

AN EFFICIENT RESIDUAL POWER SERIES METHOD FOR SOLVING THE BAGLEY-TORVIK EQUATION

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ABSTRACT. This paper comprehensively analyzes the application of the Residual Power Series (RPS) method to solve the Bagley-Torvik fractional differential equation, originally formulated to model viscoelastic damping in thin plates immersed in Newtonian fluids. The method is rigorously applied to three case studies, with exact solutions verified via analytical methods such as Laplace transforms, demonstrating accuracy for polynomial solutions. Theoretical convergence analysis and numerical validation using MATLAB illustrations demonstrate the efficiency of the RPS as a tool for solving fractional initial value problems.

Keywords. Bagley-Torvik Equation, Caputo Fractional Derivative, Residual Power Series Method, Series Solution

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1. INTRODUCTION

The Bagley-Torvik equation first emerged in the 1983-1984 studies of Bagley and Torvik, motivated by the challenge of modeling the intermediate viscoelastic response observed in real-world materials commonly found in engineering and geophysics [16]. Unlike purely elastic or purely viscous media, these materials exhibit a memory of past deformations that lies between the integer-order derivative descriptions of classical mechanics. The equation elegantly models a thin rigid plate, attached to a spring, oscillating within a viscous fluid [13]. What distinguishes it is the incorporation of a Caputo fractional derivative of order $\frac{3}{2}$, which mathematically bridges the gap between integer-order viscous and elastic behaviors [14]. This fractional-order term captures the nuanced, frequency-dependent dissipation characteristic of materials that are neither fully solid nor fully fluid [2].

Power series are a cornerstone of mathematical analysis, providing a powerful way to represent functions as infinite sums of powers [3]. In pure mathematics, they provide exact representations of elementary functions (such as exponential,

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trigonometric, and logarithmic functions) and facilitate the analysis of more complex non-elementary functions. In practical computing and science, truncated power series deliver fast, accurate approximations for everything from solving differential equations to numerical integration. Across physics, chemistry, and engineering, they simplify nonlinear problems by expanding around key points (like equilibrium), often keeping just the first few terms for linear approximations crucial in stability analysis, perturbations, and control systems from classical mechanics to quantum theory [11]. In fact, by converting complex behaviors into manageable polynomials, power series make tough problems solvable while offering clear error estimates through remainder analysis, bridging pure theory with real-world computation.

The residual power series (RPS) method was originally introduced as an efficient analytical–numerical tool for determining the coefficients of power series in order to solve fuzzy differential equations [12]. Following its initial development, the method has been successfully extended to a wide range of problems, including systems of initial-value problems (IVPs), fractional-order epidemic models, and nonlinear partial differential equations [10]. Unlike traditional approaches such as the Taylor series which often require extensive symbolic differentiation, or the Adomian decomposition method which depends on computing Adomian polynomials and inversion steps, the RPS approach is computationally efficient [9]. It employs the residual condition $R_n(0) = 0$ to iteratively determine series coefficients, ensuring rapid convergence without the need for linearization, perturbation, or restrictive assumptions. By combining analytical rigor with numerical practicality, the RPS method constitutes a general and computationally efficient framework for solving differential equations in applied mathematics, physics, and engineering [1].

This paper introduces and rigorously evaluates the Residual Power Series (RPS) method as precise analytical tool for solving the fractional Bagley–Torvik equation, particularly in cases involving polynomial or smooth forcing terms. We first develop a tailored fractional power-series formulation under the Caputo derivative framework, establishing clear recurrence relations for practical implementation. Through benchmark test problems, we demonstrate that RPS recovers known exact solutions.

2. FRACTIONAL CALCULUS PRELIMINARIES

In this section we recall some basic notions and properties related to the Caputo fractional derivative that will be used throughout the paper.

Definition 2.1. Let $0 < \alpha < 1$ and let y be an absolutely continuous function on $[0, T]$. The *Caputo fractional derivative* of order α of y is defined by

$${}^C D_t^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} y'(\tau) d\tau = (I^{1-\alpha} y')(t), \quad t > 0. \quad (2.1)$$

where the Gamma function $\Gamma(\cdot)$ is defined for $\Re(z) > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad (2.2)$$

Definition 2.2. The Caputo time fractional derivative of order α of $u(x, t)$ is defined as [6, 8]

$$D_t^\alpha u(x, t) = \begin{cases} \frac{1}{\Gamma(m - \alpha)} \int_0^t (t - \xi)^{m - \alpha - 1} \frac{\partial^m u(x, \xi)}{\partial t^m} d\xi, & m - 1 < \alpha < m, \\ \frac{\partial^m u(x, t)}{\partial t^m}, & \alpha = m \in \mathbb{N}. \end{cases}$$

Lemma 2.3. Let $m > -1$. The Caputo derivative of the function t^m is given by

$${}^C D_t^\alpha (t^m) = \frac{\Gamma(m + 1)}{\Gamma(m + 1 - \alpha)} t^{m - \alpha}, \quad t > 0. \quad (2.3)$$

In particular we obtain ${}^C D_t^\alpha(1) = 0$, ${}^C D_t^\alpha(t) = \frac{t^{1 - \alpha}}{\Gamma(2 - \alpha)}$.

The main properties of the Caputo derivative are as follows

- Linearity:

$${}^C D_t^\alpha (a y(t) + b z(t)) = a {}^C D_t^\alpha y(t) + b {}^C D_t^\alpha z(t), \quad a, b \in \mathbb{R}$$

- Consistency with the first derivative:

$$\lim_{\alpha \rightarrow 1} {}^C D_t^\alpha y(t) = y'(t)$$

- Homogeneity of constants: For any constant $c \in \mathbb{R}$,

$${}^C D_t^\alpha c = 0$$

3. PROBLEM SETTING

In this section, we present the physical background of Bagley–Torvik equation which models the motion $y(t)$ of an immersed plate as:

$$\begin{cases} Ay''(t) + B {}^C D_t^{3/2} y(t) + Cy(t) = f(t), & t > 0 \\ y(0) = y_0, & y'(0) = y_1 \end{cases} \quad (3.1)$$

where ${}^C D_t^{3/2}$ denotes the Caputo fractional derivative of order $3/2$.

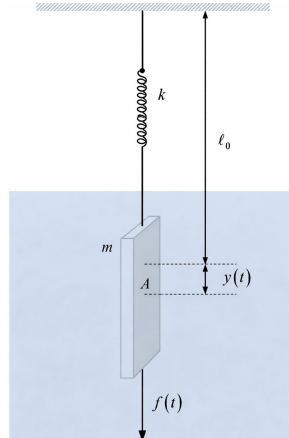


FIGURE 1. Rigid plate immersed in a Newtonian fluid, connected by a massless spring

Physical parameters satisfy $A = m$ (plate mass), $C = k$ (spring stiffness), and $B = A\sqrt{\mu\rho}/\sqrt{\pi}$ (viscoelastic damping), where μ is fluid viscosity, ρ its density, and $\sqrt{\pi}$ normalizes the fractional 3/2-order damping from unsteady Stokes flow boundary layers around an oscillating plate [16]. This particular fractional order emerges naturally from the fluid mechanics of the problem, representing an intermediate damping mechanism between purely viscous and purely elastic responses.

Our implementation strategy proceeds as follows: First, we express the solution as a fractional power series expansion involving half-integer powers $t^{k/2}$, reflecting the 3/2-order derivative structure. Second, we substitute this expansion into the governing equation to define a residual function. Third, we minimize this residual by sequentially enforcing conditions on its fractional derivatives at the initial point $t = 0$. This process generates a recursive system for determining series coefficients without solving large linear systems or performing numerical integration.

4. RESIDUAL POWER SERIES METHOD

In this section, we will introduce some important definitions and theorems related with the fractional and residual power series in the sense of the Caputo fractional derivative [7].

Definition 4.1. A power series expansion of the form

$$\sum_{m=0}^{\infty} c_m(t-t_0)^{m\alpha} = c_0 + c_1(t-t_0)^\alpha + c_2(t-t_0)^{2\alpha} + \cdots, \quad 0 \leq m-1 < \alpha \leq m, \quad t \geq t_0,$$

is called a fractional power series about $t = t_0$, where t is a variable and c_n represents the coefficients of the series [3].

Theorem 4.2. Consider the fractional power series $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ for $t \geq 0$. The following statements hold:

- (1) If the numerical series $\sum_{n=0}^{\infty} c_n b^{n\alpha}$ converges for some $b > 0$, then $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ converges for every t satisfying $0 \leq t < b$.
- (2) If the numerical series $\sum_{n=0}^{\infty} c_n d^{n\alpha}$ diverges for some $d > 0$, then $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ diverges for all $t > d$.

Proof. Assume first that $\sum_{n=0}^{\infty} c_n b^{n\alpha}$ is convergent. Then $\lim_{n \rightarrow \infty} c_n b^{n\alpha} = 0$. By the definition of the limit of a sequence (taking $\varepsilon = 1$), there exists $N \in \mathbb{N}$ such that $|c_n b^{n\alpha}| < 1$ for all $n \geq N$. Hence, for $n \geq N$ we have

$$|c_n t^{n\alpha}| = |c_n b^{n\alpha}| \left| \frac{t}{b} \right|^{n\alpha} < \left| \frac{t}{b} \right|^{n\alpha}$$

If $0 \leq t < b$, then $\left| \frac{t}{b} \right|^\alpha < 1$, and the geometric series $\sum_{n=N}^{\infty} \left| \frac{t}{b} \right|^{n\alpha}$ is comparison convergent.

By comparison, the series $\sum_{n=N}^{\infty} |c_n t^{n\alpha}|$ converges, hence $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ is absolutely convergent.

For the second assertion, assume that $t > d > 0$ and suppose, for contradiction, that $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ is convergent. By the first part, this would imply the convergence of $\sum_{n=0}^{\infty} c_n d^{n\alpha}$, which contradicts the hypothesis. Thus the latter series must diverge. $\square$

Theorem 4.3. [15] *For the fractional power series $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ with $t \geq 0$, exactly one of the following mutually exclusive situations occurs:*

- (1) *the series converges only at $t = 0$;*
- (2) *the series converges for every $t \geq 0$;*
- (3) *there exists a constant $R > 0$ such that the series converges whenever $0 \leq t < R$ and diverges for all $t > R$.*

Definition 4.4. Suppose that $u(x, t)$ has a multiple fractional power series representation at $t = t_0$ of the form [1]

$$u(x, t) = \sum_{m=0}^{\infty} f_m(x)(t - t_0)^{m\alpha}, \quad x \in I, \quad t_0 \leq t \leq t_0 + R.$$

If $D_t^{m\alpha} u(x, t)$, $m = 0, 1, 2, \dots$ are continuous on $I \times (t_0, t_0 + R)$, then

$$f_m(x) = \frac{D_t^{m\alpha} u(x, t_0)}{\Gamma(m\alpha + 1)}.$$

Theorem 4.5. *Suppose that y has a fractional power series representation at $t = t_0$ of the form*

$$y(t) = \sum_{i=0}^{\infty} a_i (t - t_0)^{i\alpha}$$

If $y(t) \in C[t_0, t_0 + R)$, and $D^{i\alpha} y(t) \in C(t_0, t_0 + R)$, for $i = 0, 1, 2, \dots$, then coefficients a_i will be in the form $a_i = \frac{\mathcal{D}_0^{i\alpha} y(t_0)}{\Gamma(n\alpha + 1)}$, where R is radius of convergence, and

$$\mathcal{D}^{i\alpha} = (\mathcal{D}^\alpha)^i = \underbrace{\mathcal{D}^\alpha \circ \mathcal{D}^\alpha \circ \dots \circ \mathcal{D}^\alpha}_{i \text{ times}}$$

In this study, the Residual Power Series (RPS) method begins by assuming the solution can be represented as a fractional power series (FPS) expansion:

$$y(t) = \sum_{k=0}^{\infty} a_k t^{k/2}, \quad (4.1)$$

which have the Caputo derivative

$${}^C D_t^{3/2} (t^{k/2}) = \frac{\Gamma(\frac{k}{2} + 1)}{\Gamma(\frac{k}{2} - \frac{1}{2})} t^{\frac{k}{2} - \frac{3}{2}} \quad \text{which is finite at } t = 0 \text{ provided } \frac{k}{2} \geq \frac{3}{2}$$

the first and second derivatives are respectively

$$y'_n(t) = \sum_{k=0}^n a_k \frac{k}{2} t^{k/2-1}, \quad y''_n(t) = \sum_{k=0}^n a_k \frac{k(k-2)}{4} t^{k/2-2}$$

Initial conditions lead to $y(0) = a_0 = y_0$, $y'(0) = a_2 = y_1$ and finite $y''(0)$ impose $a_1 = a_3 = 0$, yielding the constrained expansion:

$$y(t) = a_0 + a_2 t + \sum_{k=4}^{\infty} a_k t^{k/2} \quad (4.2)$$

In fact, this approach expands solutions of (3.1) through the truncated series

$$y_n(t) = a_0 + a_2 t + \sum_{k=4}^n a_k t^{k/2} \quad (4.3)$$

the n^{th} partial sum $y_n(t)$ produces the residual function

$$R_n(t) = Ay''_n(t) + B {}^C D_t^{3/2} y_n(t) + Cy_n(t) - f(t) \quad (4.4)$$

The residual power series (RPS) condition requires

$$D_t^{k/2} R_n(0) = 0, \quad k = 0, 1, \dots, n-4, \quad (4.5)$$

which generates a recursive system for $\{a_k\}$. Thus, while (4.1) is the representation, equations (4.4)–(4.5) constitute the RPS method proper.

Remark 4.6. Some literature absorbs the Gamma function factors $\Gamma(k/2+1)$ into the coefficients via $a_k = b_k/\Gamma(k/2+1)$, where b_k are coefficients of the normalized FPS $y(t) = \sum b_k t^{k/2}/\Gamma(k/2+1)$. Both forms are mathematically equivalent, but the unnormalized version (4.1) yields simpler recurrence relations for the Bagley–Torvik equation.

For $k \geq 4$, we have

$$y''_n(t) = \sum_{k=4}^n a_k \frac{k(k-2)}{4} t^{k/2-2}$$

$${}^C D_t^{3/2} (t^{k/2}) = \frac{(\frac{k}{2})!}{(\frac{k}{2} - \frac{3}{2})!} t^{\frac{k}{2} - \frac{3}{2}}$$

Substituting these expressions into the fundamental equation of (3.1) and collecting the coefficients of $t^{n/2-2}$ from all terms (y'' , ${}^C D^{3/2} y$, y), we obtain

$$A \cdot a_{n+4} \frac{(n+4)(n+2)}{4} + B \cdot a_{n+3} \frac{\Gamma(\frac{n+5}{2})}{\Gamma(\frac{n+2}{2})} + C \cdot a_n = f_n$$

using the identity $\Gamma(z+1) = z\Gamma(z)$, the ratio simplifies to

$$\frac{\Gamma(\frac{n+5}{2})}{\Gamma(\frac{n+2}{2})} = \frac{n+3}{2} \cdot \frac{n+1}{2}$$

and solving for a_{n+4} yields the recurrence relation

$$a_{n+4} = -\frac{4}{A(n+4)(n+2)} \left[Ba_{n+3} \frac{n+3}{2} + Ca_n - f_n \right]$$

where f_n denotes the coefficient of $t^{n/2}$ in the fractional power series representation of the forcing term $f(t)$

$$f(t) = \sum_{k=0}^{\infty} f_k t^{k/2}$$

Consequently, one can say that this technique provides a forward recursive construction of the coefficients: once the initial values a_0, a_1, a_2, a_3 are determined from the initial or compatibility conditions, the above recurrence uniquely generates a_{n+4} from previously computed terms for every $n > 4$. Hence, the series solution can be computed iteratively to any prescribed order.

5. NUMERICAL IMPLEMENTATION

To analyze the accuracy of the truncated residual power series approximation, we examine how the truncation error depends on the expansion order. On the other hand, next result shows how obtaining an upper bound for the approximation error in terms of fractional Taylor remainder under Caputo derivative framework.

Theorem 5.1 (RPS Error Bound). *Let $y_n(t) = \sum_{k=0}^n a_k t^{k/2}$ be the n -term RPS approximation to the solution $y(t)$ of the Bagley-Torvik equation (3.1). Then there exists a nonnegative constant M_n such that for all $t \in [0, T]$:*

$$|y(t) - y_n(t)| \leq \frac{M_n}{\Gamma((n+1)/2 + 1)} T^{(n+1)/2} \quad (5.1)$$

Moreover, the approximation converges uniformly to $y(t)$ on $[0, T]$ as n tends to ∞ .

Proof. The proof utilizes the generalized Taylor formula for Caputo fractional derivatives [2]. For any $\alpha > 0$ and $n \in \mathbb{N}$, if $y(t)$ possesses fractional derivatives up to order $(n+1)\alpha$, then

$$y(t) = \underbrace{\sum_{k=0}^n \frac{D^{k\alpha} y(0)}{\Gamma(k\alpha + 1)} t^{k\alpha}}_{y_n(t) \text{ (RPS approximation)}} + \underbrace{\frac{1}{\Gamma((n+1)\alpha)} \int_0^t (t-\tau)^{(n+1)\alpha-1} D^{(n+1)\alpha} y(\tau) d\tau}_{R_n(t) \text{ (remainder/error term)}} \quad (5.2)$$

setting $\alpha = 1/2$, the basic fractional order of our transformed system, yields:

$$y(t) = \sum_{k=0}^n \frac{D^{k/2} y(0)}{\Gamma(k/2 + 1)} t^{k/2} + \frac{1}{\Gamma((n+1)/2)} \int_0^t (t-\tau)^{(n-1)/2} D^{(n+1)/2} y(\tau) d\tau.$$

the RPS method constructs $y_n(t)$ such that $D^{k/2} y_n(0) = D^{k/2} y(0)$ for $k = 0, 1, \dots, n$. Therefore, the approximation error equals the Taylor remainder

$$y(t) - y_n(t) = R_n(t) = \frac{1}{\Gamma((n+1)/2)} \int_0^t (t-\tau)^{(n-1)/2} D^{(n+1)/2} y(\tau) d\tau$$

since $y(\cdot)$ is considered sufficiently smooth in the fractional sense, take absolute values and assume $|D^{(n+1)/2}y(\tau)| \leq M_n$ for $\tau \in [0, t]$

$$\begin{aligned}
|y(t) - y_n(t)| &\leq \frac{M_n}{\Gamma((n+1)/2)} \int_0^t (t-\tau)^{(n-1)/2} d\tau \\
&\leq \frac{M_n}{\Gamma((n+1)/2)} \frac{2}{n+1} t^{(n+1)/2} \\
&\leq \frac{2M_n}{(n+1)\Gamma((n+1)/2)} t^{(n+1)/2} \\
&\leq \frac{M_n}{\Gamma((n+1)/2 + 1)} t^{(n+1)/2}, \text{ using } \Gamma(z+1) = z\Gamma(z) \text{ with } z = \frac{n+1}{2}
\end{aligned}$$

The constant $M_n = \max_{\tau \in [0, T]} |D^{(n+1)/2}y(\tau)|$ provides a uniform bound on $[0, T]$. $\square$

Remark 5.2.

- The constant M_n can be chosen as

$$M_n = \max_{\tau \in [0, T]} \left| D_t^{(n+1)/2} y(\tau) \right|$$

where D_t^α denotes the Caputo fractional derivative of order α .

- In numerical computations, the bound (5.1) can be used as a stopping criterion: choose n such that

$$\frac{M_n}{\Gamma((n+1)/2 + 1)} T^{(n+1)/2} < \varepsilon_{\text{tol}},$$

where ε_{tol} is the desired tolerance.

To conclude this section, the RPS algorithm for computing coefficients a_k via residual error minimization can be summarized as follows

Step 1: Truncated series: Define the n th approximate solution

$$y_n(t) = \sum_{k=0}^n a_k t^{k/2}$$

Step 2: Residual function: Substitute into the Bagley–Torvik equation

$$R_n(t) = Ay_n''(t) + BD_t^{3/2}y_n(t) + Cy_n(t) - \varphi(t)$$

Step 3: Minimization conditions: Impose

$$D_t^{k/2}R_n(0) = 0, \quad k = 0, 1, \dots, n-4$$

These conditions ensure the residual vanishes identically as n tends to ∞ .

Step 4: Recursive solution: Solve sequentially for $a_4, a_5, \dots, a_n$.

In order to demonstrate the validity and effectiveness of the proposed numerical approximation, we present four illustrative test problems involving various forms of non-Homogeneous Bagley–Torvik equation. For each of these problems, the exact solution is available and is obtained analytically [4].

Example 1. Consider the fractional Bagley–Torvik equation:

$$\begin{cases} y''(t) + {}^C D^{3/2}y(t) + y(t) = t^2 + 2 + 4\sqrt{\frac{t}{\pi}} \\ y(0) = 0, \quad y'(0) = 0 \end{cases} \quad (5.3)$$

Assume a fractional power series solution:

$$y(t) = \sum_{k=0}^{\infty} a_k t^{k/2}.$$

From $y(0) = 0$, we have $a_0 = 0$. For finite $y'(0)$, the series for

$$y'(t) = \sum_{k=0}^{\infty} \frac{k}{2} a_k t^{k/2-1}$$

must not contain t^{-1} terms, implying $a_1 = 0$. Evaluating at $t = 0$ yields $a_2 = 0$. Similarly, finite $y''(0)$ requires $a_3 = 0$. Thus:

$$y(t) = \sum_{k=4}^{\infty} a_k t^{k/2}.$$

for $n = 4$ Approximation, we have the 4-term approximation:

$$y_4(t) = a_4 t^2$$

then $y_4''(t) = 2a_4$, and

$$D^{3/2}y_4(t) = D^{3/2}[a_4 t^2] = a_4 \frac{\Gamma(3)}{\Gamma(3-3/2)} t^{1/2} = a_4 \frac{2}{\Gamma(3/2)} t^{1/2} = a_4 \frac{4}{\sqrt{\pi}} t^{1/2}$$

thus, the residual function is

$$\begin{aligned} R_4(t) &= y_4''(t) + D^{3/2}y_4(t) + y_4(t) - f(t) \\ &= 2a_4 + a_4 \frac{4}{\sqrt{\pi}} t^{1/2} + a_4 t^2 - \left(2 + \frac{4}{\sqrt{\pi}} t^{1/2} + t^2\right) \\ &= (2a_4 - 2) + \frac{4}{\sqrt{\pi}}(a_4 - 1)t^{1/2} + (a_4 - 1)t^2 \end{aligned}$$

Consequently, we determine the coefficient using the RPS condition $R_4(0) = 0$:

$$2a_4 - 2 = 0 \quad \Rightarrow \quad a_4 = 1.$$

Substituting $a_4 = 1$ into $R_4(t)$ yields $R_4(t) \equiv 0$ for all t , meaning the approximation exactly satisfies the differential equation. Thus, the 4-term RPS solution is:

$$y_4(t) = t^2,$$

which coincides with the exact solution of (5.3). No higher-order terms are needed ($a_k = 0$ for $k > 4$), demonstrating that RPS recovers polynomial solutions exactly in finite steps.

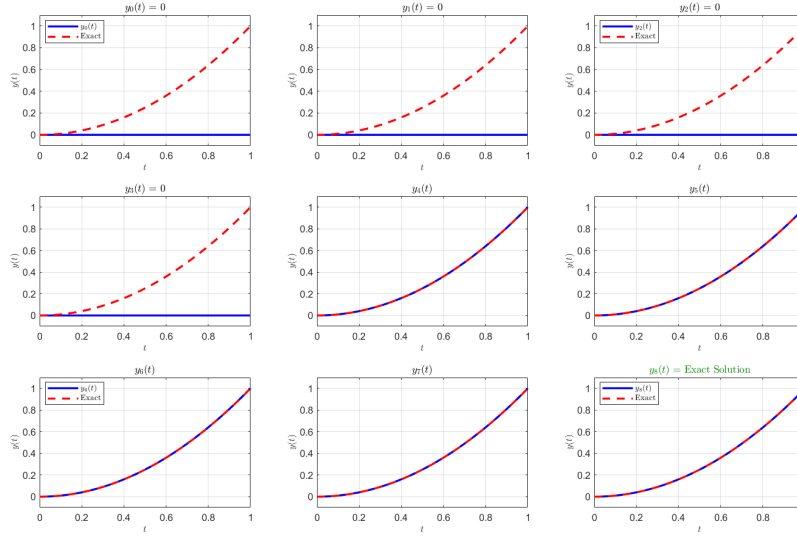


FIGURE 2. RPS approximations vs exact solution of the example 1

Example 2. Applying the same steps we can get the exact solution of the following fractional Bagley–Torvik examples

$$\begin{cases} y''(t) + {}^C D^{3/2} y(t) + y(t) = t^3 + 5t + \frac{8t^{3/2}}{\sqrt{\pi}} \\ y(0) = 0, \quad y'(0) = -1 \end{cases} \quad (5.4)$$

Assume a fractional power series solution $y(t) = \sum_{k=0}^{\infty} a_k t^{k/2}$.

From the initial condition $y(0) = 0$, we have $a_0 = 0$. For finite $y'(0)$, the series for

$$y'(t) = \sum_{k=0}^{\infty} \frac{k}{2} a_k t^{k/2-1}$$

must not contain t^{-1} terms, implying $a_1 = 0$. Evaluating at $t = 0$ yields $y'(0) = a_2 = -1$. Similarly, finite $y''(0)$ requires $a_3 = 0$. Thus the series begins as:

$$y(t) = -t + \sum_{k=4}^{\infty} a_k t^{k/2}$$

For the 4-term approximation ($n = 4$), we include terms up to $k = 4$

$$y_4(t) = a_2 t^1 + a_4 t^2 = -t + a_4 t^2.$$

Therefore

$$\begin{aligned} y_4''(t) &= 2a_4, \\ {}^C D^{3/2} y_4(t) &= {}^C D^{3/2} [-t] + {}^C D^{3/2} [a_4 t^2] \end{aligned}$$

For the linear term t , since $1 < 3/2$, we use the Caputo definition with $n = 2$

$${}^C D^{3/2} t = \frac{1}{\Gamma(1/2)} \int_0^t \frac{\frac{d^2}{d\tau^2}(\tau)}{(t-\tau)^{1/2}} d\tau = 0, \text{ since } \frac{d^2}{d\tau^2}(\tau) = 0$$

for the quadratic term

$${}^C D^{3/2} t^2 = \frac{\Gamma(3)}{\Gamma(3-3/2)} t^{1/2} = \frac{2}{\Gamma(3/2)} t^{1/2} = \frac{4}{\sqrt{\pi}} t^{1/2}, \text{ since } \Gamma(3/2) = \sqrt{\pi}/2$$

thus

$${}^C D^{3/2} y_4(t) = a_4 \frac{4}{\sqrt{\pi}} t^{1/2}$$

the forcing term is:

$$f(t) = t^3 + 5t + \frac{8}{\sqrt{\pi}} t^{3/2}$$

Then, the residual function for $n = 4$ is expressed as

$$\begin{aligned} R_4(t) &= y_4''(t) + {}^C D^{3/2} y_4(t) + y_4(t) - f(t) \\ &= 2a_4 + a_4 \frac{4}{\sqrt{\pi}} t^{1/2} + (-t + a_4 t^2) - \left(t^3 + 5t + \frac{8}{\sqrt{\pi}} t^{3/2} \right) \\ &= 2a_4 + \frac{4a_4}{\sqrt{\pi}} t^{1/2} - 6t + a_4 t^2 - t^3 - \frac{8}{\sqrt{\pi}} t^{3/2} \end{aligned}$$

imposing $R_4(0) = 0$ yields $2a_4 = 0$ which means that $a_4 = 0$.

Thus $y_4(t) = -t$ is the 4-term approximation. However, it does not yet match the exact solution, indicating higher terms are needed.

In the same systematic manner, we find $a_5 = 0$.

The 6-term approximation is expressed as $y_6(t) = -t + a_6 t^3$, such that

$$\begin{aligned} y_6''(t) &= 6a_6 t, \\ {}^C D^{3/2} y_6(t) &= {}^C D^{3/2} [-t + a_6 t^3] \\ &= 0 + a_6 \frac{\Gamma(4)}{\Gamma(4-3/2)} t^{3/2} \\ &= a_6 \frac{6}{\Gamma(5/2)} t^{3/2} = a_6 \frac{8}{\sqrt{\pi}} t^{3/2}, \text{ since } \Gamma(5/2) = 3\sqrt{\pi}/4 \end{aligned}$$

Now compute the residual

$$\begin{aligned} R_6(t) &= (6a_6 t) + \left(a_6 \frac{8}{\sqrt{\pi}} t^{3/2} \right) + (-t + a_6 t^3) - \left(t^3 + 5t + \frac{8}{\sqrt{\pi}} t^{3/2} \right) \\ &= (6a_6 - 6)t + \frac{8}{\sqrt{\pi}} (a_6 - 1) t^{3/2} + (a_6 - 1) t^3. \end{aligned}$$

Setting $a_6 = 1$ makes $R_6(t) = 0$ for all t (Noting that $R_6(0) = 0$ gives no new constraint since all terms vanish at $t = 0$). Thus, the 6-term RPS solution is exact

$$y_6(t) = -t + t^3 = t^3 - t$$

The RPS method recovers the exact polynomial solution $y(t) = t^3 - t$ with only 6 terms. No higher-order terms are required.

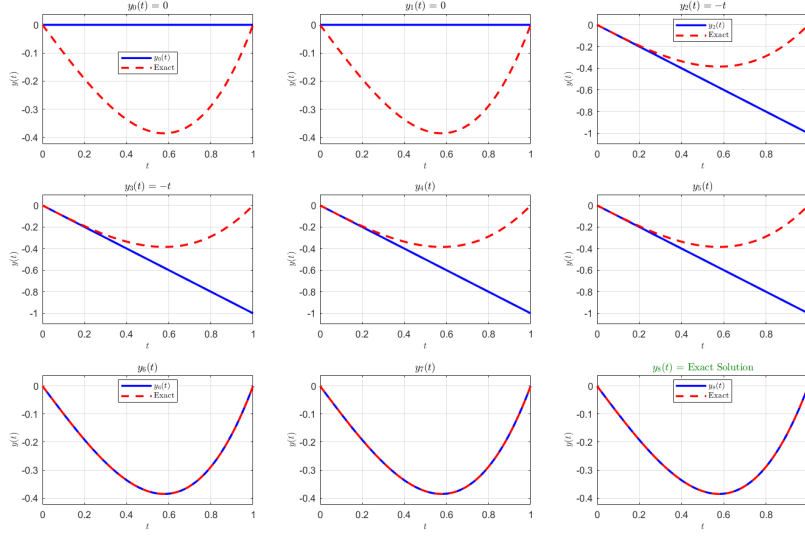


FIGURE 3. RPS approximations vs exact solution of the example 2

Example 3. The RPS method similarly gives the exact solution for

$$\begin{cases} {}^C D^{3/2} y(t) + y(t) = t^4 - 8t + \frac{64t^{5/2}}{5\sqrt{\pi}} \\ y(0) = 0, \quad y'(0) = -8 \\ \text{Exact solution: } y(t) = t^4 - 8t \end{cases} \quad (5.5)$$

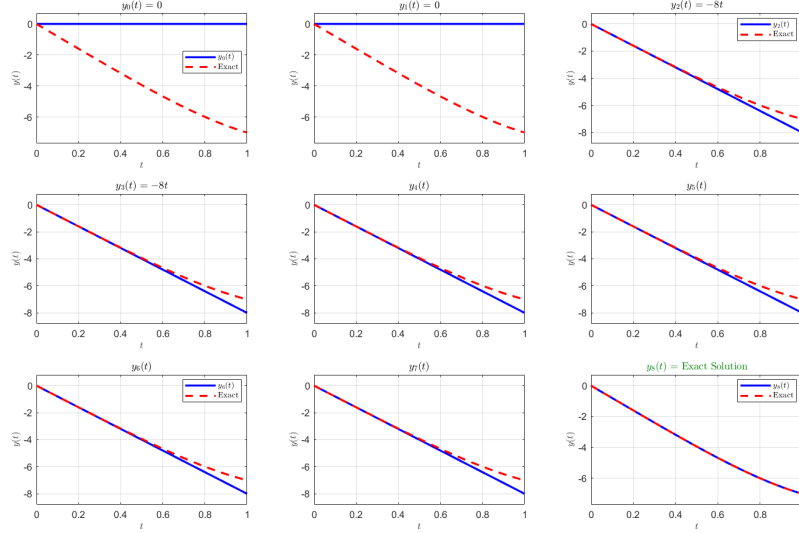


FIGURE 4. RPS approximations vs exact solution of the example 3

The previous examples demonstrated the RPS method for problems with *polynomial* exact solutions (these examples are analytically solved using Laplace transform method in [4]). We now present a non-polynomial case to further validate the method's convergence beyond trivial polynomial recovery.

Example 4. Consider the fractional Bagley-Torvik type equation

$$y''(x) + D^{3/2}y(x) - y(x) = h(x), \quad x > 0 \quad (5.6)$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 1,$$

where the forcing term is

$$h(x) = \sqrt{x} E_{1, \frac{3}{2}}(x),$$

and $E_{\alpha, \beta}(z)$ denotes the two-parameter Mittag-Leffler function

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

the exact solution of this problem is $y(x) = e^x$

We seek a fractional power-series solution

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k/2}$$

The initial conditions give

$$a_0 = y(0) = 1, \quad a_2 = y'(0) = 1,$$

while finiteness of $y''(0)$ requires

$$a_1 = a_3 = 0.$$

The forcing term can be expressed as the following expansion

$$h(x) = x^{1/2} \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(m + 3/2)} = \sum_{m=0}^{\infty} \frac{x^{\frac{2m+1}{2}}}{\Gamma(m + 3/2)}$$

Writing $h(x) = \sum_{k=0}^{\infty} b_k x^{k/2}$, we obtain

$$b_{2m+1} = \frac{1}{\Gamma(m + 3/2)}, \quad m = 0, 1, 2, \dots, \quad b_k = 0 \text{ for even } k.$$

The first few coefficients are

$$b_1 = \frac{1}{\Gamma(3/2)} = \frac{2}{\sqrt{\pi}}, \quad b_3 = \frac{1}{\Gamma(5/2)} = \frac{4}{3\sqrt{\pi}}, \quad b_5 = \frac{1}{\Gamma(7/2)} = \frac{8}{15\sqrt{\pi}}$$

Substituting the series into the Bagley-Torvik equation (5.6) and using the expressions for $y''(x)$ and $D^{3/2}y(x)$ in the half-power basis, we equate coefficients of $x^{k/2}$ to obtain the residual equation

$$a_{k+4} \frac{k+4}{2} \left(\frac{k+4}{2} - 1 \right) + a_{k+3} \frac{\Gamma(\frac{k+3}{2} + 1)}{\Gamma(\frac{k}{2} + 1)} - a_k = b_k, \quad k \geq 0$$

consequently, we get the recurrence relation for a_{k+4}

$$a_{k+4} = \frac{b_k - a_{k+3} \frac{\Gamma(\frac{k+3}{2} + 1)}{\Gamma(\frac{k}{2} + 1)} + a_k}{\frac{k+4}{2} \left(\frac{k+4}{2} - 1 \right)}, \quad k \geq 0$$

which yields sequentially the coefficients

$$\begin{aligned} a_4 &= \frac{0 - 0 + 1}{2} = \frac{1}{2}, & a_5 &= \frac{\frac{2}{\sqrt{\pi}} - \frac{1}{2} \cdot \frac{4}{\sqrt{\pi}} + 0}{\frac{15}{4}} = 0, \\ a_6 &= \frac{0 - 0 + 1}{6} = \frac{1}{6}, & a_7 &= \frac{\frac{4}{3\sqrt{\pi}} - \frac{1}{6} \cdot \frac{8}{\sqrt{\pi}} + 0}{\frac{35}{4}} = 0, \\ a_8 &= \frac{0 - 0 + \frac{1}{2}}{12} = \frac{1}{24} \end{aligned}$$

Truncating the series at $n = 8$ yields the approximate solution

$$y_8(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

which is precisely the fourth-degree Taylor polynomial of the exact solution e^x . Higher truncations recover successively more terms of the exponential series.

Furthermore, theorem (5.1) guarantees for every $x \in [0, T]$

$$|e^x - y_n(x)| \leq \frac{M_n}{\Gamma((n+1)/2 + 1)} T^{(n+1)/2}, \quad \text{where } M_n = \max_{[0, T]} |D^{(n+1)/2} e^x| \leq e^T$$

for $T = 1$, we get

$$|e^x - y_n(x)| \leq \frac{e}{\Gamma((n+3)/2)}$$

which decays super-exponentially as n tends to infinity.

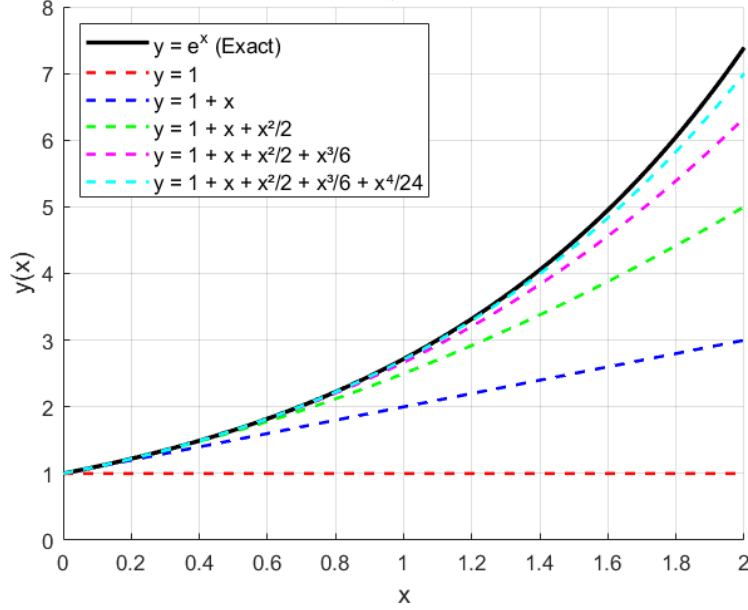


FIGURE 5. RPS approximations vs exact solution of the example 4

This example confirms that the RPS method converges to a non-polynomial (exponential) solution, validating the theoretical error bound and demonstrating the algorithm's applicability as showing in figure (5).

6. CONCLUSION

The Residual Power Series (RPS) method has been rigorously applied to solve the fractional Bagley-Torvik equation, demonstrating its effectiveness as an accurate and computationally efficient analytical tool. Through carefully constructed case studies that include both polynomial and non-polynomial exact solutions, we have shown that the method achieves rapid convergence with remarkably few approximation terms. The half-power series representation

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k/2}$$

naturally accommodates the fractional derivative $D^{3/2}$, while the residual minimization conditions $D^{k/2}R_n(0) = 0$ provide a systematic recursive framework for coefficient determination. This framework requires only elementary operations; no numerical integration or special-function evaluations beyond initial expansions are needed, even when forcing terms involve Mittag-Leffler functions. The accompanying theoretical convergence analysis, validated through MATLAB implementations, shows super-exponential error decay at rates of $O(1/\Gamma((n+3)/2))$. This confirms that the RPS method combines low computational cost with systematic error reduction, establishing it as a particularly suitable and versatile approach for initial value problems in fractional calculus. The method offers spectral accuracy

and analytical transparency for a broad class of fractional differential equations arising in viscoelastic materials, anomalous diffusion, and related applications.

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