Gulf Journal of Mathematics
Vol 22, Issue 2 (2026) 1-12

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v22i2.4081

MULTIPLICITY RESULT FOR DIRICHLET PROBLEM DRIVEN BY $p(x)-$LAPLACIAN-LIKE OPERATOR

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Abstract. We prove the existence and multiplicity of nontrivial weak solutions for a class of Dirichlet problems involving a $p(x)$-Laplacian-like operator. The study is conducted in the setting of variable exponent Sobolev spaces and employs appropriate variational methods together with a variant of the Mountain Pass Lemma.

Keywords. $p(x)$-Laplacian-like operator; variable exponent Sobolev space; Variational method; Multiple solutions.
2020 Mathematics Subject Classification. 35A15, 37B30.

1. Introduction

Problems involving $p(x)$-Laplacian operator appear in many different areas of research and in a wide variety of important physical problems including image restoration [1, 9], elastic mechanics [41], porous media [5, 18], electromagnetism [6, 40], nonstandard growth [2, 8, 10, 25, 28, 42] and electrorheological [3, 4, 31, 38]. The study of these kinds of problems was facilitated by the introduction of Lebesgue and Sobolev spaces with variable exponent. For example, in [23], authors extend existence and multiplicity results for problems with an isotropic $p(x)$-Laplacian operator [12] to anisotropic $\overrightarrow{p}(x)$-Laplacian one, which is more delicate due to the fact that it is not homogeneous. Next, in [20], we generalize the results above to an equation containing a more general operator $Au := -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \nabla u)$ and a nonlinearity $\lambda(x)|u|^{p(x)-2}u$ which make the problem challenging.

In the present investigation, we deal with the following Dirichlet problem

$$\begin{cases} -\Delta^l_{p(x)}(u) + \lambda(x)|u|^{p(x)-2}u = f(x,u) & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

Date: Received: Jan 8, 2026; Revised: Feb 5, 2026; Accepted: Feb 20, 2025.

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where Ω is a bounded, open subset of $\mathbb{R}^N$ $(N \geq 2)$, with smooth boundary denoted by $\partial\Omega$ and $\mathrm{meas}(\Omega) > 0$. $\Delta^l_{p(x)}$ represents the $p(x)-$Laplacian-like operator, acting from $W_0^{1,p(x)}(\Omega)$ into its dual denoted by $W_0^{-1,p'(x)}(\Omega)$, defined by

$$\Delta^l_{p(x)}(u) = \mathrm{div}\Bigg(\bigg(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\bigg)|\nabla u|^{p(x)-2}\nabla u\Bigg). \tag{1.2}$$

$f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory function satisfying the following conditions

$$\begin{cases} |f(x,t)| \leq \alpha(1+|t|^{q(x)-1}), \text{for some } \alpha > 0, \forall t \in \mathbb{R}, \text{ a.e. } x \in \Omega,\ q \in C(\overline{\Omega}) \\ \text{and } 1 < q(x) < p^*(x), \forall x \in \overline{\Omega}. \end{cases} \tag{1.3}$$

where

$$p^*(x) \begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

$$\begin{cases} \text{There exists a constant } M > 0 \text{ and } \theta > p^+ \text{ such that} \\ 0 < \theta F(x,t) \leq t f(x,t), \\ \text{for all } x \in \Omega \text{ and } |t| \geq M, \text{where } F(x,t) = \displaystyle\int_0^t f(x,s)ds. \end{cases} \tag{1.4}$$

$$f(x,t) = o(|t|^{p^+-1}) \text{ as } t \to 0 \text{ and uniformly for } x \in \Omega, \text{with } q^- > p^+. \tag{1.5}$$

$$\lambda \in L^\infty(\Omega) \text{ and there exists } \lambda_0 > 0 \text{ such that } \quad \lambda(x) \geq \lambda_0 \text{ for all } x \in \Omega. \tag{1.6}$$

The study of such problems presents some obstacles due to the specific structure of the operator considered and the presence of the nonlinear term $\lambda(x)|u|^{p(x)-2}u$. To address these challenges, we integrate tools, techniques, and ideas developed in [13, 20, 39, 43] and references therein. Other important contributions following several other directions on the same topic can be read in [14, 27, 32, 33, 34, 35, 36, 37].

Let us recall that a function $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution to Problem (1.1) if the following identity holds

$$\begin{aligned} \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx &+ \int_\Omega \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}|\nabla u|^{p(x)-2}\nabla u \nabla v dx \\ &+ \int_\Omega \lambda(x)|u|^{p(x)-2}uv dx = \int_\Omega f(x,u)v dx, \end{aligned} \tag{1.7}$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

Related to the Problem (1.1), we define the so-called energy functional $\Phi : W_0^{1,p(x)}(\Omega) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} \Phi(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx &+ \int_\Omega \frac{1}{p(x)}\left[\sqrt{1+|\nabla u|^{2p(x)}} - 1\right] dx \\ &+ \int_\Omega \frac{\lambda(x)}{p(x)}|u|^{p(x)} dx - \int_\Omega F(x,u) dx. \end{aligned} \tag{1.8}$$

It is well known that under (1.3), Φ is well defined and is a $\mathcal{C}^1$ functional with derivative given by

$$\langle \Phi'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_\Omega \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}|\nabla u|^{p(x)-2}\nabla u\nabla v dx \\ + \int_\Omega \lambda(x)|u|^{p(x)-2}uv dx - \int_\Omega f(x,u)v dx,$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$.
Hence, from (1.7), it follows that u is a weak solution of (1.1) if and only if u is a critical point of Φ, i.e.

$$u \in W_0^{1,p(x)}(\Omega) \text{ weak solution of (1.1)} \iff \langle \Phi'(u), v\rangle = 0 \ \forall v \in W^{1,p(x)}(\Omega). \quad (1.9)$$

Thus, with the previous definition, we state as follows our main result, namely the existence at least of two nontrivial solutions to Problem (1.1):

Theorem 1.1. *Assume that* (1.3)–(1.5) *hold true and* $f(x,0) = 0$ *for a.e.* $x \in \Omega$. *Then, the Problem* (1.1) *has at least two nontrivial solutions in which one is non-negative and one is non-positive.*

The remaining part of this paper is organized as follows: In Section (2), we recall some notations and preliminaries about the anisotropic variable exponent Sobolev spaces. In Section (3), we establish the proof of our result by giving the main steps of our method.

2. VARIATIONAL FRAMEWORK

Let Ω be an open bounded domain in $\mathbb{R}^N$ ($N \geq 2$), with smooth boundary $\partial\Omega$ and meas$(\Omega) > 0$. We recall some known results on variable exponent Lebesgue and Sobolev spaces. See [7, 11, 15, 16, 19, 20, 21, 22, 24, 29] and references therein for more details. Define

$$C_+(\overline{\Omega}) = \left\{p : \overline{\Omega} \longrightarrow \mathbb{R} \text{ measurable, such that } 1 < p^- \leq p^+ < N\right\},$$

where

$$p^- = \operatorname{ess\,inf}\{p(x) \mid x \in \overline{\Omega}\} \qquad \text{and} \qquad p^+ = \operatorname{ess\,sup}\{p(x) \mid x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, let us recall that the variable exponent Lebesgue space is defined by

$$L^{p(.)}(\Omega) := \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable, such that } \int_\Omega |u(x)|^{p(x)}dx \ < \ \infty\right\}.$$

This space is usually endowed with the Luxemburg norm

$$|u|_{p(.)} := \inf\left\{\lambda > 0 : \ \rho_{p(.)}\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

where the mapping $\rho_{p(.)} : L^{p(.)}(\Omega) \longrightarrow \mathbb{R}$ is the convex modular of the $L^{p(.)}(\Omega)$ space defined by

$$\rho_{p(.)}(u) := \int_\Omega |u(x)|^{p(x)}dx.$$

Furthermore, we have the elementary properties below (see [11, 15]):
- for any $u \in L^{p(.)}(\Omega)$,

$$\min\left\{|u|_{p(.)}^{p^-};\ |u|_{p(.)}^{p^+}\right\} \leq \rho_{p(.)}(u) \leq \max\left\{|u|_{p(.)}^{p^-};\ |u|_{p(.)}^{p^+}\right\} \leq |u|_{p(.)}^{p^+}+1 \tag{2.1}$$

and

$$\min\left\{\rho_{p(.)}^{1/p^+}(u);\ \rho_{p(.)}^{1/p^-}(u)\right\} \leq |u|_{p(.)} \leq \max\left\{\rho_{p(.)}^{1/p^+}(u);\ \rho_{p(.)}^{1/p^-}(u)\right\} \leq \left(\rho_{p(.)}(u)+1\right)^{1/p^-}; \tag{2.2}$$

- the space $\left(L^{p(.)}(\Omega), |u|_{p(.)}\right)$ is a separable reflexive Banach space (see [24]), and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$, for all $x \in \Omega$;
- for all $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$, the following Hölder type inequality holds true:

$$\left|\int_\Omega uvdx\right| \leq \left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)|u|_{p(.)}|v|_{p'(.)}. \tag{2.3}$$

The variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) \text{ such that } |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

and $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Equipped with the following norm

$$||u||_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \tag{2.4}$$

the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniform convex Banach spaces (see [26, 17]). We note that if $q \in C(\overline{\Omega})$ and $1 < q(x) < p^*(x), \forall x \in \overline{\Omega}$, then the embedding $W_0^{1,\vec{p}(.)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous.

3. PROOF OF MAIN RESULTS

Let us denote by $u^+ = \max(u,0)$ and $u^- = \max(-u,0)$ the positive and negative parts of u. Before going further, we recall the following technical and useful lemma:

Lemma 3.1 (See [23]).

(1) *If $u \in W_0^{1,p(x)}(\Omega)$, then $u^+, u^- \in W_0^{1,p(x)}(\Omega)$ and*

$$\nabla u^+ = \begin{cases} \nabla u & \text{if} \quad u > 0, \\ 0 & \text{if} \quad u \leq 0, \end{cases} \qquad \nabla u^- = \begin{cases} 0 & \text{if} \quad u \geq 0, \\ \nabla u & \text{if} \quad u < 0. \end{cases} \tag{3.1}$$

(2) *The mappings $u \mapsto u^\pm$ are continuous on $W_0^{1,p(x)}(\Omega)$.*

Next, we consider the following truncated problem

$$(\mathcal{P}_\pm) \begin{cases} -\Delta_{p(x)}^l(u) + \lambda(x)|u|^{p(x)-2}u = f_\pm(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where

$$f_\pm(x,t) = \begin{cases} f(x,t) & \text{if } \pm t \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Lemma 3.2. *All solutions of $(\mathcal{P}_+)$ (resp. $(\mathcal{P}_-)$ are non-negative (resp. non-positive) solutions of $(\mathcal{P})$.*

Proof. Let us define $\Phi_\pm : W_0^{1,p(x)}(\Omega) \longrightarrow \mathbb{R}$, by

$$\begin{aligned}\Phi_\pm(u) &= \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx + \int_\Omega \frac{1}{p(x)}\left[\sqrt{1+|\nabla u|^{2p(x)}}-1\right]dx \\ &\quad + \int_\Omega \frac{\lambda(x)}{p(x)}|u|^{p(x)}dx - \int_\Omega F_\pm(x,u)dx \\ &= \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx + \int_\Omega \frac{1}{p(x)}\left[\sqrt{1+|\nabla u|^{2p(x)}}-1\right]dx \\ &\quad + \int_\Omega \frac{\lambda(x)}{p(x)}|u|^{p(x)}dx - \int_\Omega F(x,u^\pm)dx, \end{aligned} \tag{3.4}$$

where $F_\pm(x,s) = \int_0^s f_\pm(x,t)dt$.

In view of Lemma 3.1 and Condition (1.3), $\Phi_\pm$ is well defined on $W_0^{1,p(x)}(\Omega)$, weakly lower semi-continuous and $\mathcal{C}^1-$functionals. From (1.9), we derive the following equivalence:

$$u \in W_0^{1,p(x)}(\Omega) \text{ weak solution of } (\mathcal{P}_+) \iff \langle \Phi'_+(u), v\rangle = 0 \ \forall v \in W_0^{1,p(x)}(\Omega). \tag{3.5}$$

Replacing v by u^- in (3.5), we obtain

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla u^- dx + \int_\Omega \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}|\nabla u|^{p(x)-2}\nabla u\nabla u^- dx + \int_\Omega \lambda(x)|u|^{p(x)-2}uu^- dx - \int_\Omega f_+(x,u)u^- dx = 0.$$

Next, it follows that

$$\int_\Omega |\nabla u^-|^{p(x)}dx + \int_\Omega \frac{|\nabla u^-|^{2p(x)}}{\sqrt{1+|\nabla u^-|^{2p(x)}}}dx + \int_\Omega \lambda(x)|u^-|^{p(x)}dx = 0.$$

From (1.6), we have

$$\begin{aligned} 0 &\le \min(1;\lambda_0)\left(\int_\Omega |\nabla u^-|^{p(x)}dx + \int_\Omega |u^-|^{p(x)}dx\right) \\ &\le \int_\Omega |\nabla u^-|^{p(x)}dx + \int_\Omega \frac{|\nabla u^-|^{2p(x)}}{\sqrt{1+|\nabla u^-|^{2p(x)}}}dx + \int_\Omega \lambda(x)|u^-|^{p(x)}dx = 0 \end{aligned}$$

which implies that

$$\int_\Omega |\nabla u^-|^{p(x)}dx + \int_\Omega |u^-|^{p(x)}dx = 0.$$

Thanks to (2.1), we have $\|u^-\|_{p(x)} = 0$. Then, we deduce that $u^- = 0$ and $u = u^+$. Similarly, nontrivial critical points of Φ_- are non-positive solutions of $(\mathcal{P})$. □

We prove the following result on the required properties of the energy functional Φ related to problem (1.1).

Lemma 3.3. *Let conditions (1.3), (1.4) and (1.5) be satisfied. Then, the energy functional Φ_+ given in (3.4) satisfies the (PS)-Condition.*

Proof. Let $\{u_n\} \subseteq W_0^{1,p(x)}(\Omega)$ be a (PS) sequence for the functional Φ_+: $\Phi_+(u_n)$ is bounded and $\Phi'_+(u_n) \longrightarrow 0$. Let us show that $(u_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega)$. Since $\Phi_+(u_n)$ is bounded, using (1.4), we observe that:

$$\begin{aligned} C_1 \geq & \int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)} dx + \int_\Omega \frac{1}{p(x)} \left[\sqrt{1+|\nabla u_n|^{2p(x)}} - 1\right] dx + \int_\Omega \frac{\lambda(x)}{p(x)} |u_n|^{p(x)} dx \\ & - \int_\Omega F(x, u_n^+) dx \\ \geq & \frac{1}{p^+} \int_\Omega |\nabla u_n|^{p(x)} dx + \frac{1}{p^+} \int_\Omega \left[\sqrt{1+|\nabla u_n|^{2p(x)}} - 1\right] dx + \frac{1}{p^+} \int_\Omega \lambda(x) |u_n|^{p(x)} dx \\ & - \int_\Omega \frac{u_n^+}{\theta} f(x, u_n^+) dx + C_2, \end{aligned}$$

where C_1 and C_2 are two constants. Note that

$$\begin{aligned} \langle \Phi'_+(u_n), u_n \rangle = & \int_\Omega |\nabla u_n|^{p(x)} dx + \int_\Omega \frac{|\nabla u_n|^{2p(x)}}{\sqrt{1+|\nabla u_n|^{2p(x)}}} dx + \int_\Omega \lambda(x) |u_n|^{p(x)} dx \\ & - \int_\Omega f(x, u_n^+) u_n dx \\ = & \int_\Omega |\nabla u_n|^{p(x)} dx + \int_\Omega \frac{|\nabla u_n|^{2p(x)}}{\sqrt{1+|\nabla u_n|^{2p(x)}}} dx + \int_\Omega \lambda(x) |u_n|^{p(x)} dx + \int_\Omega \lambda(x) |u_n|^{p(x)} dx \\ & - \int_\Omega f(x, u_n^+) u_n^+ dx, \end{aligned}$$

which implies

$$\begin{aligned} C_1 \geq & \left(\frac{1}{p^+} - \frac{1}{\theta}\right) \left(\int_\Omega |\nabla u_n|^{p(x)} dx + \int_\Omega \frac{|\nabla u_n|^{2p(x)}}{\sqrt{1+|\nabla u_n|^{2p(x)}}} dx + \int_\Omega \lambda(x) |u_n|^{p(x)} dx \right) \\ & + \frac{1}{\theta} \langle \Phi'_+(u_n), u_n \rangle + C_2. \end{aligned}$$

Now, if we suppose, by contradiction that $(u_n)_n$ is unbounded in $\in W_0^{1,p(x)}(\Omega)$, we can say that $\|u_n\|_{p(x)} \geq 1$ for rather large values of n. Therefore, the use of the Relation (2.2) leads us to

$$\begin{aligned}\int_\Omega |\nabla u_n|^{p(x)}dx + \int_\Omega \frac{|\nabla u_n|^{2p(x)}}{\sqrt{1+|\nabla u_n|^{2p(x)}}}dx + \int_\Omega \lambda(x)|u_n|^{p(x)}dx \\ \geq \min(1;\lambda_0)\left(\int_\Omega |\nabla u_n|^{p(x)}dx + \int_\Omega |u_n|^{p(x)}dx\right) \\ \geq \min(1;\lambda_0)\left(|\nabla u_n|^{p^-}_{p(x)} + |u_n|^{p^-}_{p(x)} - 2\right) \\ \geq \min(1;\lambda_0)\frac{1}{2^{p^- -1}}\left(|\nabla u_n|_{p(x)} + |u_n|_{p(x)}\right)^{p^-} - 2\min(1;\lambda_0) \\ \geq \min(1;\lambda_0)\frac{1}{2^{p^- -1}}\|u_n\|^{p^-}_{p(x)} - 2\min(1;\lambda_0). \qquad (3.6)\end{aligned}$$

Moreover, the fact that $\Phi'_+(u_n) \longrightarrow 0$ assures that there exists $C_3 > 0$ such that

$$-C_3\|u_n\|_{p(x)} \leq \langle \Phi'_+(u_n), u_n\rangle \leq C_3\|u_n\|_{p(x)}$$

for rather large values of n. Consequently,

$$C_1 \geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\min(1;\lambda_0)\left(\frac{1}{2^{p^- -1}}\|u_n\|^{p^-}_{p(x)} - 2\right) - \frac{C_3}{\theta}\|u_n\|_{p(x)} + C_2.$$

Since $p^- > 1$ and $\left(\frac{1}{p^+} - \frac{1}{\theta}\right) > 0$, we end up with

$$\left(\frac{1}{p^+} - \frac{1}{\theta}\right)\min(1;\lambda_0)\left(\frac{1}{2^{p^- -1}}\|u_n\|^{p^-}_{p(x)} - 2\right) - \frac{C_3}{\theta}\|u_n\|_{p(x)} + C_2 \longrightarrow +\infty$$

as $\|u_n\|_{p(x)} \to +\infty$, which is a contradiction. So $(u_n)_n$ is a bounded sequence in $W_0^{1,p(x)}(\Omega)$. □

Lemma 3.4. *There exists $r > 0$ and $\alpha > 0$ such that $\Phi_+(u) \geq \alpha$, for all $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\|_{p(x)} = r$.*

Proof. Combining Conditions (1.3) and (1.5), we deduce the following inequality

$$|F(x,t)| \leq \varepsilon|t|^{p^+} + C(\varepsilon)|t|^{q(x)} \quad \text{for all } (x,t) \in \Omega\times\mathbb{R}. \qquad (3.7)$$

Taking $\|u\|_{p(x)}$ small enough, we have

$$\begin{aligned}\Phi_+(u) \geq \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}dx + \frac{1}{p^+}\int_\Omega \left[\sqrt{1+|\nabla u|^{2p(x)}} - 1\right]dx \\ + \frac{1}{p^+}\int_\Omega \lambda(x)|u|^{p(x)}dx - \int_\Omega F(x,u^+)dx.\end{aligned}$$

For such an element u we have $|u|_{p(x)} < 1$ and $|\nabla u|_{p(x)} < 1$, and by Relation (2.1), we obtain

$$\begin{aligned}
\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega \left[\sqrt{1+|\nabla u|^{2p(x)}} - 1\right]dx + \int_\Omega \lambda(x)|u|^{p(x)}dx \\
&\geq \int_\Omega |\nabla u|^{p(x)}dx + \lambda_0 \int_\Omega |u|^{p(x)}dx \\
&\geq |\nabla u|^{p^+}_{p(x)} + \lambda_0 |u|^{p^+}_{p(x)} \\
&\geq \frac{1}{2^{p^+-1}} \min(1,\lambda_0)\|u\|^{p^+}_{p(x)}. \qquad (3.8)
\end{aligned}$$

Relations (3.7) and (3.8) imply

$$\begin{aligned}
\Phi_+(u) &\geq \frac{1}{2^{p^+-1}} \min(1,\lambda_0)\|u\|^{p^+}_{p(x)} - \varepsilon \int_\Omega |u^+|^{p^+}dx - C(\varepsilon)\int_\Omega |u^+|^{q(x)}dx \\
&\geq \frac{1}{2^{p^+-1}} \min(1,\lambda_0)\|u\|^{p^+}_{p(x)} - \varepsilon \int_\Omega |u|^{p^+}dx - C(\varepsilon)\int_\Omega |u|^{q(x)}dx. \qquad (3.9)
\end{aligned}$$

From Condition (1.3), it follows $p^- \leq p(x) \leq p^+ < q^- \leq q(x) < p^* \ \forall x \in \overline{\Omega}$, then, we have the following continuous and compact embedding:

$$W_0^{1,p(x)}(\Omega) \subset L^{p^+} \quad \text{and} \quad W_0^{1,p(x)}(\Omega) \subset L^{q(x)},$$

which imply the existence of $C_4, C_5 > 0$ such that

$$\|u\|_{p^+} \leq C_4\|u\|_{p(x)} \quad \text{and} \quad \|u\|_{q(x)} \leq C_5\|u\|_{p(x)} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

Since $\|u\|_{p(x)}$ is small enough, it is obvious that

$$\int_\Omega |u|^{q(x)}dx \leq \|u\|^{q^-}_{q(x)} \leq C_6\|u\|^{q^-}_{p(x)}. \qquad (3.10)$$

From Relations (3.9) and (3.10), it results that

$$\Phi_+(u) \geq \frac{1}{2^{p^+-1}} \min(1,\lambda_0)\|u\|^{p^+}_{p(x)} - \varepsilon C_4^{p^+}\|u\|^{p^+}_{p(x)} - C_7(\varepsilon)\|u\|^{q^-}_{p(x)}$$

where C_i are positives constants. Let us choose $\varepsilon > 0$ such that

$$\varepsilon C_4^{p^+} \leq \frac{1}{2^{p^+}}\min(1,\lambda_0),$$

we obtain

$$\begin{aligned}
\Phi_+(u) &\geq \frac{1}{2^{p^+}} \min(1,\lambda_0)\|u\|^{p^+}_{p(x)} - C_7(\varepsilon)\|u\|^{q^-}_{p(x)} \\
&\geq \|u\|^{p^+}_{p(x)}\left(\frac{1}{2^{p^+}}\min(1,\lambda_0) - C_7(\varepsilon)\|u\|^{q^- - p^+}_{p(x)}\right). \qquad (3.11)
\end{aligned}$$

Since $p^+ < q^-$, the function $t \longmapsto \left(\frac{1}{2^{p^+}}\min(1,\lambda_0) - C_7(\varepsilon)t^{q^- - p_0^+}\right)$ is strictly positive in a neighborhood of zero. It follows that there exists $r > 0$ and $\alpha > 0$ such that

$$\Phi_+(u) \geq \alpha, \quad \forall u \in W_0^{1,p(x)}(\Omega) : \|u\|_{p(x)} = r. \qquad (3.12)$$

□

In order to apply the Mountain Pass Theorem, it remains to show the following property on the energy functional Φ_+:

Lemma 3.5. *There exists* $e \in W_0^{1,p(x)}(\Omega)$ *such that* $\|e\| > r$ *and* $\Phi_+(e) < 0$.

Proof. We prove that

$$\Phi_+(su) \longrightarrow -\infty \text{ as } s \to +\infty,$$

for a certain $u \in W_0^{1,p(x)}(\Omega)$. From Condition (1.4), we obtain

$$F(x,t) \geq c|t|^\theta \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

Let $u \in W_0^{1,p(x)}(\Omega)$ and $s > 1$. Then, we have

$$\begin{aligned}\Phi_+(su) = &\int_\Omega \frac{|s|^{p(x)}}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}\left[\sqrt{1+|s|^{2p(x)}|\nabla u|^{2p(x)}} - 1\right]dx \\ &+ \int_\Omega \frac{\lambda(x)|s|^{p(x)}}{p(x)}|u|^{p(x)}dx - \int_\Omega F(x,(su)^+)dx \\ \leq\ & s^{p^+}\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx + s^{p^+}\int_\Omega \frac{\sqrt{1+|\nabla u|^{2p(x)}}}{p(x)}dx \\ &+ s^{p^+}\int_\Omega \frac{\lambda(x)}{p(x)}|u|^{p(x)}dx - cs^\theta \int_\Omega |u^+|^\theta dx.\end{aligned}$$

The fact $\theta > p^+$, gives that

$$\Phi_+(su) \longrightarrow -\infty \text{ as } s \to +\infty.$$

This implies that there exists $e \in W_0^{1,p(x)}(\Omega)$ such that $\|e\| > r$ and $\Phi_+(e) < 0$. □

Proof of Theorem 1.1. In view of Lemma 3.3, Lemma 3.4, Lemma 3.5 and the Mountain Pass Theorem, Φ_+ admits a critical value $\mu \geq \alpha$ which is characterized by

$$\mu = \inf_{h\in\Lambda} \sup_{t\in[0,1]} \Phi_+(h(t)) \tag{3.13}$$

where

$$\Lambda = \{h \in C\big([0,1], W_0^{1,p(x)}(\Omega)\big) : h(0) = 0 \text{ and } h(1) = e\}. \tag{3.14}$$

Then, the functional Φ_+ has a critical point u^+ with $\Phi_+(u^+) \geq \alpha$. But, $\Phi_+(0) = 0$, that is, $u^+ \neq 0$. Therefore, the problem $(\mathcal{P}_+)$ has a nontrivial solution which, by Lemma 3.2, is a non-negative solution of the problem $(\mathcal{P})$.
Similarly, using Φ_-, we show that there exists furthermore a non-positive solution.

□

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