

GRAM DETERMINANT FOR CELL REPRESENTATIONS

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ABSTRACT. We study a new family of cell modules over the Fuss–Catalan algebras, focusing on the irreducibility of modules labeled by certain words. We define diagrammatic bases for these modules, compute their dimensions, define associated Gram matrices, and derive closed-form expressions for their determinants. The analysis of the determinants reveal necessary and sufficient conditions for the irreducibility of these family of modules.

Keywords. Diagram algebras, cell modules, Gram determinant.
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1. INTRODUCTION

Fuss–Catalan algebras, introduced by Bisch and Jones [1], are graphical algebras that arise in the study of subfactors and their intermediate subalgebras. They generalize the well-known Temperley–Lib algebra and are constructed using non-intersecting diagrams subject to additional coloring rules, which are controlled by the parameters $a, b \in \mathbb{C}^*$. The name is derived from their connection to the Fuss–Catalan numbers, which enumerate certain classes of colorable, non-intersecting diagrams and extend classical Catalan calculus. These algebras play a central role in various areas of mathematics and mathematical physics, including planar algebras, representation theory, and integrable lattice models [4, 11]. As a class of cellular algebra, Fuss–Catalan algebra has a rich structural framework first developed by Graham and Lehrer [6]. Cellular algebras are equipped with a cell datum that encodes the collection of natural bases and cell modules. These modules serve as the building blocks of the representation theory of algebra and are key to understanding its semisimplicities, branching rules, and homological properties. For the Fuss–Catalan algebra $FC_n(a, b)$ a combinatorial description of cellular modules was established in [7], where it was shown that the algebra allows a cellular structure compatible with its diagrammatic representation. Furthermore, these algebras were proved to be semi-hereditary when $a, b \neq 0$, satisfy the axioms of the towers of recollement [3]. An essential problem in the study of cell modules is to determine when they are irreducible, a property closely related to the non-degeneracy of bilinear forms defined through diagrammatic gluing. This

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form, whose matrix representation is the Gram matrix, contains the algebraic operation on the modules, as well as, encapsulates the action of the algebra on its modules and determines the semisimplicity of the representation. The Gram determinant plays a pivotal role when it is non-zero, the module is irreducible, when it vanishes, the module is reducible or even decomposable. This connection was formally captured in [6, Theorem 3.8] and further explored in specific algebraic contexts such as Brauer algebras, partition algebras [10], and Hecke algebras [5]. The cellular structure of $FC_n(a, b)$ has been investigated from several perspectives. In particular Semisimplicity for a specified class of cell modules over $FC_n(a, b)$ was established in [8]. Moreover, authors [9] introduced a dual basis for a different family of cell representations, which plays an important role in the structural analysis of these modules.

In this work, we investigate the cell module with label $\mu_n = aaaa(abba)^{(n-6)/2}$ if n is even whereas $\mu_n = aaaa(abba)^{(n-7)/2}ab$ if n is odd. This label corresponds to the admissible word whose diagrammatic realization forces nontrivial recoupling, making it a natural test case for Gram determinant computations and primitive idempotent construction. In Section 2, we review basic concepts of cellular algebras and examine the cellular structure for the algebras $FC_n(a, b)$. We define the cell modules of $FC_n(a, b)$, and determine their dimensions. In Section 3, for any $n \geq 6$, we introduce a family of diagrams that represents a basis of the cell module $\Delta_n(\mu_n)$ corresponding to a distinguished label μ_n , and study the bilinear forms naturally associated with these modules. By constructing the associated Gram matrix and computing its determinant, we identify the values of the parameters a and b for which this class of modules is irreducible.

2. PRELIMINARIES.

We begin by recalling basic concepts from [1]. An (n, n) -planar diagram D consists of $2n$ points placed on two parallel horizontal lines, with n points on each line. Each point is joined to exactly one other point by a non-intersecting string. A string may connect two points on the same line or two points on different lines. In the former case, the string is called a non-through string, while in the latter case it is called a through string. The length of D , denoted by $\ell(D)$, counts the vertical through strings in a diagram D . Fix $2n$ points on both the upper and lower lines. We color these points alternately by the colors a and b in the repeating pattern $abbaabbaabba \dots s$, where the final color s is a for even n and b for odd n . A diagram is said to be colored if every string joins two points with identical color. We denote by B_n the set of all planar $(2n, 2n)$ -diagrams satisfying this coloring condition. The composition of two diagrams $D_1, D_2 \in B_n$ is defined by their usual concatenation. Specifically, D_1D_2 is constructed by placing D_1 on top of D_2 , identifying the lower line of D_1 with the lower line of D_2 , and then removing this middle line. During this process, closed loops may appear. Suppose that r_1 closed loops of color a and r_2 closed loops of color b are formed. Removing these loops yields a diagram $D_3 \in B_n$ and the product is defined by $D_1D_2 = a^{r_1}b^{r_2}D_3$, where $a, b \in \mathbb{C}$. The upper half diagram of D is known as the initial part while the lower half is known as the final part of D .

Definition 2.1 ([6, Definition 1.1]). For a commutative ring R with identity, let A be an associative R -algebra equipped with cell datum $(\Lambda, \mathcal{M}, C, \iota)$. Then A is cellular algebra if the following axioms are satisfied:

- (A1) The set Λ is finite and partially ordered. For each $\lambda \in \Lambda$ there is a finite set $\mathcal{M}(\lambda)$. For all $\lambda \in \Lambda$, the diagrams $C_{S,T}^\lambda$ is an R -basis for the algebra A , where $(S, T) \in \mathcal{M}(\lambda) \times \mathcal{M}(\lambda)$.
- (A2) There is an R -linear anti automorphism map ι such that $\iota^2 = \text{id}$ and sends $C_{S,T}^\lambda$ to $C_{T,S}^\lambda$.
- (A3) If $S, T \in \mathcal{M}(\lambda)$ and $\lambda \in \Lambda$ then, for each element $a \in A$, the product $aC_{S,T}^\lambda$ is given by

$$\sum_{U \in \mathcal{M}(\lambda)} y_a(U, S) C_{U,T}^\lambda \pmod{A(< \lambda)}$$

where the scalars $y_a(U, S) \in R$ do not depend on T , and $A(< \lambda)$ is the R -submodule of A that has generating set $\{C_{S_1, T_1}^\mu \mid \mu < \lambda; S_1, T_1 \in \mathcal{M}(\mu)\}$.

Definition 2.2. We denote the full set of labels by Λ_n , and for each $\lambda \in \Lambda_n$, we define the full set of distinct initial parts for basis diagrams in FC_n carrying a label λ by $\mathcal{M}_n(\lambda)$. For $S, T \in \mathcal{M}_n(\lambda)$, we define $C_{S,T}^\lambda$ to be the diagram obtained by placing S on top of T after reflecting T upside down. Finally, we define an involution ι on FC_n by mapping each diagram to its upside-down reflection.

Theorem 2.3 ([7, Theorem 4.5]). *Consider the cell datum $(\Lambda_n, \mathcal{M}_n(\lambda), \mathcal{B}_n, \iota)$ that is defined in Definition 2.2, then the Fuss-Catalan algebras $\text{FC}_n(a, b)$ are cellular.*

Definition 2.4 ([6, Definition 2.1]). For all $\lambda \in \Lambda$, we introduce a left A -module $\Delta(\lambda)$. As an R -module, $\Delta(\lambda)$ is free with basis $\{C_S \mid S \in \mathcal{M}(\lambda)\}$. The action of A on $\Delta(\lambda)$ is given by the following rule

$$aC_S = \sum_{U \in \mathcal{M}(\lambda)} y_a(U, S) C_U, (a \in A, S \in \mathcal{M}(\lambda)),$$

with $y_a(U, S) \in R$ that are given in Definition 2.1. It is called the cell representation of A associated with $\lambda \in \Lambda$.

Definition 2.5. Let $\mathcal{M}_n(\lambda)$ be the full set of basis elements for the cell module $\Delta_n(\lambda)$ of the algebra FC_n . For each $\lambda \in \Lambda_n$, we describe the action of $D \in \text{FC}_n$ on $S \in \mathcal{M}_n(\lambda)$ by

$$D \cdot S = \begin{cases} DS, & \text{if } \ell(DS) = \ell(S), \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.6 ([2, Section 2]). *Suppose $\lambda = a^{l_1} b^{l_2} \dots s^{l_p}$ is a label for the cell module $\Delta_n(\lambda)$, where $s = a$ if n is even, and $s = b$ if n is odd. The dimension of this module is*

$$\dim(\Delta_n(\lambda)) = \begin{cases} \frac{m}{3(k-r)+m} \binom{3(k-r)+m}{k-r} & \text{if } k \geq r \\ 0 & \text{if } k < r, \end{cases}$$

such that $\ell = \sum_1^p l_i$, $m = \frac{1}{2}(3\ell - 2p + 4)$, moreover, $r = \frac{1}{2}(\ell - p + 1)$ for even $n = 2k$ and $r = \frac{1}{2}(\ell - p)$ for odd $n = 2k + 1$.

3. GRAM MATRICES AND THEIR DETERMINANTS

In this part, we are going to investigate the cell modules $\Delta_n(\mu_n)$ over $\text{FC}_n(a, b)$ for $n \geq 6$, and

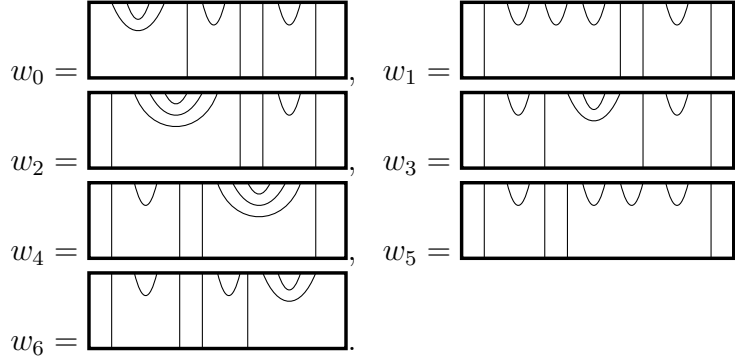
$$\mu_n = \begin{cases} a^4(abba)^{k-3} & \text{if } n = 2k \\ a^4(abba)^{k-3}ab & \text{if } n = 2k + 1. \end{cases} \quad (3.1)$$

We first describe an explicit basis of planar diagrams for these modules and compute their dimensions. We then introduce a bilinear form on $\Delta_n(\mu_n)$, which allows us to write down the associated Gram matrices explicitly. Finally, by computing the determinants of these Gram matrices, we determine the values of the parameters a and b for which the modules $\Delta_n(\mu_n)$ are irreducible.

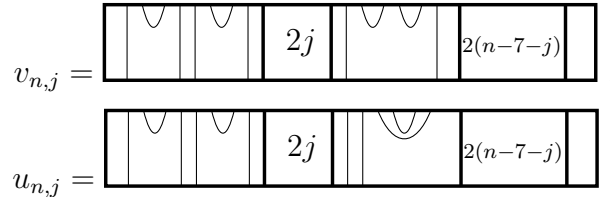
Proposition 3.1. *We have $\dim \Delta_n(\mu_n) = 2n - 5$, where $n \geq 6$.*

Proof. For even $n = 2k$ with $k \geq 3$, we have $\mu_n = a^4(abba)^{k-3}$ with length $4(k-3) + 4$. We rewrite μ_n in the form $\mu_n = a^3\tau a$, where $\tau = (a^2b^2)^{k-3}$. Let τ_a and τ_b represent number of a^2 's and b^2 's in τ respectively. Thus, $\tau_a = \tau_b = k - 3$, and μ_n has parts equal to $p = \tau_a + \tau_b + 1 = 2(k-3) + 1$. Using Proposition 2.6, we have $r = k - 1$, $l = 2n - 8$ and $m = 2n - 5$. Hence, $\dim(\Delta_n(\mu_n)) = \frac{m}{3+m} \binom{3+m}{1} = m = 2n - 5$. Similarly, we can prove the result for odd n . \square

Definition 3.2. We define the initial diagrams w_0, w_1, \dots, w_6 as follows



In addition, we define the initial diagrams $v_{n,j}$ and $u_{n,j}$ for $j = 0, 1, \dots, n-7$ as follows



Definition 3.3. We introduce a notation Id_n for the identity diagram of FC_n . This diagram consists of $2n$ through strings, that is, $\ell(\text{Id}_n) = 2n$. In addition, the label of this diagram is $(abba)^{\frac{n}{2}}$ for even n , and $(abba)^{\frac{n-1}{2}}ab$ for odd n .

Definition 3.4. We define the stacking of two initial diagrams S_1 and S_2 by $S_1 \otimes S_2$.

For example, if $S_1 = \begin{array}{|c|c|} \hline \text{---} \\ \hline \end{array}$ and $S_2 = \begin{array}{|c|c|c|} \hline \text{---} \\ \hline \end{array}$, then the stacking diagrams $S_1 \otimes S_2 = \begin{array}{|c|c|c|} \hline \text{---} \\ \hline \end{array}$.

Proposition 3.5. For $i = 0, 1, \dots, 6$, define $w_{i,n-6} = w_i \otimes \text{Id}_{n-6}$. Then the cell module $\Delta_n(\mu_n)$ is spanned by the set

- (i) $\{w_0, w_1, \dots, w_6\}$ for $n = 6$.
- (ii) $\{w_{0,n-6}, \dots, w_{6,n-6}, v_{n,0}, u_{n,0}, \dots, v_{n,n-7}, u_{n,n-7}\}$ for $n > 6$.

Proof. (i) For $n = 6$, by Proposition 3.1, we have $\dim \Delta_6(\mu_6) = 7$. It is clear that each diagram in the set $\{w_0, w_1, \dots, w_6\}$ has label a^4 .

(ii) For $n > 6$, we can see that number of diagrams in the set

$$\mathcal{M}(\mu_n) = \{w_{0,n-6}, \dots, w_{6,n-6}, v_{n,0}, u_{n,0}, \dots, v_{n,n-7}, u_{n,n-7}\}$$

is $2n - 5$. Now, we will explain that all diagrams in $\mathcal{M}(\mu_n)$ distinguished with label μ_n . Note that, for $0 \leq i \leq 6$, the through strings of w_i gives the label a^4 . Further, using Definition 3.3, Id_{n-6} has label $(abba)^{\frac{n-6}{2}}$ for even n and $(abba)^{\frac{n-7}{2}}$ for odd n . Hence, $w_{i,n-6}$ has label μ_n . The diagram $v_{n,j} = f_1 \otimes f_2$ where

$$f_1 = \begin{array}{|c|c|c|} \hline \text{---} \\ \hline \end{array}, f_2 = \begin{array}{|c|c|c|c|} \hline 2j \quad \text{---} \quad 2(n-7-j) \\ \hline \end{array}.$$

The label of f_1 is a^4 . The subdiagram f_2 has $2(n-6)$ through strings starts with color a . From Definition 3.3, f_2 has label identical to the label of the identity diagram Id_{n-6} . Therefore, the diagram $v_{n,j}$ has label μ_n . Consider the diagram $u_{n,j} = f_1 \otimes f_3$ where f_1 is as defined above and

$$f_3 = \begin{array}{|c|c|c|c|} \hline 2j \quad \text{---} \quad 2(n-7-j) \\ \hline \end{array}.$$

Similarly, we can show that f_3 has $2(n-6)$ through strings starts with color a and its label is μ_n . \square

We now introduce the bilinear forms associated with the cell modules of FC_n . By [6, Theorem 3.8], the irreducibility of a cell module is characterized by the non-degeneracy of its Gram matrix.

Definition 3.6. Suppose S_1 and S_2 are basis elements for a cell module $\Delta_n(\lambda)$. We describe a bilinear form $\langle -, - \rangle_{n,\lambda}: \Delta_n(\lambda) \times \Delta_n(\lambda) \rightarrow \mathbb{C}$ as follows, to compute $\langle S_1, S_2 \rangle$, turn the diagram S_2 upside down and place it on top of S_1 . If the resulting composite diagram has the same number of through strings as S_1 , then $\langle S_1, S_2 \rangle = a^{d_1} b^{d_2}$ where d_1 and d_2 are the numbers of loops with color a and loops of color b created in the composition. If the number of through strings is different, we set $\langle S_1, S_2 \rangle = 0$. We extend this definition to all of $\Delta_n(\lambda)$ bilinearity. When there is no risk of confusion, we omit the indices λ, n from the notation.

Definition 3.7. Assume that $\Delta_n(\lambda)$ is a cell module with basis (x_1, x_2, \dots, x_k) . We have a Gram matrix related to this module given by $G_n(\lambda)_{i,j} = \langle x_i, x_j \rangle$, for $i, j = 1, 2, \dots, k$.

Proposition 3.8. Fix the basis of the cell module $\Delta_n(\mu_n)$ that was determined in Proposition 3.5. The associated Gram matrix is given by

(i) When $n = 6$, with respect to the ordered basis (w_0, \dots, w_6) , we get

$$G(\mu_6) = b^2 \begin{pmatrix} ab & b & 1 & 0 & 0 & 0 & 0 \\ b & ab & a & b & 0 & 0 & 0 \\ 1 & a & ab & 1 & 0 & 0 & 0 \\ 0 & b & 1 & ab & 1 & b & 0 \\ 0 & 0 & 0 & 1 & ab & a & 1 \\ 0 & 0 & 0 & b & a & ab & b \\ 0 & 0 & 0 & 0 & 1 & b & ab \end{pmatrix}. \quad (3.2)$$

(ii) When $n > 6$, with respect to the ordered basis

$$(w_{0,n-6}, \dots, w_{6,n-6}, v_{n,0}, u_{n,0}, \dots, v_{n,n-7}, u_{n,n-7}),$$

we get

$$G(\mu_n) = \begin{bmatrix} H & E_s \\ E_s^T & Y_t \end{bmatrix}, \quad (3.3)$$

where $H = G(\mu_{n-1})$, $E_s^T = \begin{bmatrix} 0 & \dots & 0 & sb^2 \\ 0 & \dots & 0 & b^2 \end{bmatrix}_{2 \times (2n-7)}$, $Y_t = \begin{bmatrix} ab^3 & tb^2 \\ tb^2 & ab^3 \end{bmatrix}$, moreover, $(s, t) = (a, b)$ for even n and $(s, t) = (b, a)$ for odd n .

Proof. (i) For $n = 6$, it is easy to show that (3.2) satisfies $G(\mu_6)_{ij} = \langle w_i, w_j \rangle$ for all $i, j = 0, \dots, 6$,

(ii) The basis for $\Delta_{n-1}(\mu_{n-1})$ can be identified by deleting the last pair of through strings from the diagram basis

$$\{w_{0,n-6}, \dots, w_{6,n-6}\}, \text{ if } n = 7,$$

and

$$\{w_{0,n-6}, \dots, w_{6,n-6}, v_{n,0}, u_{n,0}, \dots, v_{n,n-8}, u_{n,n-8}\}, \text{ if } n > 7.$$

Furthermore, the presence of these two through strings does not affect the value of the bilinear form, so, they can be omitted from the computation, and hence, $H = G(\mu_{n-1})$. Applying the definition of the bilinear form, then the entries of E_s^T are

$$\begin{aligned} \langle v_{n,n-7}, w_{i,n-6} \rangle &= \langle u_{n,n-7}, w_{i,n-6} \rangle = 0, \text{ where } i = 0, 1, \dots, 6. \\ \langle v_{n,n-7}, v_{n,j} \rangle &= \langle u_{n,n-7}, v_{n,j} \rangle = 0, \text{ where } j = 0, 1, \dots, n-8. \\ \langle v_{n,n-7}, u_{n,j} \rangle &= \langle u_{n,n-7}, u_{n,j} \rangle = 0, \text{ where } j = 0, 1, \dots, n-9. \\ \langle v_{n,n-7}, u_{n,n-8} \rangle &= sb^2, \quad \langle u_{n,n-7}, u_{n,n-8} \rangle = b^2. \end{aligned}$$

Similarly, one can compute the entries of Y_t :

$$\langle v_{n,n-7}, v_{n,n-7} \rangle = \langle u_{n,n-7}, u_{n,n-7} \rangle = ab^3, \quad \langle v_{n,n-7}, u_{n,n-7} \rangle = tb^2, \quad \square$$

Proposition 3.9. *The Gram matrix $G(\mu_n)$ associated to the cell module $\Delta_n(\mu_n)$ has determinant given by*

$$\begin{cases} a^3b^{17}(a^2-1)(b^2-1)^2(a^2-3) & \text{for } n = 6, \\ a^3b^{21}(b^2-1)^3 \times c & \text{for } n = 7, \\ a^3b^{21}(b^2-1)^3 \times c \times d^{\frac{n-7}{2}}, & \text{for odd } n > 7, \\ a^3b^{27}(b^2-1)^2(b^2-2)(a^2-1) \times c \times d^{\frac{n-8}{2}}, & \text{for even } n > 7, \end{cases} \quad (3.4)$$

where $c = (a^3 + a^2 - 2a - 1)(a^3 - a^2 - 2a + 1)$, and $d = a^2b^{10}(a^2 - 2)(b^2 - 2)$.

Proof. By Proposition 3.8, we can compute $\det G(\mu_6)$ and $\det G(\mu_7)$. For $n > 7$, using Proposition 3.8(ii), we get

$$G(\mu_n) = \begin{pmatrix} G(\mu_{n-1}) & E_s \\ E_s^T & Y_t \end{pmatrix} = \left(\begin{array}{cc|c} G(\mu_{n-2}) & E_t & \\ \hline E_t^T & Y_s & E_s \\ \hline & & \\ E_s^T & & Y_t \end{array} \right).$$

Extending the block matrices E_s, E_t^T, Y_s, E_s^T and Y_t into the equation above, we see that

$$G(\mu_n) = \left(\begin{array}{ccc|cc} G(\mu_{n-2}) & & & & 0 \\ 0 & 0 & \cdots & 0 & tb^2 \\ 0 & 0 & \cdots & 0 & b^2 \\ \hline & & & 0 & sb^2 \\ & & & 0 & b^2 \end{array} \begin{array}{c} R_4 \\ R_3 \\ R_2 \\ R_1 \end{array} \right)$$

where R_1, R_2, R_3 and R_4 refer to the last four rows in the matrix $G(\mu_n)$ such that R_1 represents the last row and R_2 represents the second to last row and so on. Note that, $ab = st$, and set $x = sb^2(\frac{t^2-1}{t})$. Replacing R_3 by $(-1/t)R_4 + R_3$, R_1 by $(-1/s)R_2 + R_1$ and R_2 by $\frac{-sb^2}{x}R_3 + R_2$, we get the matrix $H_n = \begin{pmatrix} H_{1,n} & H_{2,n} \\ H_{3,n} & P_n \end{pmatrix}$,

where $H_{1,n} \cong G(\mu_{n-1})$, $P_n = \begin{pmatrix} ab^3(\frac{t^2-2}{t^2-1}) & tb^2(\frac{t^2-2}{t^2-1}) \\ 0 & tb^2(\frac{s^2-1}{s}) \end{pmatrix}$ and $H_{3,n}$ is the zero matrix.

Now,

$$\det H_n = \det G(\mu_n) = \det(H_{1,n} - H_{2,n}P_n^{-1}H_{3,n})\det(P_n),$$

but $H_{3,n}$ is zero matrix and $\det H_{1,n} = \det G(\mu_{n-1})$. Then

$$\begin{aligned} \det G(\mu_n) &= \det G(\mu_{n-1}) \times \det P_n \\ &= \det G(\mu_{n-2}) \times \det P_n \times \det P_{n-1} \\ &\vdots \\ &= \det G(\mu_7) \times \det P_n \times \det P_{n-1} \times \cdots \times \det P_8. \end{aligned}$$

We have $\det P_n \times \det P_{n-1} = \det P_{n-2} \times \det P_{n-3} = \cdots = a^2b^{10}(a^2 - 2)(b^2 - 2)$. Thus, $\det G(\mu_n)$ is

$$\begin{cases} \det G(\mu_7) \times d^{\frac{n-7}{2}}, & \text{if } n \text{ is odd,} \\ \det G(\mu_7) \times d^{\frac{n-8}{2}} \times \det P_8, & \text{if } n \text{ is even.} \end{cases}$$

□

Using [6, Theorem 3.8], we conclude

Corollary 3.10. *Let $f(a) = (a^3 + a^2 - 2a - 1)(a^3 - a^2 - 2a + 1)$. The cell modules $\Delta_n(\mu_n)$ of $FC_n(a, b)$ is irreducible for the specified parameters a and b :*

- (i) $b \neq \pm 1$, for all $n \geq 6$.
- (ii) $b \neq \pm\sqrt{2}$, for all $n \geq 8$.
- (iii) $a \neq \pm 1$, for all even $n \geq 6$.
- (iv) $a \neq \pm\sqrt{2}$, for all $n \geq 9$.
- (v) $a \neq \pm\sqrt{3}$, for $n = 6$.
- (vi) a is not a root of $f(a) = 0$, for all $n \geq 7$.

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