepsilon^k).$$

The double integrals over Ω_{21} and Ω_{22} are expressed using homotopy and the recursive coefficients c_{2i}: $c_{2,0} = 1, \quad c_{2i} = \dfrac{f_{2i}}{H_{2y}} + \dfrac{1}{H_{2x}} \sum_{j=1}^{i-1} c_{2j} g_{2,i-j}, \quad i \in \mathbb{N}, \quad$ and

$$\iint_{\Omega_{21}\cup\Omega_{22}} d\omega_{2i}$$
$$= -\sum_{j=1}^{k-1} \varepsilon^j \int_0^{a_3} \Delta_{2j}(x,y) I_{2j}(y,u)dy + \sum_{l+r+j=i} \frac{\varepsilon^i}{l!} \int_0^{a_{3,l}} \Delta_{2j} I_{2l}(y,u)dy + \mathcal{O}(\varepsilon^k).$$

Collecting all terms, we get

$$d_{22}(\varepsilon,h)=\sum_{i=1}^{k-1}\varepsilon^i\Bigg[\int_{\widehat{A_2A_3}}\omega_{2i}+\sum_{r+t=i}\sum_{m=1}^{t}\frac{1}{t!}\frac{\partial^{m-1}g_{2r}}{\partial y^{m-1}}(0,a_3)B^{(3)}_{t,m} \\ -\sum_{r+t=i}\sum_{m=1}^{t}\frac{1}{t!}\frac{\partial^{m-1}f_{2r}}{\partial x^{m-1}}(a_2,0)B^{(2)}_{t,m}-\sum_{j+r=i}\int_0^{a_3}\Delta_{2j}(x,y)I_{2j}(y,u)dy \\ +\sum_{l+r+j=i}\frac{1}{l!}\int_0^{a_{3,l}}\Delta_{2j}(x,y)I_{2l}(y,u)dy\Bigg]+\mathcal{O}(\varepsilon^k). \tag{4.1}$$

Alternatively, by Faà di Bruno's formula and Taylor expansion,

$$d_{22}(\varepsilon,h)=\sum_{i=0}^{k-1}\left[\sum_{m=1}^{i}\frac{\partial^m H_2}{\partial y^m}(0,a_3)B^{(3)}_{i,m}-\sum_{m=1}^{i}\frac{\partial^m H_2}{\partial x^m}(a_2,0)B^{(2)}_{i,m}\right]\varepsilon^i+\mathcal{O}(\varepsilon^k).$$

Comparing powers of ε in (4.1) and the above expansion gives recursive relations for the coefficients a_{3j} in terms of a_{2j}. Proof for $l=3$ and $l=4$ are very similar to that of $l=2$. □

Remark 4.3. Also for $i\in\{1,2,3,4\}$, we have $d_{i+1,i}(\varepsilon,h)=H_{i+1}(A_{i+1,\varepsilon}(h))-H_i(A_{i+1,\varepsilon}(h))$. Using Faàdi Bruno's formula for differentiation of composite functions and Taylor's expansion, we have

$$d_{i+1,i}(\varepsilon,h)=\sum_{j=0}^{k-1}\frac{1}{j!}\frac{\partial^j d_{i+1,i}(\varepsilon,h)}{\partial\varepsilon^j}(0,h)\varepsilon^j+\mathcal{O}(\varepsilon^k) \\ =\sum_{j=1}^{k-1}\left(\sum_{m=1}^{j}\left(\frac{\partial^m}{\partial z^m}(H_{i+1}-H_i)(A_{i+1})\right)\prod_{l=1}^{j}B^{(i+1)}_{j,m}\right)\varepsilon^j+\mathcal{O}(\varepsilon^k),$$

where, $z=x$ if $i\in\{2,4\}$ and $z=y$ if $i\in\{1,3\}$ and $A_5=A_1$. Adding the expressions from lemma 4.1, 4.2, we can find the expression
$d(\varepsilon,h)=\sum_{k=1}^{\infty}M_k(h)\varepsilon^k=\sum_{i=1}^{4}d_{ii}(\varepsilon,h)+d_{i+1,i}(\varepsilon,h)$. Also the quantities a_{1j} represents the Melnikov functions $M_j(h)$ for all $j\in\mathbb{N}$.

Now all above lemmas are unified as

Lemma 4.4. *Consider the perturbed four-zone piecewise smooth system* (2.1) *under assumptions* (A_1)*–*(A_2)*. Let* $A_i(\varepsilon,h)$*,* $i=1,2,3,4$*, be the successive intersection points of the perturbed periodic orbit with the switching lines, namely* $A_1(\varepsilon,h)=(0,a_1(\varepsilon,h))$*,* $A_2(\varepsilon,h)=(a_2(\varepsilon,h),0)$*,* $A_3(\varepsilon,h)=(0,a_3(\varepsilon,h))$*,*
$A_4(\varepsilon,h)=(a_4(\varepsilon,h),0)$*, where* $a_i(\varepsilon,h)=a_i+\sum_{j=1}^{k-1}a_{i,j}\varepsilon^j+\mathcal{O}(\varepsilon^k)$*,* $i\in\{1,2,3,4\}$*.*
Define $d_{ii}(\varepsilon,h)=H_i\big(A_{i+1}(\varepsilon,h)\big)-H_i\big(A_i(\varepsilon,h)\big)$*,* $i\in\{1,2,3,4\}$ *with the convention* $A_5=A_1$*. Then for each* $i\in\{1,2,3,4\}$*,* $d_{ii}(\varepsilon,h)=\sum_{n=0}^{k-1}d_{ii,n}(h)\,\varepsilon^n+\mathcal{O}(\varepsilon^k)$*,*

where for $n \in \mathbb{N}$,

$$d_{ii,n}(h) = \int_{\widehat{A_iA_{i+1}}} \omega_{in} + \sum_{\substack{r+t=n \\ r,t\geq 1}} \int_0^{a_{i+1}} \Delta_{ir}(x,y)\mathcal{K}_{i,t}(y)\,dy + \sum_{\substack{l+r+t=n \\ l,r,t\geq 1}} \frac{1}{l!} \int_0^{a_{i+1,l}} \Delta_{ir}(x,y)\mathcal{K}_{i,t}(y)\,dy,$$

and $\mathcal{K}_{i,t}(y) = \displaystyle\int_y^{a_{i+1}} \frac{H_{iy}}{H_{ix}}\big(x_0(y),u\big)\, c_{i,t}\big(x_0(y),u\big)\,du$, *with the recursive coefficients* $c_{i,0} = 1,\ c_{i,t} = \dfrac{f_{i,t}}{H_{iy}} + \dfrac{1}{H_{ix}} \displaystyle\sum_{j=0}^{t-1} c_{i,j} g_{i,t-j},\ t \in \mathbb{N}.$
Moreover, the full displacement function $\Delta(h,\varepsilon) = \Pi(h,\varepsilon) - h$ *admits the expansion* $\Delta(h,\varepsilon) = \displaystyle\sum_{n=1}^{k-1} \varepsilon^n \sum_{i=1}^{4} d_{ii,n}(h) + \mathcal{O}(\varepsilon^k)$.

Example 4.5. Consider the piecewise smooth system

$$(\dot{x},\dot{y}) = \begin{cases} (-y,-x+1) + \displaystyle\sum_{j=1}^{n} \varepsilon^j (f_{1j}, g_{1j}), & (x,y) \in \Omega_1, \\ (-y,-x+1) + \displaystyle\sum_{j=1}^{n} \varepsilon^j (f_{2j}, g_{2j}), & (x,y) \in \Omega_2, \\ (-y,x) + \displaystyle\sum_{j=1}^{n} \varepsilon^j (f_{3j}, g_{3j}), & (x,y) \in \Omega_3, \\ (-y,x) + \displaystyle\sum_{j=1}^{n} \varepsilon^j (f_{4j}, g_{4j}), & (x,y) \in \Omega_4, \end{cases} \tag{4.2}$$

where

$$f_{ij}(x,y) = \sum_{r+s=j} a_{i,r,s} x^r y^s, \qquad g_{ij}(x,y) = \sum_{r+s=j} b_{i,r,s} x^r y^s,$$

and

$$\Omega_1 = \{x > 0, y < 0\}, \quad \Omega_2 = \{x > 0, y > 0\},$$
$$\Omega_3 = \{x < 0, y > 0\}, \quad \Omega_4 = \{x < 0, y < 0\}.$$

The unperturbed system has a saddle in the right half-plane and a center in the left half-plane, with a period annulus (see Figure 4). Each periodic orbit

$$\Gamma_h := \Gamma_1(h) \cup \Gamma_2(h) \cup \Gamma_3(h) \cup \Gamma_4(h),$$

passes through

$$A_1 = (0,-h), \quad A_2 = (1-\sqrt{1-h^2}, 0), \quad A_3 = (0,h), \quad A_4 = (-h,0),$$

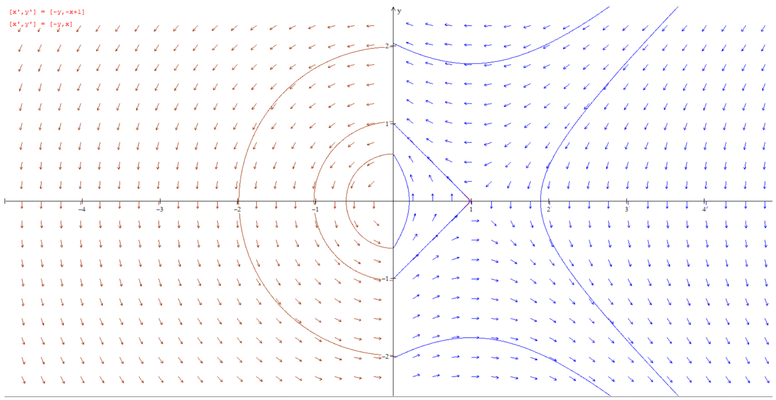

FIGURE 4. Period annulus of system 4.2 at $\varepsilon = 0$

where

$$\Gamma_1(h) : y = -\sqrt{x^2 - x + h^2}, \quad \Gamma_2(h) : y = \sqrt{x^2 - x + h^2},$$
$$\Gamma_3(h) : y = \sqrt{h^2 - x^2}, \quad \Gamma_4(h) : y = -\sqrt{h^2 - x^2}.$$

We compute the following line integrals for $i \in \{1, 2, 3, 4\}$,

$\int_{\widehat{A_i A_{i+1}}} \omega_{i1}$, where $\omega_{i1} = (a_{i10}x + a_{i01}y)\, dy - (b_{i10}x + b_{i01}y)\, dx$.

$$\begin{aligned}\int_{\widehat{A_1 A_2}} \omega_{11} = &\int_0^{1-\sqrt{1-h^2}} \left(f_{11} - g_{11} \frac{H_{1x}}{H_{1y}} \right) dx \\ =&\frac{1}{4}(a_{101} - b_{110})h^2 \ln(1 - h^2) + \frac{1}{4}(b_{110} - a_{101}) \ln(1 - h^2) \\ &- \frac{1}{2}(a_{110} + b_{101})h^2 - a_{110}(1 + \sqrt{1 - h^2}) - \frac{1}{2}(a_{101} - b_{110})h^2 \ln(1 - h) \\ &- \frac{1}{2}(b_{110} - a_{101})h + \frac{1}{2}(a_{101} - b_{110}) \ln(1 - h),\end{aligned}$$

$$\begin{aligned}\int_{\widehat{A_2 A_3}} \omega_{21} = &- \int_0^{1-\sqrt{1-h^2}} \left(f_{21} - g_{21} \frac{H_{2x}}{H_{2y}} \right) dx \\ =&- \frac{1}{4}(a_{201} - b_{210})h^2 \ln(1 - h^2) + \frac{1}{4}(b_{210} - a_{201}) \ln(1 - h^2) \\ &+ \frac{1}{2}(a_{210} + b_{201})h^2 - a_{210}(1 + \sqrt{1 - h^2}) + \frac{1}{2}(a_{201} - b_{210})h^2 \ln(1 - h) \\ &+ \frac{1}{2}(b_{210} - a_{201})h - \frac{1}{2}(a_{201} - b_{210}) \ln(1 - h),\end{aligned}$$

$$\int_{\widehat{A_3 A_4}} \omega_{31} = \int_0^{-h} \left(f_{31} - g_{31} \frac{H_{3x}}{H_{3y}} \right) dx = \left(-\frac{\pi}{4} a_{301} + \frac{1}{2} a_{310} + \frac{1}{2} b_{301} - \frac{\pi}{4} b_{310} \right) h^2,$$

$$\int_{\widehat{A_4 A_1}} \omega_{41} = - \int_0^{-h} \left(f_{41} - g_{41} \frac{H_{4x}}{H_{4y}} \right) dx = \left(\frac{\pi}{4} a_{401} - \frac{1}{2} a_{410} - \frac{1}{2} b_{401} + \frac{\pi}{4} b_{410} \right) h^2.$$

The coefficients in $c-ij,\ i\in\mathbb{N}, j\in\{0,1,2\}$ are given by Lemmas 4.1–4.2. We compute the correction terms $\displaystyle\int_0^{\alpha_{i0}}\Delta_{i1}(x,y)I_{i0}(y)\,dy$.

$$
\begin{aligned}
&\int_0^{\alpha_{10}}\Delta_{11}I_{10}dy\\
=&\int_0^{-h}(a_{101}-b_{110})\left(\int_y^{-h}\frac{u}{\sqrt{-h^2+u^2+1}}du\right)dy\\
=&\frac{1}{2}(-a_{101}+b_{110})h+\frac{1}{2}(a_{101}-b_{110})h^2\ln(1-h)+\frac{1}{2}(-a_{101}+b_{110})\ln(1-h)\\
&+\frac{1}{4}(-a_{101}+b_{110})h^2\ln(1-h^2)+\frac{1}{4}(a_{101}-b_{110})\ln(1-h^2).
\end{aligned}
$$

(Analogous explicit formulas hold for $\Delta_{21},\Delta_{31},\Delta_{41}$ and are omitted here for brevity.) From Lemmas 4.1 and 4.2, the coefficients α_{i1} satisfy

$$
\begin{aligned}
2\sqrt{1-h^2}\,\alpha_{21}&=\int_{\widehat{A_1A_2}}\omega_{11}+\int_0^{\alpha_{10}}\Delta_{11}I_{10}dy,\\
2h\alpha_{31}-2\sqrt{1-h^2}\,\alpha_{21}&=\int_{\widehat{A_2A_3}}\omega_{21}-\int_0^{\alpha_{20}}\Delta_{21}I_{20}dy,\\
-2h\alpha_{41}-2h\alpha_{31}&=\int_{\widehat{A_3A_4}}\omega_{31}+\int_0^{\alpha_{30}}\Delta_{31}I_{30}dy,\\
-2h\alpha_{11}+2h\alpha_{41}&=\int_{\widehat{A_4A_1}}\omega_{41}-\int_0^{\alpha_{40}}\Delta_{41}I_{40}dy.
\end{aligned}
$$

Hence, $M_1(h)=-2h\alpha_{11}$ is a linear combination of the eight functions

$$
h,\ h^2,\ \ln(1-h^2),\ h^2\ln(1-h^2),\ \ln(1-h),\ h^2\ln(1-h),\ \ln(1+h),\ 1+\sqrt{1-h^2}.
$$

The above 8 functions form an extended Chebyshev system on $(0,1)$. Therefore, any nontrivial linear combination has at most 7 isolated zeros. Each simple zero corresponds to a limit cycle. Then $M_1(h)$ has at most 7 zeros on $(0,1)$. Consequently, system (4.2) admits at most 7 limit cycles bifurcating from the period annulus.

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5. APPENDIX

In this appendix, we present the explicit expressions of the first three Melnikov functions for the system (3.1)
Explicit coefficients of $M_1(h)$: The first-order Melnikov function has the form $M_1(h) = A_1 h + B_1 h^{3/2} + C_1 h^{5/2}$, where

$$A_1 = -\frac{1}{2}\big(b_{110} + b_{210} + b_{310} + b_{410}\big), \tag{5.1}$$

$$B_1 = \frac{2}{3}\big(a_{110} - a_{210} - a_{310} + a_{410}\big) - \frac{1}{3}\big(b_{101} - b_{201} - b_{301} + b_{401}\big), \tag{5.2}$$

$$C_1 = \frac{2}{5}\big(a_{101} + a_{201} + a_{301} + a_{401}\big). \tag{5.3}$$

Explicit coefficients of $M_2(h)$: Assume that the first-order Melnikov function vanishes identically. Then the second-order Melnikov function can be written as $M_2(h) = A_2h + B_2h^{3/2} + C_2h^2 + D_2h^{5/2} + E_2h^3$, where the coefficients are given by $A_2 = -\frac{1}{2}\sum_{i=1}^{4} b_{i20}$, $B_2 = \frac{2}{3}\sum_{i=1}^{4}\sigma_i a_{i20} - \frac{1}{3}\sum_{i=1}^{4}\sigma_i b_{i11}$, $C_2 = \frac{1}{2}\sum_{i=1}^{4} b_{i11}$, $D_2 = \frac{2}{5}\sum_{i=1}^{4} a_{i11}$, $E_2 = \frac{2}{7}\sum_{i=1}^{4} a_{i02}$.

Explicit coefficients of $M_3(h)$: Assume that $M_1(h) \equiv 0$, $M_2(h) \equiv 0$. Then the third-order Melnikov function can be written as
$M_3(h) = A_3h + B_3h^{3/2} + C_3h^2 + D_3h^{5/2} + E_3h^3 + F_3h^{7/2}$, where
$A_3 = -\frac{1}{2}\sum_{i=1}^{4} b_{i30}$, $B_3 = \frac{2}{3}\sum_{i=1}^{4}\sigma_i a_{i30} - \frac{1}{3}\sum_{i=1}^{4}\sigma_i b_{i21}$, $C_3 = \frac{1}{2}\sum_{i=1}^{4} b_{i21}$, $D_3 = \frac{2}{5}\sum_{i=1}^{4} a_{i21}$, $E_3 = \frac{2}{7}\sum_{i=1}^{4} a_{i12}$, $F_3 = \frac{2}{9}\sum_{i=1}^{4} a_{i03}$.

Now we present the coefficient of Melnikov functions for the system (3.3).

First-order Melnikov function $M_1(h)$: We have by definition,

$$M_1(h) = \sum_{k=1}^{4} \int_{\widehat{A_kA_{k+1}}} \omega_{i(k),1},$$

where $i(k) = 1$ for $k = 1, 3$ and $i(k) = 2$ for $k = 2, 4$. Substituting

$$x = \sqrt{2h}\cos t, \quad y = \sqrt{2h}\sin t, \quad dx = -\sqrt{2h}\sin t\,dt, \quad dy = \sqrt{2h}\cos t\,dt,$$

we obtain

$$\begin{aligned}\omega_{i1}(\gamma_h(t)) &= (a_{i1}x - y + b_{i1})dy - (x + a_{i1}y + c_{i1})dx \\ &= \Big[a_{i1}(2h\cos^2 t) + 2h\sin^2 t + b_{i1}\sqrt{2h}\cos t + c_{i1}\sqrt{2h}\sin t\Big]dt.\end{aligned}$$

Integrating over the four arcs yields

$$M_1(h) = \pi h(a_{11} + a_{21}) + 2\sqrt{2h}\,(b_{11} - b_{21}) + 2\sqrt{2h}\,(c_{11} - c_{21}).$$

Hence

$$M_1(h) = A_1h + B_1\sqrt{h},$$

with

$$A_1 = \pi(a_{11} + a_{21}), \qquad B_1 = 2\sqrt{2}\,(b_{11} - b_{21} + c_{11} - c_{21}).$$

The condition $M_1(h) \equiv 0$ is

$$a_{11} + a_{21} = 0, \qquad b_{11} - b_{21} + c_{11} - c_{21} = 0.$$

Second-order Melnikov function $M_2(h)$: Assuming $M_1(h) \equiv 0$, the second-order Melnikov function is $M_2(h) = \sum_{k=1}^{4} \int_{\widehat{A_kA_{k+1}}} \big(\omega_{i(k),2} - \psi_{i(k),1}\omega_{i(k),1}\big)$,
where $\psi_{i1}(x, y) = \int_0^t \big(\partial_x f_{i1} + \partial_y g_{i1}\big) \circ \gamma_h(s)\,ds = \int_0^t (2a_{i1})\,ds = 2a_{i1}t$.
Hence $\Psi_{i2} = \omega_{i2} - 2a_{i1}t\,\omega_{i1}$. Carrying out the same trigonometric integrations

yields $M_2(h) = A_2 h + B_2 h^{3/2}$,
where

$$A_2 = \pi(a_{12} + a_{22}) - \pi(a_{11}^2 + a_{21}^2),$$

$$B_2 = \frac{4\sqrt{2}}{3}\Big[(b_{12} - b_{22} + c_{12} - c_{22}) - (a_{11}b_{11} - a_{21}b_{21} + a_{11}c_{11} - a_{21}c_{21})\Big].$$

Third-order Melnikov function $M_3(h)$: Assuming $M_1(h) \equiv M_2(h) \equiv 0$, the third-order Melnikov function is $M_3(h) = \sum_{k=1}^{4} \int_{\widehat{A_k A_{k+1}}} \big(\omega_{i(k),3} - \psi_{i(k),1}\omega_{i(k),2} - \psi_{i(k),2}\omega_{i(k),1}\big)$, with $\psi_{i2}(t) = \int_0^t \big(\partial_x f_{i2} + \partial_y g_{i2}\big) \circ \gamma_h(s)\, ds = 2a_{i2}t$. After direct but lengthy computation, one obtains $M_3(h) = A_3 h + B_3 h^{3/2} + C_3 h^2$, where

$$A_3 = \pi(a_{13} + a_{23}) - \pi(a_{11}a_{12} + a_{21}a_{22}), \qquad C_3 = \frac{\pi}{2}\Big(a_{11}^3 + a_{21}^3\Big),$$

$$B_3 = \frac{4\sqrt{2}}{3}\Big[(b_{13} - b_{23} + c_{13} - c_{23}) - (a_{11}b_{12} - a_{21}b_{22} + a_{12}b_{11} - a_{22}b_{21}) - (a_{11}c_{12} - a_{21}c_{22} + a_{12}c_{11} - a_{22}c_{21})\Big].$$

Realization of six limit cycles: Computing Higher order Melnikov function we conclude that the perturbed system exhibits up to 6 small-amplitude limit cycles bifurcating from the period annulus of the linear center.

[1] Department of Mathematics, MES Abasaheb Garware College (Autonomous), Pune, India.
Email address: krishnatmasalkar@gmail.com

[2] Department of Mathematics, Savitribai Phule Pune University, Pune-411007, India.
Email address: sdkendre@yahoo.com,drsubhash2010@gmail.com