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NON-DEGENERATE INTEGRATED SEMIGROUP AND ITS GENERATOR

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Abstract. In this paper, we unify the continuous, discrete, and intermediate instances by introducing the definition of a time-scale integrated semigroup. We investigate the connection between the infinitesimal generator of the integrated semigroup.

Keywords. Integrated semigroups, Infinitesimal generators, Time scales calculus
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1. Introduction

Stefan Hilger first proposed time scale theory in his doctoral thesis in 1988 in order to combine the two methods of dynamic modeling—differential equations and difference equations. We refer to [8, 11, 13, 17] for additional advancements on the time scale.

Differential equations are solved using the notion of semigroups of bounded linear operators. Markov processes, ergodic theory, and partial differential equations all immediately benefited from semigroup theory. The space of all bounded linear operators on a Banach space X is denoted by $B(X, X)$. Then a family of $\{T(t)\}$ is called a semigroup of operators if:

(1) $T(0) = I$
(2) $T(s + t) = T(s)T(t)$
(3) $T(t)x \xrightarrow[t\to 0^+]{} x$.

The name semigroup comes from the fact that the family $\{T(t)\}_{t\geq 0}$ forms a semigroup in the algebraic sense. Since E. Hille [19], K. Yoshida [[29, 30], Pazy [23], Nathan [22] and Toukmati [27] established the characterization of generators of C_0-semigroups in the 1940s, semigroups of linear operators and its neighboring areas have developed into a beautiful abstract theory. Moreover, the fact that mathematically this abstract theory has many direct and important applications

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in partial differential equations enhances its importance as a necessary discipline in both functional analysis and differential equations.

In [4], the authors introduced the notion of the N_F^α-conformable semigroup, that is,

(1) $T(0) = I$
(2) $T(G_\alpha^{-1}(t+s)) = T(G_\alpha^{-1}(t))T(G_\alpha^{-1}(s))$.
(3) $T(t)x \xrightarrow[t\to 0^+]{} x$,

where $F : [0, +\infty) \times (0, 1] \to (0, \infty)$ and $G'_\alpha(t) = F(t, \alpha)$.
In [18], the authors studied the semigroup on time scales $\mathbb{T}_0^+ = [0, +\infty)$, and apply its results to the abstract Cauchy problem.

Motivation: The present work focuses on the study of one of the most important mathematical tools in solving "well-posed" problems in the theory of evolution equations on time scales and the theory of stochastic processes on time scales, namely integrated semi-groups of linear operators on time scales and their applications to the theory of partial differential equations on time scales.

The outline of this paper is organized as follows : In section 2, attention will be paid to the statement and the introduction of the basic definitions and properties of the time scales theory and delta N_F^α-derivative. Section 3, will be devoted to give a new definition of integrated semigroup on time scales and the relationship between the generator of a non-degenerate integrated semigroup.

2. Preliminaries and Tools

The theory of time Scales is a mathematical framework introduced by Stefan Hilger in 1988 to unify and generalize both continuous and discrete analysis. It achieves this by defining a "time scale," which can be any closed set of real numbers, and developing a new calculus that works for these arbitrary sets. This allows for unified treatment of differential equations (continuous time) and difference equations (discrete time) and has applications in fields like biology, economics, and physics. The definition of the forward and backward jump operators is stated in [11, 14] by

Definition 2.1. [11] Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = inf\{s \in \mathbb{T} : s > t\}$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$: is defined by

$$\rho(t) = sup\{s \in \mathbb{T} : s < t\}.$$

And the graininess operator $\mu : \mathbb{T} \to \mathbb{R}^+$ defined by $\mu(t) = \sigma(t) - t$.

Some of the definitions are in the following table

t right-scattered	$t < \sigma(t)$
t right-dense	$t = \sigma(t)$ and $\sigma(t) < \sup \mathbb{T}$
t left-scattered	$\rho(t) < t$
t left-dense	$\rho(t) = t$ and $\rho(t) > \inf \mathbb{T}$
t dense	$\rho(t) = t = \sigma(t)$ and $\sigma(t) < \sup \mathbb{T}$ and $\rho(t) > \inf \mathbb{T}$

Let $F : \mathbb{T} \times (0, 1] \to (0, \infty)$ be a given rising function with respect to the first argument, and let $\alpha \in (0, 1]$.

Definition 2.2. Assume that $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^\kappa$. The delta N_F^α-derivative of f at t noted by $f_F^\Delta(t)$ is defined by

$$\Big|\big(f(\sigma(t)) - f(s)\big)F(t, \alpha) - f_F^\Delta(t)(\sigma(t) - s)\Big| \leq \epsilon|\sigma(t) - s|.$$

In the neighbourhood $U \subset \mathbb{T}$ of t.
And put

$$f_F^{2\Delta}(t) := (f_F^\Delta)^\Delta(t).$$

Remark 2.3. If f is differentiable at t, then

$$f^\sigma(t) := f(\sigma(t)) = f(t) + \frac{\mu(t)}{F(t, \alpha)} f^\Delta(t)$$

Theorem 2.4. [7] *Assume $f; g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then:*

(1) *$f + g$ is conformable differentiable with $(f + g)_F^\Delta(t) = f_F^\Delta(t) + g_F^\Delta(t)$;*
(2) *fg is conformable differentiable with $(fg)_F^{(\alpha)}(t) = f_F^\Delta(t)g(\sigma(t)) + f(t)g_F^\Delta(t)$;*
(3) *$\frac{1}{f}$ is conformable differentiable with $(\frac{1}{f})_F^\Delta(t) = -\frac{f_F^\Delta(t)}{f(t)f^\sigma(t)}$ for which $ff^\sigma \neq 0$;*
(4) *$\frac{f}{g}$ is conformable differentiable with $(\frac{f}{g})_F^\Delta(t) = \frac{f_F^\Delta(t)g(t) - f(t)g_F^\Delta(t)}{g(t)g(\sigma(t))}$, $gg^\sigma \neq 0$.*

Theorem 2.5. [7] *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. If f is N_F^α-differentiable at t, then f is continuous at t.*

Definition 2.6. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function. f belongs to $C^1(\mathbb{T}; X)$ if and only if f is delta N_F^α-derivative on $\mathbb{T}$ and if the map $t \to N_F^\alpha f(t)$ is continuous.

Definition 2.7. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function. f belongs to $C^2(\mathbb{T}; X)$ if and only if f is delta $N_F^{2\alpha}$-derivative on $\mathbb{T}$ and if the map $t \to N_F^{2\alpha} f(t)$ is continuous.

Definition 2.8. Assume that $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Then, the delta N_F^α-integral of f, for $\alpha \in (0, 1]$, is defined by

$$\int f(t)\Delta_F^\alpha t = \int f(t)F(t, \alpha)\Delta t.$$

If $H^\Delta(t) = f(t)F(t, \alpha)$. The Cauchy integral is defined by

$$\int_s^t f(\tau)\Delta_F^\alpha \tau = H(t) - H(s).$$

Then we have some properties [11]:

1. $\displaystyle\int_a^b \big(f(t)+g(t)\big)\Delta_F^\alpha \tau = \int_a^b f(t)\Delta_F^\alpha \tau + \int_a^b g(t)\Delta_F^\alpha \tau.$
2. $\displaystyle\int_a^b \beta f(t)\Delta_F^\alpha \tau = \beta \int_a^b f(t)\Delta_F^\alpha \tau.$
3. $\displaystyle\int_a^b f(t)\Delta_F^\alpha \tau = \int_a^c f(t)\Delta_F^\alpha \tau + \int_c^b f(t)\Delta_F^\alpha \tau.$

For $F : \mathbb{T} \times (0,1] \to (0,\infty)$, put

$$M_0(t) = \int_0^t F(s,\alpha)\Delta s = \int_0^t \Delta_F^\alpha s.$$

Definition 2.9. A C_0-semigroup over E is a family $\{T(t)\}_{t\in\mathbb{T}_0^+} \subset B(E)$ satisfying the following properties :
1. $T(0) = I$;
2. $t \to T(t)$ is strongly continuous, for all $t \in \mathbb{T}_0^+$;
3. $T(t)T(s) = T\Big(M_0^{-1}\big(M_0(t) + M_0(s)\big)\Big)$.

Not that (3) is equivalent to $T(M_0^{-1}(t))T(M_0^{-1}(s)) = T\Big(M_0^{-1}\big(t+s\big)\Big)$.

3. INTEGRATED SEMIGROUP ON TIME SCALES

Put

$$M_f(t) = \int_0^t f(t)\Delta_F^\alpha s$$

and

$$C^n = \{x \in X, \text{ such that } S(t)x \in C^n(\mathbb{T}_0^+; X)\}.$$

Definition 3.1. A family $S(t)\}_{t\in\mathbb{T}_0^+} \subset B(E)$ is an integrated semigroup over E, if the following properties are verified: An integrated semigroup over E is a family $\{S(t)\}_{t\in\mathbb{T}_0^+} \subset B(E)$ satisfying the following properties :
1. $S(0) = 0$;
2. $t \to S(t)$ is strongly continuous, for all $t \in \mathbb{T}_0^+$;
3. $\displaystyle S(t)S(s) = \int_0^t \bigg(S\Big(M_0^{-1}\big(M_0(\tau) + M_0(s)\big)\Big) - S(\tau)\bigg)\Delta_F^\alpha \tau.$

Remark 3.2. If $\{S(t)\}_{t\ge 0}$ is an integrated semigroup, then
1. $S(0)S(s) = 0$ for all $s \in \mathbb{T}$.
2. $S(t)S(0) = 0$ for all $t \in \mathbb{T}$.

Proposition 3.3. *By assuming $\{S(t)\}_{t\in\mathbb{T}_0^+}$ as an integrated semigroup, then the following properties hold true*
1. For all $x \in C^1$

$$S(s)S^\Delta(t)x = S\Big(M_0^{-1}\big(M_0(t) + M_0(s)\big)\Big)x - S(t)x, \ \forall s,t \in \mathbb{T}_0^+.$$

2. For all $x \in C^2$

$$S^\Delta(t)x = S^{2\Delta}(0)S(t)x + S^\Delta(0)x, \forall t \in \mathbb{T}_0^+.$$

3. For all $x \in C^2$

$$S^{2\Delta}(0)S(t)x = S(t)S^{2\Delta}(0)x, \forall t \in \mathbb{T}_0^+.$$

Proof. : Let $s, t \in \mathbb{T}_0^+$.
1.

$$\begin{aligned} S(s)S^{\Delta}(t)x = S^{\Delta}(t)S(s)x &= \big(S(t)S(s)\big)^{\Delta}x \\ &= \left(\int_0^t \left(S\Big(M_0^{-1}\big(M_0(\tau)+M_0(s)\big)\Big) - S(\tau)\right)\Delta_F^{\alpha}\tau\right)^{\Delta} x \\ &= S\Big(M_0^{-1}\big(M_0(t)+M_0(s)\big)\Big)x - S(t)x. \end{aligned}$$

2.

$$\begin{aligned} S^{2\Delta}(s)S(t)x &= \big(S^{\Delta}(s)S(t)\big)^{\Delta}x \\ &= \left(S\Big(M_0^{-1}\big(M_0(s)+M_0(t)\big)\Big) - S(s)\right)^{\Delta} x \\ &= S^{\Delta}\Big(M_0^{-1}\big(M_0(s)+M_0(t)\big)\Big)x - S^{\Delta}(s)x \end{aligned}$$

For $s = 0$, we get

$$S^{\Delta}(t)x = S^{2\Delta}(0)S(t)x + S^{\Delta}(0)x.$$

3. For $x \in C^2$, we have

$$\begin{aligned} S(s)S^{2\Delta}(t)x &= \big(S(s)S^{\Delta}(t)\big)^{\Delta}x \\ &= \left(S\Big(M_0^{-1}\big(M_0(t)+M_0(s)\big)\Big) - S(t)\right)^{\Delta} x \\ &= S^{\Delta}\Big(M_0^{-1}\big(M_0(t)+M_0(s)\big)\Big)x - S^{\Delta}(t)x \end{aligned}$$

For $s = 0$, we get

$$S(s)S^{2\Delta}(0)x = S^{\Delta}(s)x - S^{\Delta}(0)x.$$

And By (2), we get

$$S(s)S^{2\Delta}(0)x = S^{2\Delta}(0)S(s)x.$$

This completes the proof. □

Example 3.4. Let $\{T(t), t \in \mathbb{T}_0^+\}$ be a C_0-semigroup on time scales [9]. The $\{S(t), t \in \mathbb{T}_0^+\}$ which is defined by:

$$S(t) = \int_0^t T(s)F(s,\alpha)\Delta s = \int_0^t T(s)\Delta_F^{\alpha} s$$

is an integrated semigroup.

Definition 3.5. The set $\mathcal{N}_F$, is the generating space of the integrated semigroup $S(t)\}_{t\in\mathbb{T}_0^+}$. With

$$\mathcal{N}_F = \{x \in X, \text{ such that } S(t)x = 0, \forall t \in \mathbb{T}_0^+.\}$$

Proposition 3.6. *Let $\{S(t)\}_{t\in\mathbb{T}_0^+}$ be an integrated semigroup. Then*

$$\mathcal{N}_F = \mathcal{N}_F^1$$

where $\mathcal{N}_F^1 = \{x \in C^1 : S^\Delta(0)x = 0.\}$

Proof. Let $x \in \mathcal{N}_F$. Then $S(t)x = 0$, for all $t \in \mathbb{T}_0^+$. therefore $S^\Delta(t)x = 0$, for all $t \in \mathbb{T}_0^+$. Thus $S^\Delta(0)x = 0$, and $x \in \mathcal{N}_F^1$. Consequently $\mathcal{N}_F \subset \mathcal{N}_F^1$.
Let $x \in \mathcal{N}_F^1$, then $S^\Delta(0)x = 0$. By applying Proposition 3.3 (1), we have

$$S(t)S^\Delta(s)x = S\Big(M_0^{-1}\big(M_0(s) + M_0(t)\big)\Big)x - S(s)x$$

For $s = 0$, we get

$$S(t)S^\Delta(0)x = S(t)x.$$

Thus $S(t)x = 0$ for all $t \in \mathbb{T}_0^+$. Consequently $\mathcal{N}_F^1 \subset \mathcal{N}_F$.
This completes the proof. □

Definition 3.7. The integrated semigroup $\{S(t)\}_{t\in\mathbb{T}_0^+}$ is said to be non-degenerate if $\mathcal{N}_F = \{0\}$. Otherwise, $\{S(t)\}_{t\in\mathbb{T}_0^+}$ is supposed to be a degenerate integrated semigroup.

Proposition 3.8. *An integrated semigroup $\{S(t)\}_{t\in\mathbb{T}_0^+}$ is non-degenerate if and only if we have $S^\Delta(0)x = x$ for all $x \in C^1$.*

Proof. Let $\{S(t)\}_{t\in\mathbb{T}_0^+}$ be non-degenerate integrated semigroup. Then

$$S(t)x = 0, \forall t \in \mathbb{T}_0^+ \Rightarrow x = 0. \tag{3.1}$$

Let $x \in C^1$. By Proposition 3.3 (1) for s=0, we get

$$S(t)\big(S^\Delta(0)x - x\big) = 0.$$

Then, by Eq. (3.1), we get $S^\Delta(0)x = x$.
Let $\{S(t)\}_{t\in\mathbb{T}_0^+}$ be an integrated semigroup such that $S^\Delta(0)x = x$, for all $x \in C^1$.
Let $x \in \mathcal{N}_F$. Then

$$\begin{aligned} x \in \mathcal{N}_F^1 &\Rightarrow S^\Delta(0)x = 0 \\ &\Rightarrow x = 0 \\ &\Rightarrow \mathcal{N}_F = \{0\}. \end{aligned}$$

Thus $\{S(t)\}_{t\in\mathbb{T}_0^+}$ is a non-degenerate integrated semigroup. □

Theorem 3.9. *If we consider $\{S(t)\}_{t\in\mathbb{T}_0^+}$ as a non-degenerate integrated semigroup. Then $\{S^\Delta(t)\}_{t\in\mathbb{T}_0^+}$ is a C_0-semigroup over C^1.*

Proof. For all $x \in C^1, t \in \mathbb{T}_0^+$, $t \to S^\Delta(t)x$ is continuous.
By applying Proposition 3.8, we get

$$S^\Delta(0) = 1.$$

Taking into account Proposition 3.3 (1), it implies

$$S(s)S^\Delta(t)x = S\Big({M_0}^{-1}\big(M_0(t) + M_0(s)\big)\Big)x - S(t)x$$

Then, we get

$$S^{\Delta}(t)S^{\Delta}(s) = S\Big(M_0^{-1}\big(M_0(t) + M_0(s)\big)\Big).$$

This completes the proof. $\square$

Definition 3.10. A generator of a non-degenerate integrated semigroup $\{S(t)\}_{t\in\mathbb{T}_0^+}$ is designated as a linear operator.

$$A : D(A) \subset X \to X$$

defined by:

$$S^{\Delta}(t)x = S(t)Ax + x$$

For all $t \in \mathbb{T}_0^+$ and $x \in C^1$.

Proposition 3.11. *Let $A : D(A) \subset X \to X$ be the generator of a non-degenerate integrated semigroup $\{S(t)\}_{t\in\mathbb{T}_0^+}$. Then for $x \in C^2$*

$$Ax = S^{2\Delta}(0)x$$

Proof. By Definition $S^{\Delta}(t)x - x = S(t)Ax$. According to the Proposition 3.3

$$S^{2\Delta}(0)S(t)x + S^{\Delta}(0)x - x = S(t)Ax$$

By applying Proposition 3.8 and Proposition 3.3, we get

$$S(t)S^{2\Delta}(0)x = S^{2\Delta}(0)S(t)x = S(t)Ax.$$

Then

$$S(t)\big(Ax - S^{2\Delta}(0)x\big) = 0.$$

Thus

$$Ax = S^{2\Delta}(0)x.$$

This completes the proof. $\square$

Proposition 3.12. *Let $A : D(A) \subset X \to X$ be the generator of a non-degenerate integrated semigroup $\{S(t)\}_{t\in\mathbb{T}_0^+}$, such that $S(t)x \in D(A)$ for all $x \in C^1$. Then*

$$AS(t)x = S(t)Ax.$$

Proof. According to the Proposition 3.3 and Proposition 3.11

$$AS(t)x = S^{2\Delta}(t)x = S^{\Delta}(t)x - S^{\Delta}(0)x.$$

In view of Proposition 3.8, we obtain

$$AS(t)x = S^{2\Delta}(t)x = S^{\Delta}(t)x - x.$$

Then $AS(t)x = S(t)Ax$. Hence

$$AS(t)x = S^{2\Delta}(0)S(t)x = S(t)S^{2\Delta}(0)x = S(t)Ax.$$

$\square$

Proposition 3.13. *Let $A : D(A) \subset X \to X$ be the generator of a non-degenerate integrated semigroup $\{S(t)\}_{t\in\mathbb{T}_0^+}$. For all $x \in X$, we have $\int_0^t S(s)x\Delta_F^{\alpha}s \in D(A)$ and*

$$A\int_0^t S(s)x\Delta_F^{\alpha}s = S(t)x - M_0(t)x$$

Proof. By virtue of Definition 3.1, we get for all $r, s \in \mathbb{T}_0^+$

$$S(r)S(s) = \int_0^r \Big(S\Big(M_0^{-1}\big(M_0(\tau) + M_0(s)\big)\Big) - S(\tau)\Big)\Delta_F^\alpha \tau.$$

It follows that

$$\int_0^t S(r)S(s)\Delta_F^\alpha s = \int_0^t \left(\int_0^r \Big(S\Big(M_0^{-1}\big(M_0(\tau) + M_0(s)\big)\Big) - S(\tau)\Big)\Delta_F^\alpha \tau\right)\Delta_F^\alpha s$$

$$S(r)\int_0^t S(s)\Delta_F^\alpha s = \int_0^r \left(\int_0^t \Big(S\Big(M_0^{-1}\big(M_0(\tau) + M_0(s)\big)\Big) - S(\tau)\Big)\Delta_F^\alpha s\right)\Delta_F^\alpha \tau$$

$$\begin{aligned}
S^\Delta(r)\big(\int_0^t S(s)x\Delta_F^\alpha s\big) &= \int_0^t \Big(S\Big(M_0^{-1}\big(M_0(r) + M_0(s)\big)\Big) - S(r)\Big)x\Delta_F^\alpha s \\
&= \int_0^t \Big(S\Big(M_0^{-1}\big(M_0(r) + M_0(s)\big)\Big) - S(s) + S(s) - S(r)\Big)x\Delta_F^\alpha s \\
&= S(t)S(r)x - S(r)\int_0^t x\Delta_F^\alpha s + \int_0^t S(s)x\Delta_F^\alpha s \\
&= S(r)\big(S(t)x - M_0(t)x\big) + \int_0^t S(s)x\Delta_F^\alpha s.
\end{aligned}$$

Finally, one easily has

$$A\int_0^t S(s)x\Delta_F^\alpha s = S(t)x - M_0(t)x.$$

Hence, the proof of the proposition is fulffiled. □

Lemma 3.14. *Let $A : D(A) \subset X \to X$ be the generator of a non-degenerate integrated semigroup $\{S(t)\}_{t\in\mathbb{T}_0^+}$ and $\varphi : \mathbb{T}_0^+ \to X$ be a continuous map such that $\varphi > 0$ and*

$$\int_0^t \varphi(s)\Delta_F^\alpha s \in D(A), \forall t \in \mathbb{T}_0^+.$$

If $A\int_0^t \varphi(s)\Delta_F^\alpha s = \varphi(t)$ then $\varphi(t) = 0$.

Proof. Put $g(s) = S(t-s)\int_0^s \varphi(\tau)\Delta_F^\alpha \tau$. Then

$$\begin{aligned}
g^\Delta(s) &= (S(t-s))^\Delta \int_0^s \varphi(\tau)\Delta_F^\alpha \tau + S(t-s)\big(\int_0^s \varphi(\tau)\Delta_F^\alpha \tau\big)^\Delta \\
&= -S(t-r)A\int_0^s \varphi(\tau)\Delta_F^\alpha \tau - \int_0^s \varphi(\tau)\Delta_F^\alpha \tau + S(t-s)\varphi(s) \\
&= -\int_0^s \varphi(\tau)\Delta_F^\alpha \tau.
\end{aligned}$$

Therefore $-\int_0^t\int_0^s \varphi(\tau)\Delta_F^\alpha \tau\Delta_F^\alpha t = g(t) - g(0) = 0$. Which implies that $\varphi = 0$. □

Theorem 3.15. *Let $\{S_1(t)\}_{t\in\mathbb{T}_0^+}$ and $\{S_2(t)\}_{t\in\mathbb{T}_0^+}$ be two integrated semigroups whose generator is the same linear operator $A : D(A) \subset X \to X$. Then for all $t \in \mathbb{T}_0^+$, we have $S_1(t) = S_2(t)$.*

Proof. suppose that $S_1(t) > S_2(t)$ and put $\varphi(t) = S_1(t) - S_2(t)$. Then by the virtue of Proposition 3.13, we have for $x \in X$

$$\begin{aligned} A\int_0^t \varphi(s)x\Delta_F^\alpha s &= \int_0^t S_1(s)x\Delta_F^\alpha s - \int_0^t S_2(s)x\Delta_F^\alpha s \\ &= S_1(t)x - M_0(t)x - S_2(t)x + M_0(t)x = \varphi(t)x. \end{aligned}$$

Hence $A\int_0^t \varphi(s)\Delta_F^\alpha s = \varphi(t), \forall t \in \mathbb{T}_0^+$. Thus by Lemma 3.14, we conclude that $S_1(t) = S_2(t)$. This completes the proof. □

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