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A SIMPLER PROOF OF THE EQUIVALENCE BETWEEN THE BACKWARDS ITÔ INTEGRAL AND THE MCSHANE BACKWARDS ITÔ INTEGRAL

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ABSTRACT. The equivalence between the backwards Itô integral and the McShane backwards Itô integral was previously obtained via the stochastic M-integral. This paper provides a simpler proof of this equivalence that avoids the M-integral entirely and is developed within the deterministic framework of the backwards McShane integral. By leveraging Henstock's Lemma and absolute continuity properties, the proposed approach simplifies the theoretical structure, improves conceptual transparency, and offers a foundation that is readily adaptable to other forms of stochastic integration.

Keywords. Lebesgue integral, backwards McShane integral, McShane backwards Itô integral, backwards Itô integral
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1. Introduction

Stochastic integration plays a central role in modern probability theory and its applications, with the Itô integral at its core for modeling semimartingales in finance, physics, and engineering ([6], [16], [17]). However, classical constructions rely heavily on measure-theoretic machinery, leading to alternate approaches based on gauge (Henstock–Kurzweil) integrals that recast integration in terms of Riemann sums and fine divisions ([8], [12]). This framework is now called the Henstock-Kurzweil approach or simply the Henstock approach.

Henstock's approach is well known for redefining integrals, whether deterministic or stochastic integrals. See, for example, [9], [10], [13], [19], [20], and [21]. In earlier work, Arcede and Cabral [1] introduced a stochastic integral with respect to a standard Brownian motion known as the backwards Itô integral, formulated using partial divisions tagged at right endpoints—hence the term "backwards". They established its equivalence to a McShane-style (full division) formulation, called the McShane backwards Itô integral, by employing the M-integral defined by [4], a stochastic integral that serves as an intermediate bridge. Their proof

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proceeds by first connecting the backwards Itô integral to the M-integral and then linking the M-integral to the McShane backwards Itô integral. Their result is established through a comprehensive and technically sophisticated proof.

In this paper, we present a simpler proof of the equivalence between the two integrals discussed by Arcede and Cabral in [1], bypassing the stochastic M-integral altogether. Our approach relies solely on the deterministic backwards McShane integral. By exploiting Henstock's Lemma and the absolute continuity property of this integral, we provide a more straightforward argument that avoids additional stochastic constructions and substantially simplifies the proof in [1].

2. Preliminaries

In this section, we provide several concepts required for the main results, primarily from [1] and [3]. Within this paper, $(\Omega, \mathcal{G}, \mathbb{P})$ refers to a probability space. A collection of σ-subalgebras $\{\mathcal{G}_t : 0 \leq t \leq T\}$ of $\mathcal{G}$ is called a backwards filtration if for every $s < t$, it satisfies $\mathcal{G}_s \supseteq \mathcal{G}_t$. In addition, $\{\mathcal{G}_t : 0 \leq t \leq T\}$ is called a standard backwards filtration if it fulfills the criteria specified below: (1) Every $\mathbb{P}$-null set in $\mathcal{G}$ belongs to $\mathcal{G}_T$; and (2) for every $t \in [0,T]$, $\mathcal{G}_t = \bigcap_{s<t} \mathcal{G}_s$. For convenience, we denote $\{\mathcal{G}_t : 0 \leq t \leq T\}$ simply as $\{\mathcal{G}_t\}$. Let $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}, \mathbb{P})$ be a standard backwards filtered space. A function $f : \Omega \times [0,T] \to \mathbb{R}$ is called a process if $f(., t)$ is a random variable for all $t \in [0,T]$. We say that a process f is adapted to a standard backwards filtration $\{\mathcal{G}_t\}$ if f_t is $\mathcal{G}_t$-measurable for all $t \in [0,T]$. For $p \geq 1$, a random variable $\tau : \Omega \to \mathbb{R}$ belongs to $L^p(\Omega, \mathbb{R})$ if $\int_\Omega |\tau|^p \, d\mathbb{P} < \infty$. For $\tau \in L^1(\Omega, \mathbb{R})$, the expectation of random variable τ is denoted by $\mathbb{E}(\tau)$. Let $\mathcal{H} \subseteq \mathcal{G}$ be a σ-subalgebra. A conditional expectation of a random variable τ given σ-subalgebra $\mathcal{H}$ is a $\mathcal{H}$-measurable random variable, denoted by $\mathbb{E}(\tau \mid \mathcal{H})$.

A finite collection of non-overlapping subintervals $P = \{[u_i, v_i]\}_{i=1}^n$ of $[0,T]$ is called a partial division of $[0,T]$. In addition, P is called a full division of $[0,T]$ if $\cup_{i=1}^n [u_i, v_i] = [0,T]$.

Let δ be a positive function defined on $[0,T]$. A finite collection of interval–point pairs $D = \{([u_i, \xi_i], \xi_i)\}_{i=1}^n$ is called a backwards partial division of $[0,T]$ if $\{[u_i, \xi_i]\}_{i=1}^n$ is a finite collection of non-overlapping subintervals of $[0,T]$. An interval–point pair $([u, \xi], \xi)$ is said to be backwards δ-fine if $[u, \xi] \subseteq (\xi - \delta(\xi), \xi]$, with $[u, \xi] \subseteq [0,T]$. We call $D = \{([u_i, \xi_i], \xi_i)\}_{i=1}^n$ a backwards δ-fine partial division of $[0,T]$ if D is a backwards partial division of $[0,T]$ and for each i, an interval-point pair $([u_i, \xi_i], \xi_i)$ is backwards δ-fine. Let $\eta > 0$ be given. We call $D = \{([u_i, \xi_i], \xi_i)\}_{i=1}^n$ a backwards (δ, η)-fine partial division of $[0,T]$ if it fails to cover $[0,T]$ by at most length η.

Let δ be a positive function defined on $[0,T]$, $[u, v] \subseteq [0,T]$, and $\xi \in [0,T]$. An interval-point pair $([u, v], \xi)$ is called McShane δ-fine if $[u, v] \subseteq (\xi - \delta(\xi), \xi + \delta(\xi))$. A finite collection of interval–point pairs $D' = \{([w_j, z_j], \xi_j)\}_{j=1}^m$ is called a McShane δ-fine partial division of $[0,T]$ if $\{[w_j, z_j]\}_{j=1}^m$ is a partial division of $[0,T]$ and for each j, an interval-point pair $([w_j, z_j], \xi_j)$ is McShane δ-fine. In addition, D' is called a McShane δ-fine division of $[0,T]$ if $\cup_{j=1}^m [w_j, z_j] = [0,T]$.

Let δ be a positive function on $[0,T]$ and $\eta > 0$ be given. We call a full division D'' a (δ,η)-fine division of $[0,T]$ if $D'' = D_1 \cup D_2$, where D_1 is a backwards (δ,η)-fine partial division of $[0,T]$ and $D_2 = \{([x_k,y_k],\xi_k)\}_{k=1}^r$ is a McShane δ-fine partial division of $[0,T]$ such that $(D_2)\sum_{k=1}^r (y_k - x_k) \leq \eta$.

Let δ be a positive function on $[0,T]$ and $\eta > 0$ be given. A division Q is called a (δ,η)-fine partial division of $[0,T]$ if Q is a subset of some (δ,η)-fine division Q' of $[0,T]$.

In the subsequent section, we will state the definitions of the backwards Itô integral and the McShane backwards Itô integral as defined in [1]. Some results are also presented without proof.

Definition 2.1. [1]Let $(\Omega,\mathcal{G},\{\mathcal{G}_t\},\mathbb{P})$ be the standard backwards filtered probability space and let $f : \Omega \times [0,T] \to \mathbb{R}$ be a process adapted to the standard backwards filtration $\{\mathcal{G}_t\}$ and $f_t \in L^2(\Omega,\mathbb{R})$ for all $t \in [0,T]$. The process f is said to be backwards Itô integrable on $[0,T]$ if there exists $F \in L^2(\Omega,\mathbb{R})$ such that for every $\varepsilon > 0$, there exist a positive function δ on $[0,T]$ and a positive number η such that for any backwards (δ,η)-fine partial division $P = \{([x_j,\rho_j],\rho_j)\}_{j=1}^r$ of $[0,T]$, we have

$$\mathbb{E}\left(\left|(P)\sum_{j=1}^r f_{\rho_j}(B_{\rho_j} - B_{x_j}) - F\right|^2\right) < \varepsilon. \tag{2.1}$$

Denote by $\Phi_{\mathrm{BI}}(\mathbb{R})$ the collection of all backwards Itô integrable processes on $[0,T]$. According to [1], the backwards Itô integral satisfies the Itô isometry.

Definition 2.2. [1] Let $(\Omega,\mathcal{G},\{\mathcal{G}_t\},\mathbb{P})$ be the standard backwards filtered probability space and $f : \Omega\times[0,T] \to \mathbb{R}$ be a process adapted to the standard backwards filtration $\{\mathcal{G}_t\}$ and $f_t \in L^2(\Omega,\mathbb{R})$ for all $t \in [0,T]$. The process f is said to be McShane backwards Itô integrable on $[0,T]$ if there exists $F \in L^2(\Omega,\mathbb{R})$ such that for every $\varepsilon > 0$, there exist a positive function δ on $[0,T]$ and a positive number η such that for any (δ,η)-fine division $P = \{([w_k,z_k],\xi_k)\}_{k=1}^m$ of $[0,T]$, we have

$$\mathbb{E}\left(\left|(P)\sum_{k=1}^m \mathbb{E}(f_{\xi_k} \mid \mathcal{G}_{z_k})(B_{z_k} - B_{w_k}) - F\right|^2\right) < \varepsilon. \tag{2.2}$$

Denote by $\Xi_{\mathrm{MB}}(\mathbb{R})$ the collection of all McShane backwards Itô integrable processes on $[0,T]$. The following result is also crucial. We state it below.

Lemma 2.3. [1] *Let $f \in \Xi_{\mathrm{MB}}(\mathbb{R})$ and let $D = \{([x_i,y_i],\rho_i)\}_{i=1}^n$ be a (δ,η)-fine division of $[0,T]$. We have*

$$\mathbb{E}\left(\left|(D)\sum_{i=1}^n \mathbb{E}(f_{\rho_i} \mid \mathcal{G}_{y_i})(B_{y_i} - B_{x_i})\right|^2\right) = (D)\sum_{i=1}^n \mathbb{E}\left(\mathbb{E}(f_{\rho_i} \mid \mathcal{G}_{y_i})^2\right)(y_i - x_i).$$

Now, we also recall the definition of the M-integral in the following.

Definition 2.4. [4] Let $f : \Omega \times [0,T] \to \mathbb{R}$ be a process such that $f_t \in L^1(\Omega, \mathbb{R})$ for all $t \in [0,T]$. The process f is said to be M-integrable on $[0,T]$ if for every $\varepsilon > 0$, there exists a positive function δ on $[0,T]$ such that for any two McShane δ-fine divisions $D' = \{([a_i, b_i], \xi_i)\}_{i=1}^n$ and $D'' = \{([a_i, b_i], \rho_i)\}_{i=1}^n$ of $[0,T]$, we have

$$\sum_{i=1}^{n} \mathbb{E}\left(|(f_{\xi_i} - f_{\rho_i})(b_i - a_i)|\right) < \varepsilon. \tag{2.3}$$

Denote by $\Upsilon_{\mathrm{M}}(\mathbb{R})$ the collection of all M-integrable processes on $[0,T]$.

Let $f : \Omega \times [0,T] \to \mathbb{R}$ be a process with $f_t \in L^2(\Omega, \mathbb{R})$ for all $t \in [0,T]$. According to [1], if $f \in \Phi_{\mathrm{BI}}(\mathbb{R})$ then by Itô isometry, $\mathbb{E}(f_t^2)$ is Lebesgue integrable on $[0,T]$. By using the dominated convergence theorem, it follows that $f^2 \in \Upsilon_{\mathrm{M}}(\mathbb{R})$, see [7]. Moreover, since $f^2 \in \Upsilon_{\mathrm{M}}(\mathbb{R})$ then by using Cauchy criterion combined with a suitable refinement construction shows that $f \in \Xi_{\mathrm{MB}}(\mathbb{R})$. This part is not straightforward and requires a fairly detailed argument. Conversely, if $f \in \Xi_{\mathrm{MB}}(\mathbb{R})$, then, by applying Henstock's Lemma of McShane backwards Itô integral, we have $f \in \Phi_{\mathrm{BI}}(\mathbb{R})$.

A schematic outline of the proof in [1] is given in the Figure 1 below.

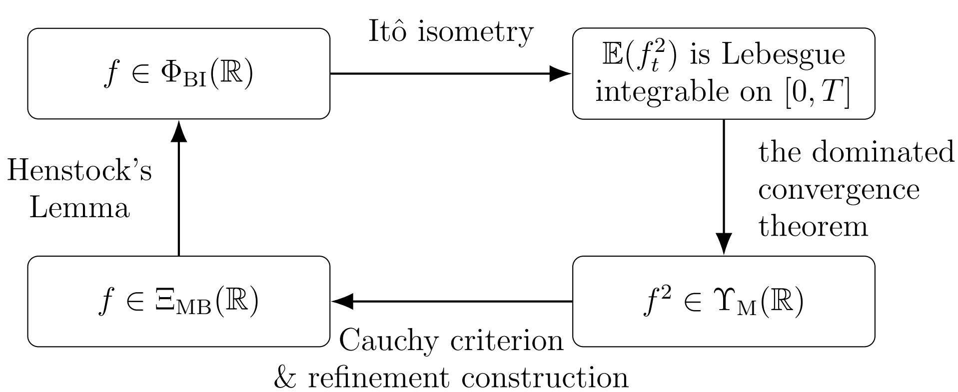

FIGURE 1. A schematic outline of the proof.

Furthermore, we recall the definition of the backwards McShane integral and certain properties proved in [3].

Definition 2.5. [3] A function $h : [0,T] \to \mathbb{R}$ is said to be backwards McShane integrable on $[0,T]$ to $A \in \mathbb{R}$ if for every $\varepsilon > 0$, there exist a positive function δ of $[0,T]$ and a positive number η such that for any (δ, η)-fine division $P = \{([w_k, z_k], \beta_k)\}_{k=1}^m$ of $[0,T]$, we have

$$\left| (P) \sum_{k=1}^{m} h(\beta_k)(z_k - w_k) - A \right| < \varepsilon. \tag{2.4}$$

Denote by $\Psi_{\mathrm{BM}}(\mathbb{R})$ the collection of all backwards McShane integrable functions on $[0,T]$.

Let $h \in \Psi_{\mathrm{BM}}(\mathbb{R})$. We can define a function $G : [0,T] \to \mathbb{R}$ by $G(t) = \int_t^T h(u)\,du$ for $0 \leq t < T$. The function G is called a primitive of h on $[0,T]$.

Now, one can construct an additive interval function, which is defined by $G(u, v) = G(u) - G(v)$.

Theorem 2.6 (Henstock's Lemma). [3] *If $h \in \Psi_{\mathrm{BM}}(\mathbb{R})$ and $G(x,y) = \int_x^y h(t)\,dt$ for all $[x,y] \subseteq [0,T]$, then for every $\varepsilon > 0$, there exist a positive function δ on $[0,T]$ and a positive number η such that for any (δ,η)-fine partial division $P = \{([a_j,b_j],\rho_j)\}_{j=1}^r$ of $[0,T]$, we have*

$$\left|(P)\sum_{j=1}^{r} h(\rho_j)(b_j - a_j) - (P)\sum_{j=1}^{r} G(a_j,b_j)\right| < \varepsilon. \tag{2.5}$$

According to [3], if $g \in \Psi_{\mathrm{BM}}(\mathbb{R})$ with the primitive G, then G is absolutely continuous on $[0,T]$. Moreover, the backwards McShane integral and the Lebesgue integral are equivalent.

3. Main results

In the previous proof in [1], the required steps are technically involved. First, establishing that $f^2 \in \Upsilon_{\mathrm{M}}(\mathbb{R})$ from the Lebesgue integrability of $\mathbb{E}(f_t^2)$ is not straightforward, as it requires applying the dominated convergence theorem to a sequence of step-function approximations. Second, passing from the M-integrability of f^2 to the McShane backwards Itô integrability of f on $[0,T]$ is nontrivial, necessitating a refinement construction to produce new divisions suitable for exploiting the M-integrability of f^2. We provide a simpler proof of the equivalence between the backwards Itô integral and the McShane backwards Itô integral, focusing on the implication from backwards Itô integrability to McShane backwards Itô integrability, while the reverse direction follows the earlier proof. By applying the equivalence between the Lebesgue integral and the backwards McShane integral as proved in [3] and by exploiting Henstock's Lemma and the absolute continuity property of the backwards McShane integral, we streamline the argument and simplify many of the technical steps.

In what follows, we first establish Lemma 3.1, which serves as a key tool in proving the equivalence, and then state our main result formally.

Lemma 3.1. *Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space and let $\mathcal{H} \subseteq \mathcal{G}$ be a σ-subalgebra. Let $\tau : \Omega \to \mathbb{R}$ be a random variable such that $\tau \in L^2(\Omega, \mathbb{R})$.*

$$\mathbb{E}\left((\mathbb{E}(\tau \mid \mathcal{H}))^2\right) \leq \mathbb{E}(\tau^2). \tag{3.1}$$

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$. Since g is convex, by Jensen inequality (see, e.g., [6]), we have

$$g\big(\mathbb{E}(\tau \mid \mathcal{H})\big) \leq \mathbb{E}\big(g(\tau) \mid \mathcal{H}\big) \qquad \text{a.s.}$$

Hence,

$$\big(\mathbb{E}(\tau \mid \mathcal{H})\big)^2 \leq \mathbb{E}\big(\tau^2 \mid \mathcal{H}\big) \qquad \text{a.s.}$$

By the property of conditional expectation (see, e.g., [1]), we have

$$\mathbb{E}\left(\left(\mathbb{E}(\tau \mid \mathcal{H})\right)^2\right) \leq \mathbb{E}\big(\mathbb{E}(\tau^2 \mid \mathcal{H})\big) = \mathbb{E}(\tau^2).$$

□

A schematic outline of the new proof is given in the Figure 2 below.

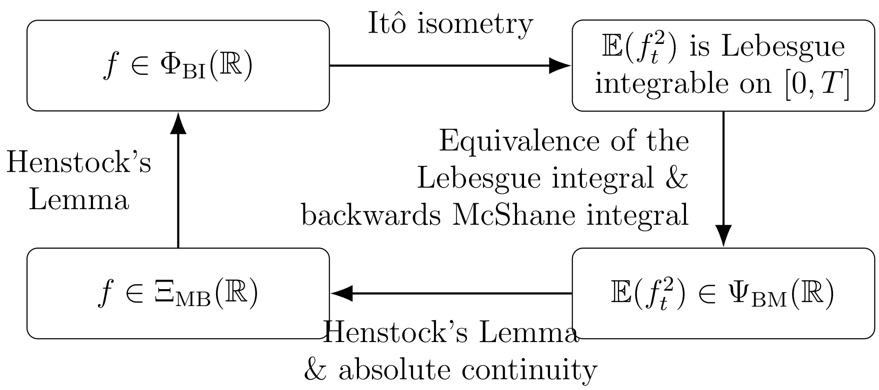

FIGURE 2. A schematic outline of the new proof.

We prove that the backwards Itô integral and the McShane backwards Itô integral are equivalent.

Theorem 3.2. *A process $f \in \Phi_{\mathrm{BI}}(\mathbb{R})$ if and only if $f \in \Xi_{\mathrm{MB}}(\mathbb{R})$ and the value of both integrals are equal.*

Proof. Since the necessity follows from [1], we only show the sufficiency. Let $\varepsilon > 0$ be given. Since $f \in \Phi_{\mathrm{BI}}(\mathbb{R})$, there exists $F \in L^2(\Omega, \mathbb{R})$ such that there exist a positive function δ' and a positive number η' such that for any backwards (δ', η')-fine partial division $P' = \{([z_p, \varphi_p], \varphi_p)\}_{p=1}^{q}$ of $[0,T]$, we have

$$\mathbb{E}\left(\left|(P')\sum_{p=1}^{q} f_{\varphi_p}\big(B_{\varphi_p} - B_{z_p}\big) - F\right|^2\right) < \frac{\varepsilon}{4}. \tag{3.2}$$

Since $f \in \Phi_{\mathrm{BI}}(\mathbb{R})$, by Itô isometry in [1], it implies that $\mathbb{E}(f_t^2)$ is Lebesgue integrable on $[0,T]$. It follows that $\mathbb{E}(f_t^2) \in \Psi_{\mathrm{BM}}(\mathbb{R})$ as proved in [3]. Let G be the primitive of $\mathbb{E}(f_t^2)$. By Theorem 2.6, there exist a positive function δ'' and a positive number η'' such that for any (δ'', η'')-fine partial division $P'' = \{([x_k, y_k], \gamma_k)\}_{k=1}^{l}$ of $[0,T]$, we have

$$\left|(P'')\sum_{k=1}^{l} \mathbb{E}(f_{\gamma_k}^2)(y_k - x_k) - (P'')\sum_{k=1}^{l} G(x_k, y_k)\right| < \frac{\varepsilon}{8}. \tag{3.3}$$

Since $\mathbb{E}(f_t^2) \in \Psi_{\mathrm{BM}}(\mathbb{R})$, it follows that G is absolutely continuous on $[0,T]$ by [3]. Hence, there exists a positive number η''' such that for any finite collection of non-overlapping subintervals $\{[\alpha_e,\beta_e]\}_{e=1}^{d}$ with $\sum_{e=1}^{d}(\beta_e-\alpha_e)\le\eta'''$, we have

$$\left|\sum_{e=1}^{d} G(\alpha_e,\beta_e)\right| < \frac{\varepsilon}{8}. \tag{3.4}$$

Choose $\delta(\xi)=\min\{\delta'(\xi),\delta''(\xi)\}$ for all $\xi\in[0,T]$ and $\eta=\min\{\eta',\eta'',\eta'''\}$. Let $D=\{([a_j,b_j],\rho_j)\}_{j=1}^{m}$ be an arbitrary (δ,η)-fine division of $[0,T]$ such that $D=D_1\cup D_2$, $D_1=\{([w_r,\tau_r],\tau_r)\}_{r=1}^{s}$ is a backwards (δ,η)-fine partial division of $[0,T]$, and $D_2=\{([u_i,v_i],\xi_i)\}_{i=1}^{n}$ is a McShane δ-fine partial division of $[0,T]$ where

$$(D_2)\sum_{i=1}^{n}(v_i-u_i)\le\eta.$$

This implies that D_2 is a (δ,η)-fine partial division of $[0,T]$ which is also a (δ'',η'')-fine partial division of $[0,T]$.

Therefore, by inequality 3.3, we have

$$\left|(D_2)\sum_{i=1}^{n}\mathbb{E}(f_{\xi_i}^2)(v_i-u_i)-(D_2)\sum_{i=1}^{n}G(u_i,v_i)\right|<\frac{\varepsilon}{8}. \tag{3.5}$$

Since G is absolutely continuous on $[0,T]$ and $\{[u_i,v_i]\}_{i=1}^{n}$ is a finite collection of non-overlapping subintervals of $[0,T]$ with $\sum_{i=1}^{n}(v_i-u_i)\le\eta$, it follows from inequality 3.4 that

$$\left|(D_2)\sum_{i=1}^{n}G(u_i,v_i)\right|<\frac{\varepsilon}{8}. \tag{3.6}$$

Now, by Cauchy-Schwarz inequality, we know

$$\begin{aligned}&\mathbb{E}\left(\left|(D)\sum_{j=1}^{m}\mathbb{E}(f_{\rho_j}\mid\mathcal{G}_{b_j})(B_{b_j}-B_{a_j})-F\right|^2\right)\\&\le 2\mathbb{E}\left(\left|(D_1)\sum_{r=1}^{s}f_{\tau_r}(B_{\tau_r}-B_{w_r})-F\right|^2\right)+2\mathbb{E}\left(\left|(D_2)\sum_{i=1}^{n}\mathbb{E}(f_{\xi_i}\mid\mathcal{G}_{v_i})(B_{v_i}-B_{u_i})\right|^2\right).\end{aligned} \tag{3.7}$$

Since $f\in\Phi_{\mathrm{BI}}(\mathbb{R})$, by the inequality (3.2), the first term in the inequality (3.7) can be made small enough

$$\mathbb{E}\left(\left|(D_1)\sum_{r=1}^{s}f_{\tau_r}(B_{\tau_r}-B_{w_r})-F\right|^2\right)<\frac{\varepsilon}{4}.$$

By Lemma 2.3, Lemma 3.1, inequality (3.5), and inequality (3.6), the second term in inequality (3.7) becomes

$$\begin{aligned}
\mathbb{E}\left(\left|(D_2)\sum_{i=1}^{n}\mathbb{E}(f_{\xi_i}\mid\mathcal{G}_{v_i})(B_{v_i}-B_{u_i})\right|^2\right) &= (D_2)\sum_{i=1}^{n}\mathbb{E}\left((\mathbb{E}(f_{\xi_i}\mid\mathcal{G}_{v_i}))^2\right)(v_i-u_i)\\
&\leq (D_2)\sum_{i=1}^{n}\mathbb{E}(f_{\xi_i}^2)(v_i-u_i)\\
&\leq \left|(D_2)\sum_{i=1}^{n}\mathbb{E}(f_{\xi_i}^2)(v_i-u_i)-(D_2)\sum_{i=1}^{n}G(u_i,v_i)\right|\\
&\quad+\left|(D_2)\sum_{i=1}^{n}G(u_i,v_i)\right|\\
&= \frac{\varepsilon}{4}.
\end{aligned}$$

Therefore,

$$\mathbb{E}\left(\left|(D)\sum_{j=1}^{m}\mathbb{E}(f_{\rho_j}\mid\mathcal{G}_{b_j})(B_{b_j}-B_{a_j})-F\right|^2\right)<\varepsilon.$$

□

4. CONCLUSION

In this paper, we have presented a simpler proof of the equivalence between the backwards Itô integral and the McShane backwards Itô integral. In contrast to the approach in [1], which relies on the stochastic M-integral, our method uses the deterministic backwards McShane integral. By building on fundamental properties of the backwards McShane integral, together with Henstock's Lemma and the absolute continuity of the primitives, our approach not only simplifies the argument but also makes the steps of the proof more straightforward and transparent. Moreover, the framework established here provides a natural starting point for extending these results to other stochastic integrals within the Henstock-type integration framework.

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