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# A SEMI-PARAMETRIC L-MOMENTS ESTIMATION METHOD FOR COPULAS UNDER RIGHT-CENSORING

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Abstract. We present a new semi-parametric estimator for Archimedean copula models in the present research. Based on the classical copula L-moment estimation in complete data, we improve this approach for data that is subject to right-censorship. The estimation relies on conditional moment methods, accounts for various censoring patterns, and is based on the empirical survival copula. Moreover, the suggested estimator's asymptotic normality is proven, offering theoretical support for the proposed estimator. A simulation study is conducted to illustrate the performance and robustness of the estimator, highlighting its advantages in modeling the dependence structure in censored bivariate survival settings.

*Keywords.* Semi-parametric estimation; L-moments method; Archimedean copula; Empirical survival copula; Right-censoring.
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## 1. Introduction

Understanding dependence structures in multivariate survival data is an important problem in applied statistics, especially in the presence of censoring. In such contexts, Copula models provide an adaptable foundation for simulating lifetimes' combined behavior while separating their dependence structure from their marginal distributions. Among them, Archimedean copulas have gained popularity due to their tractable mathematical form and their ability to capture various dependency patterns [22, 8, 10, 28, 19]. Assume that two survival periods, denoted by $(T_1, T_2)$, have continuous marginal survival functions, $S_1(t_1)$ and $S_2(t_2)$. The following is a representation of their joint survival function:

$$S(t_1, t_2) = \tilde{C}(S_1(t_1), S_2(t_2)), \tag{1.1}$$

where the survival copula can be identified by $\tilde{C}$ [15, 4, 17]. This formulation allows modeling dependence while accommodating censored observations [6, 14, 27]. When the data are subject to right-censoring, as is frequently the case in survival analysis, the estimation of copula parameters becomes more complex.

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Several likelihood-based methods have been developed in the literature, such as maximum likelihood [10], pseudo-likelihood, inference function for margins (IFM) [16, 17], and minimum distance methods [28]. However, these approaches often require evaluating the copula density, which may lead to numerical instability, especially in the presence of censoring or in high-dimensional settings [19, 30].

Moment-based strategies, notably L-moments [14], offer robust parameter estimation less sensitive to outliers and heavy tails, with applications in hydrology, actuarial science, and dependence modeling [27, 5, 21, 12]. The $k$-th L-moment is a functional of the distribution function $F_X$ [14].

$$\lambda_k = \mathbb{E}\big[X\, P_{k-1}(F_X(X))\big] = \int_{\mathbb{R}} x\, P_{k-1}\big(F_X(x)\big)\, dF_X(x), \tag{1.2}$$

where $P_k(u)$ denotes the shifted Legendre polynomial defined in [2]. In the context of copula models, the L-moment theory was extended to complete data, leading to an efficient estimation approach that avoids the direct use of copula densities. More precisely, the $k$-th copula L-moment is defined as follows:

$$\lambda_k = \int_0^1 t\, P_{k-1}\big(K_C(t)\big)\, dK_C(t), \quad k = 1, 2, \ldots$$

Here, $K_C(t)$ stands for the distribution function of the random variable $C$, while $P_{k-1}$ corresponds to the shifted Legendre polynomial of degree $k-1$.

In bivariate survival analysis, Archimedean copulas provide a flexible framework to capture the dependence between survival times $(T_1, T_2)$, despite censorship. Following Genest and Rivest (1993) [6], if there exists a family of Archimedean copulas linking with marginals $S_1 = u_1$ and $S_2 = u_2$, it is possible to define:

$$\tilde{C}(u_1, u_2) = \varphi^{-1}\big(\varphi(u_1) + \varphi((u_2)\big), \tag{1.3}$$

while $\tilde{C}$ follows the Kendall distribution represented by:

$$K_\theta(t) = t - \frac{\varphi_\theta(t)}{\varphi'_\theta(t)}, \qquad \text{with density} \quad k'_\theta(t) = \frac{\varphi''_\theta(t)\varphi_\theta(t)}{[\varphi'_\theta(t)]^2}, \quad t \in (0, 1],$$

depending on the unknown copula parameter $\theta$. Building on this representation, the Archimedean copula framework allows for the joint modeling of marginal survival functions, making it particularly suitable for censored data. It provides the foundation for expressing the theoretical L-moments:

$$\lambda_k(\theta) = \int_0^1 s\, P_{k-1}\Big(t - \frac{\varphi_\theta(t)}{\varphi'_\theta(t)}\Big) \frac{\varphi''_\theta(t)\, \varphi_\theta(t)}{[\varphi'_\theta(t)]^2}\, dt, \quad k = 1, 2, \ldots \tag{1.4}$$

This formulation establishes a direct link between the copula generator, marginal survival functions, and moment-based measures of dependence, providing a solid foundation for semi-parametric estimation under right-censoring. Based on this formulation, Benatia et al. [2] proposed an estimator $\hat{\theta}_{\mathrm{CLM}}$ obtained by inverting the L-moment mapping:

$$\hat{\theta}_{\mathrm{CLM}} = \lambda^{-1}(\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_r), \tag{1.5}$$

where $\hat{\lambda}_k$ is the empirical L-moment of order $k$ computed from the sample. Therefore, if it exists, $\lambda^{-1}$ represents the translation of $\lambda$'s reversed form. The estimation reduces to solving a nonlinear system linking observed moments to the model parameter $\theta$. In survival analysis, right censoring occurs when event times $T_j$ $(j = 1, 2)$ are only known to exceed censoring times $C_j$ [20, 18, 26]. Observed data are $X_j = \min(T_j, C_j)$ with censoring indicators $\delta_j = \mathbf{1}_{\{T_j \leq C_j\}}$ [18, 26]. To handle such incomplete data, Idiou et al. [15] extended conditional copula moment (CCM) based estimation strategies using the survival copula, relying on conditional moment equations under various censoring schemes, providing a flexible and computationally efficient alternative to likelihood-based methods [26].

However, when considering higher-order moments, the associated computations and numerical solutions become increasingly complex. To overcome this limitation, we propose an innovative approach that simplifies the estimation of higher-order L-moments while maintaining feasibility and efficiency. The core idea is to derive theoretical expressions for L-moments adapted to censored observations and to match them with their empirical counterparts obtained from observed (potentially censored) data [20, 18]. This approach circumvents the drawbacks of density-based methods, without the need for complete margin distribution guidelines, and offers a robust and interpretable tool for inference in censored dependence models.

The paper is structured this way for the next. Section 2 provides the theoretical background on Archimedean survival copulas and derive the main results, including general expressions for the proposed conditional copula L-Moment (CCLM) estimator and the establishment of its asymptotic normality under right-censoring. Section 3 details the construction of the semi-parametric CCLM estimator based on the classical CCM and copula L-moment methods (CLM), with a specific illustration example for the Gumbel copula. A simulation analysis is carried out in Section 4 to assess the resilience and effectiveness of the proposed estimator. The work ends with discussions and new studies ideas in Section 5.

## 2. Main results

When the data is full and complete, the functional representation (1), provides the L-moment of the random variable $C(U)$ up to order $k$, was developed by Benatia et al. [2] from the general formula (1.2) introduced by Hosking [14]. When the data are subject to right censoring, the variable $C(U)$ cannot be observed for all units. Say that both lifespans $(T_1, T_2)$ are subject to right censoring, either reliant or separate through a censorship variable $(C_1, C_2)$. The reliance pattern of this vector is characterized by an Archimedean models $\widetilde{C}$, which can be expressed as follows: $V = \widetilde{C}\big(S_j(T_i)\big) \in [0, 1]$, where $S_j(t_i) = \mathbb{P}(T_j > t_i)$ for $j = 1, 2$ and $i = 1, 2$. In this case, the distribution function $K_C$ is replaced by its censored version recorded by:

$$K_C^{(c)}(v) = \mathbb{P}\big(\widetilde{C}(U) \leq v \,\big|\, \delta_1, \delta_2\big), \qquad v \in [0, 1],$$

where $\delta_i = \mathbf{1}_{\{T_i \le C_i\}}$ are the censoring indicators.Therefore, the result that follows may be formulated under the same assumptions and based on the conditional distribution of $\widetilde{C}$ given the censorship sequence as described in [15].

**Corollary 2.1.** *Examine a two-dimensional selection variable $(T_1, T_2)$ wherein joint relationship structure is represented by a generator $\varphi$ and $\widetilde{C}$. Given that $(C_1, C_2)$ derived from two-variate distribution used to apply both reliant or distinct suitable censorship.*

**(1)** *The $k$-th conditional copula L-moment $\lambda_k^{(c_1)}(c_1, c_2) = \lambda_k^{(c_1)}$of the variable $V$ given $T_1 > c_1$ and $T_2 > c_2$ is*

$$\lambda_k^{(c_1)} = \frac{1}{\widetilde{C}(c_1, c_2)} \int_0^{\widetilde{C}(c_1,c_2)} v\, P_{k-1}\Big(K_C^{(c)}(v \mid c_1, c_2)\Big) \frac{\varphi(v) - \varphi(\widetilde{C}(c_1, c_2))}{\big(\varphi'(v)\big)^2} \varphi''(v)\, dv, \tag{2.1}$$

*where $0 \le v \le \widetilde{C}(c_1, c_2)$ and $P_{k-1}$ denotes the shifted Legendre polynomial of degree $k-1$ associated with L-moments.*

**(2)** *The $k$-th conditional copula L-moment $\lambda_k^{(c)}(c_1, t_2) = \lambda_k^{(c_2)}$ of the variable $V$ given $T_1 > c_1$ and $T_2 > t_2$ is*

$$\lambda_k^{(c_2)} = \int_0^{\widetilde{C}\left(c_1,t_2\right)} v\, P_{k-1}\big(K_C^{(c)}(v \mid c_1, t_2)\big) \left[ -\ \varphi'(\widetilde{C}\big(c_1, t_2)\big) \frac{\varphi''(v)}{(\varphi'(v))^2} \right] dv, \tag{2.2}$$

*for $0 \le v \le \widetilde{C}(c_1, t_2)$.*

**(3)** *The $k$-th conditional copula L-moment $\lambda_k^{(c)}(t_1, c_2) = \lambda_k^{(c_3)}$ of the variable $V$ given $T_1 > t_1$ and $T_2 > c_2$ is*

$$\lambda_k^{(c_3)} = \int_0^{\widetilde{C}\left(t_1,c_2)\right)} v\, P_{k-1}\big(K_C^{(c)}(v \mid t_1, c_2)\big) \left[ -\ \varphi'(\widetilde{C}\big(t_1, c_2)\big)\big) \frac{\varphi''(v)}{(\varphi'(v))^2} \right] dv, \tag{2.3}$$

*for $0 \le v \le \widetilde{C}(t_1, c_2)$.*

*Proof.* Let us set, for brevity of notation, $\tilde{c} := \widetilde{C}(c_1, c_2)$, which is the upper bound of the conditional support of $V$. Based on formula (1), the $k$-th L-moment associated with the random variable $V$ can be written as:

$$\begin{aligned} \lambda_k(V) &:= \mathbb{E}_{V|c}\Big[V\, P_{k-1}\big(K_C^{(c)}(V)\big)\Big] \\ &= \int_0^1 v\, P_{k-1}\big(K_C^{(c)}(v \mid c_1, c_2)\big)\, dK_C^{(c)}(v \mid c_1, c_2), \quad k = 1, 2, \ldots \end{aligned} \tag{2.4}$$

The conditional distribution $K_C^{(c)}(\cdot \mid c_1, c_2)$ is supported on $[0, \tilde{c}]$, so the integration is restricted to this interval:

$$\lambda_k^{(c)}(c_1, c_2) = \int_0^{\tilde{c}} v\, P_{k-1}\big(K_C^{(c)}(v \mid c_1, c_2)\big)\, dK_C^{(c)}(v \mid c_1, c_2). \tag{2.5}$$

According to Corollary 2.1 and Theorem 2.1 of Idiou et al. [15], and under the regularity assumptions in particular $\varphi \in C^2((0, 1])$ and $\varphi'(v) \ne 0$ on $(0, 1]$, where $(T_i)_{i=1,2}$ is amenable to right censoring either reliant or distinct through a censoring vector $(C_i)_{i=1,2}$ drawn from any continuous two-variate pattern, given $T_1$ and $T_2$, the conditional pattern operation of $V$ is:

$$K_C^{(c)}(v \mid c_1, c_2) = \frac{1}{\tilde{c}} \left[ v \; - \; \frac{\varphi(v) - \varphi(\tilde{c})}{\varphi'(v)} \right], \tag{2.6}$$

for $0 \leq v \leq \tilde{c}$ (see [15]).

Differentiating (2.6) with respect to $v$ yields the conditional density $dK_C^{(c)}$ on $(0, \tilde{c}]$:

$$dK_C^{(c)}(c_1, c_2) := \frac{d}{dv} K_C^{(c)}(v \mid c_1, c_2) = \frac{1}{\tilde{c}} \frac{\big(\varphi(v) - \varphi(\tilde{c})\big)\varphi''(v)}{\big(\varphi'(v)\big)^2}, \qquad 0 < v \leq \tilde{c}. \tag{2.7}$$

The differentiation is valid due to the assumption $\varphi \in C^2$ and $\varphi' \neq 0$ on $(0, 1]$. Substituting (2.7) into (2.5) gives:

$$\lambda_k^{(c)}(c_1, c_2) = \frac{1}{\tilde{c}} \int_0^{\tilde{c}} v \, P_{k-1}\big(K_C^{(c)}(v \mid c_1, c_2)\big) \, \frac{\varphi(v) - \varphi(\tilde{c})}{\big(\varphi'(v)\big)^2} \, \varphi''(v) \, dv.$$

For the second and third cases, one proceeds identically: based on conditional distribution functions given in Theorem 2.1 of Idiou et al. [15], [25] differentiate to obtain the corresponding conditional densities for $V$ in each case. □

## 3. Semi-parametric Estimation of CCLM under Right Censoring

In the context of complete data, Benatia et al. [2] extended the L-moment framework for parameter estimation of Archimedean copulas. As practical datasets often involve censoring, Idiou et al. [15, 25] proposed a CCM based estimation using the empirical survival copula, which accommodates censored observations and establishes asymptotic normality (see Theorem 2.2). Building on this, we extend the conditional copula L-moment (CCLM) based estimator to right-censored bivariate data. Suppose $(T_1, T_2)$ represent a two-variate arbitrary vector that is double right censored via $(C_1, C_2)$ and whose dependency is represented by an Archimedean copula. Denoting $V = \widetilde{C}\big(S_1(T_1), S_2(T_2)\big)$ as the conditionally distributed vector, the $k$-th conditional copula L-moment of $V$ given $T_1 > c_1$ and $T_2 > c_2$ is expressed as. This approach follows the rigorous asymptotic analysis framework similar to that discussed in [1], highlighting the relevance of these mathematical techniques for applied probability and survival analysis.

$$\lambda_k^{(c_1)} = \frac{1}{\widetilde{C}(c_1, c_2)} \int_0^{\widetilde{C}(c_1,c_2)} v \, P_{k-1}\Big(K_C^{(c)}(v \mid c_1, c_2)\Big) \, \frac{\varphi(v) - \varphi(\widetilde{C}(c_1, c_2))}{\big(\varphi'(v)\big)^2} \, \varphi''(v) \, dv.$$

Now, suppose that for an unknown parameter $\theta$, the generator $\varphi$ depends on $\theta$, i.e., $\varphi = \varphi_\theta$. Consequently, the survival copula can be written as $\widetilde{C} = \widetilde{C}_\theta$, its corresponding distribution function as $K_C^c = K_\theta^c$, and the $k$-th L-moment of $\widetilde{C}$ as

$\lambda_k^c(\widetilde{C}) = \lambda_k^c(\theta)$. It then follows that

$$\lambda_k^{(c_1)} = \frac{1}{\widetilde{C}_\theta(c_1,c_2)} \int_0^{\widetilde{C}_\theta(c_1,c_2)} v\, P_{k-1}\Big(K_\theta^{(c)}(v \mid c_1,c_2)\Big) \frac{\varphi_\theta(v) - \varphi_\theta(\widetilde{C}_\theta(c_1,c_2))}{\big(\varphi_\theta'(v)\big)^2} \varphi_\theta''(v)\, dv,$$

For an unknown parameter $\theta \in \mathbb{R}^d$, assume that the selected sample from a bivariate random vector is $(X_1,\dots,X_n)$. FIf $j = 1,2$, and $X = (X_1, X_2)$, we have The empirical equivalent of $S_j$, known as the Kaplan-Meier estimate, is estimated as follows:

$$\hat{S}_j(x_{ji}) = \prod_{k/X'_{jk} < x_{ji}} (1 - \frac{\sum_{i=1}^n \mathbf{1}_{\{X_{ji}=X'_{jk},\delta_{ji}=0\}}}{\sum_{i=1}^n \mathbf{1}_{\{X_{ji}\geq X'_{jk}\}}})$$

by assuming $\hat{U}_{ji} = \hat{S}_j(X_{ji})$ and $\hat{U}_i = (\hat{U}_{1i}, \hat{U}_{2i})$. The survival copula $\widetilde{C}$ is estimated using the empirical survival copula introduced by Idiou et al. [15], defined as follows:

$$\begin{aligned} \widetilde{C}_n(u_1,u_2) = u_1 + u_2 - 1 + \frac{1}{n}\sum_{i=1}^n \frac{\delta_{1i}\,\delta_{2i}}{\widetilde{C}\Big(\hat{S}_1(X_{1i}), \hat{S}_2(X_{2i})\Big)} \\ \times \mathbf{1}_{\{1-F_{1n}(X_{1i})\geq u_1,\, 1-F_{2n}(X_{2i})\geq u_2\}}. \end{aligned} \tag{3.1}$$

By the way, the empirical distribution function of the random variable $\widetilde{C}(U)$ associated with $\big(\widetilde{C}_i(\hat{U}_i),\big)_{i=1,\dots,n}$ , defined as

$$K_n(t) = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{\widetilde{C}_n(\hat{U}_i))\leq t\}}, \quad t \in [0,1]$$

Using this result, the empirical version of the $k$-th conditional copula L-moment estimator under doubly censored data can be expressed as

$$\hat{\lambda}_k^{(c)} = \lambda\big(\widetilde{C}_n\big) = \frac{1}{n}\sum_{i=1}^n \widetilde{C}_n\big(\widehat{U}_i\big)\, P_{k-1}\Big(K_n\big(\widetilde{C}_n(\widehat{U}_i)\big)\Big), \quad k = 1,2,\dots$$

We denote the solution $\hat{\theta}^c_{\mathrm{CCLM}} = (\hat{\theta}_1^c, \dots, \hat{\theta}_d^c)$ of the following system:

$$\lambda_k^c(\theta_1,\dots,\theta_d) = \hat{\lambda}_k^c, \quad k = 1,\dots,d \tag{3.2}$$

the CCLM estimator for $\theta$. In the following, we present the most important part of our work, which consists in establishing the consistency and the asymptotic normality of the estimator $\hat{\theta}^c_{\mathrm{CCLM}}$. To this end, we define the following:

$$\Psi : \Theta \to \mathbb{R}^d,\ \theta \mapsto \Psi(\mathbf{u},\theta) = (\Psi_1(\mathbf{u},\theta),\dots,\Psi_d(\mathbf{u},\theta))$$

where

$$\Psi_k(\mathbf{u},\theta) := \widetilde{C}_\theta(\mathbf{u}) P_{k-1}\big(K_\theta^c(\widetilde{C}_\theta)\big) - \lambda_k^c(\theta), \qquad \text{with } \mathbf{u} = (u_1,u_2) \in [0,1]^2 \tag{3.3}$$

Let $\theta_1$ denote the true parameter value, and suppose that assumptions$[A1] - [A4]$ are satisfied.

[A1] For each $\theta$, the function

$$\Psi(\cdot,\theta) : (0,1)^2 \longrightarrow \mathbb{R}^d$$

is continuously differentiable with

$$\Psi^i = \frac{\partial \Psi}{\partial u_i}, \quad i = 1, 2$$

and there exist functions $r_i \in \mathcal{R}$, $\widetilde{r}_i \in \mathcal{R}$ and $q_i \in \mathcal{Q}$ $(i = 1, 2)$ such that

$$|\Psi(\mathbf{u},\theta)| \leq \prod_{i=1}^{2} r_i(u_i),$$

$$|\Psi^i(\mathbf{u},\theta)| \leq \tilde{r}_i(u_i) \prod_{j\neq i} r_j(u_j), \quad i = 1, 2$$

and

$$\int_{(0,1)^2} \left\{ \prod_{i=1}^{2} r_i(u_i) \right\}^2 d\widetilde{C}_\theta(\mathbf{u}) < \infty,$$

$$\int_{(0,1)^2} \left\{ q_i(u_i) r_i(u_i) \prod_{j\neq i} r_j(u_j) \right\} d\widetilde{C}_\theta(\mathbf{u}) < \infty, \quad i = 1, 2$$

where $\mathcal{Q}$ is the set of continuous functions $q : [0,1] \to (0,\infty)$ that are symmetric about $1/2$ and increasing on $[0,1/2]$ and satisfies $\int_0^1 \{q(t)\}^{-2}\, dt < \infty$ and $\mathcal{R}$ is the set of continuous functions $r : (0,1) \to (0,\infty)$ that are symmetric about $1/2$ and increasing on $(0,1/2]$.For further details on these sets, the reader is referred to [28]. This assumption leads us to refer to the proposition [3] introduced by Tsukahara [28], which played a crucial role in establishing the asymptotic normality of $Z$-estimators. We now aim to extend the framework of this proposition to accommodate right-censored survival data, by exploiting the asymptotic normality of the empirical survival copula proposed by Idiou et al. [15] . This automatically leads us to the following proposition.

**Proposition 3.1.** *assume that* (A.1) *holds. Then*

$$\sqrt{n}\left[ \int \Psi\Big(\hat{S}(x_1), \hat{S}(x_2), \theta_1\Big)\, d\widetilde{C}_n(S_1(x_1), S_2(x_2)) - \int \Psi(\boldsymbol{u},\theta_1)\, d\widetilde{C}_{\theta_1}(\boldsymbol{u}) \right]$$

*converges in distribution to* $\mathcal{N}(0,\sigma^2)$*, where*

$$\sigma^2 = \operatorname{Var}\left\{ \Psi(\xi^c) + \sum_{i=1}^{2} \int \Psi^i(\boldsymbol{u})\, \mathbf{1}\{\xi_i^c \leq u_i\}\, d\widetilde{C}(\boldsymbol{u}) \right\},$$

*and* $\xi^c$ *is a random vector with distribution function* $\widetilde{C}$.

[A2] Assume that there exists a unique parameter $\theta_1 \in \Theta \subset \mathbb{R}^d$ such that

$$\int_{[0,1]^2} \Psi(\mathbf{u};\theta_1)\, d\widetilde{C}_{\theta_1}(\mathbf{u}) = 0$$

[A3] $\Psi(\cdot;\theta)$ admits differentiation with respect to $\theta$, with the associated Jacobian matrix denoted by

$$\dot{\Psi}(\mathbf{u};\theta) := \left[\frac{\partial \Psi_k(\mathbf{u};\theta)}{\partial \theta_\ell}\right]_{d\times d}$$

satisfies:
- $\dot{\Psi}(\mathbf{u};\theta)$ is jointly continuous in $(\mathbf{u},\theta)$,
- and the Euclidean norm $|\dot{\Psi}(\mathbf{u};\theta)|$ is dominated by a $d\widetilde{C}_\theta$-integrable function.

[A4] the square matrix of order $d$

$$A_0^c := \int_{[0,1]^2} \dot{\Psi}(\mathbf{u};\theta_1)\, d\widetilde{C}_{\theta_1}(\mathbf{u})$$

is nonsingular.

**Theorem 3.2.** *Under [A1] – [A4], there exists a solution $\hat{\theta}^c_{\text{CCLM}}$ to system (3.2) such that $\hat{\theta}^c_{\text{CCLM}} \xrightarrow{\mathbb{P}} \theta_1$. Furthermore*

$$\sqrt{n}\,(\hat{\theta}^c_{\text{CCLM}} - \theta_1) \xrightarrow{D} \mathcal{N}\big(0,\, (A_0^c)^{-1}\, D_0^c\, ((A_0^c)^{-1})^T\big)\,, \quad \textit{as } n\to\infty,$$

*where*

$$D_0^c := \operatorname{var}\{\phi(\xi^c;\theta_1) + \boldsymbol{S}(\xi^c;\theta_1)\} \quad \textit{and} \quad \boldsymbol{S}(\xi^c;\theta_1) = \big(S_1(\xi^c;\theta_1),\ldots,S_d(\xi^c;\theta_1)\big).$$

*with*

$$S_k^c(\xi^c;\theta_1) := \sum_{j=1}^{2}\int_{[0,1]^2} \frac{\partial\Big(\widetilde{C}_\theta(\boldsymbol{u})\, P_{k-1}\big(K_\theta^c(\widetilde{C}_\theta)\big)\Big)}{\partial u_j}\big(\mathbf{1}\{\xi_j^c \le u_j\} - u_j\big)\, d\widetilde{C}_{\theta_1}(\boldsymbol{u}),$$

*with $k=1,\ldots,d$, and $\xi^c := (\xi_1^c,\xi_2^c)$ is a $(0,1)^2$-uniform random vector with joint distribution function $\widetilde{C}_{\theta_1}$.*

*Remark* 3.3. Following the PML and Z-estimation approaches of Genest *et al.* [10] and Tsukahara [28], Benatia *et al.* [2] proposed a consistent estimator of the asymptotic variance $A_0^{-1}D_0(A_0^{-1})^T$ constructed using the empirical variance of the associated sequence of random variables

$$\left\{\hat{A}_i^{-1}\hat{D}_i\left(\hat{A}_i^{-1}\right)^T, \quad i=1,\ldots,n\right\},$$

where

$$\hat{A}_i := \int_{[0,1]^2} \dot{\Psi}\left(\mathbf{u};\hat{\theta}_{\text{CLM}}\right)\, dC_{\hat{\theta}_{\text{CLM}}}(\mathbf{u}),$$

and

$$\hat{D}_i := \Psi\left(\hat{U}_i;\hat{\theta}_{\text{CLM}}\right) + V\left(\hat{U}_i;\hat{\theta}_{\text{CLM}}\right).$$

with

$$\hat{U}_i = (F_{1n}(T_{1i})), F_{2n}(T_{2i}))$$

We can thus readily conclude that, under right-censoring conditions, a consistent estimator of the asymptotic variance $(A_0^c)^{-1} D_0^c ((A_0^c)^{-1})^T$ can be constructed using the empirical variance of the associated sequence of random variables

$$(\hat{A}_i^c)^{-1} \hat{D}_i^c \left\{ (\hat{A}_i^c)^{-1} \right\}^T, \quad i = 1, \ldots, n$$

with

$$\hat{A}_i^c := \int_{[0,1]^2} \dot{\Psi}\left(\mathbf{u}; \hat{\theta}_{\text{CCLM}}^c\right) d\widetilde{C}_{\hat{\theta}_{\text{CCLM}}^c}(\mathbf{u}),$$
$$\hat{D}_i^c := \Psi\left(\hat{U}_i^c; \hat{\theta}_{\text{CCLM}}^c\right) + V\left(\hat{U}_i^c; \hat{\theta}_{\text{CCLM}}^c\right).$$

and

$$\hat{U}_i^c = (\hat{S}_1(X_{1i})), \hat{S}_2(X_{2i}))$$

*Proof.* (Theorem 3.2.) It can be shown that the proposed estimator is a RAZ-type estimator. To establish its consistency and asymptotic normality, we explicitly derive the asymptotic expansion of the estimator, accounting for the randomness introduced by the Kaplan-Meier estimates of the marginal survival functions. Let $\hat{S}_1$ and $\hat{S}_2$ denote the Kaplan-Meier estimators of the marginal survivals $S_1$ and $S_2$, respectively. Using the functional delta method together with the asymptotic properties of the empirical survival copula developed by Idiou et al. [15], we obtain

$$\hat{\theta} - \theta_0 = \frac{1}{n} \sum_{i=1}^{n} \psi(\mathbf{T}_i; \theta_0) + R_n,$$

where $\psi(\mathbf{T}_i; \theta_0)$ captures the influence of both the copula structure and the estimated margins, and $R_n = o_p(n^{-1/2})$ is a remainder term. The term $\psi(\mathbf{T}_i; \theta_0)$ explicitly incorporates the variability due to the Kaplan-Meier estimation, which modifies the asymptotic variance of the L-moment estimator compared to the complete data case. By verifying the regularity conditions of Theorem A.10.2 in Bickel et al. [3], and applying the central limit theorem for empirical processes with estimated marginals, we conclude that

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, \Sigma),$$

where $\Sigma$ is the asymptotic covariance matrix explicitly accounting for the contribution of the Kaplan-Meier estimated margins. This approach extends the framework of Benatia et al. [2] to the setting of right-censored data and provides a fully explicit derivation of the asymptotic distribution of the estimator. Let us first recall the functions given in 3.3 and set

$$\varphi(\theta) := \int_{I^2} \Psi(\mathbf{u}; \theta)\, d\widetilde{C}_{\theta_1}(\mathbf{u}), \quad \textit{and} \quad \varphi_n(\theta) := \frac{1}{n} \sum_{i=1}^{n} \Psi(\hat{U}_i^c; \theta).$$

where $\quad \hat{U}_i^c = \left(\hat{S}_1(X_{1i})), \hat{S}_2(X_{2i})\right)$
From assumption [A3], the following derivatives exist

$$\dot{\varphi}(\theta) = \int_{I^2} \dot{\Psi}(\mathbf{u}; \theta)\, d\widetilde{C}_{\theta_1}(\mathbf{u}), \quad \dot{\varphi}_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \dot{\Psi}(\hat{U}_i^c; \theta).$$

It must be verified that

$$\sup\{|\dot{\varphi}_n(\theta)-\dot{\varphi}(\theta)|\ :\ |\theta-\theta_1|<\delta_n\}\xrightarrow{P}0,\quad as\ n\to\infty \tag{H1}$$

For any real sequence $\delta_n\to 0$. Observe that

$$|\dot{\varphi}_n(\theta)-\dot{\varphi}_n(\theta_1)|\le\frac{1}{n}\sum_{i=1}^{n}\left|\dot{\Psi}(\hat{U}_i^c;\theta)-\dot{\Psi}(\hat{U}_i^c;\theta_1)\right|$$

Since $\dot{\Psi}$ is continuous in $\theta$, we have

$$\sup_{|\theta-\theta_1|<\delta_n}\left\{\left|\dot{\Psi}(\hat{U}_i^c;\theta)-\dot{\Psi}(\hat{U}_i^c;\theta_1)\right|\right\}=o_P(1),\quad i=1,\dots,n.$$

then

$$\sup_{|\theta-\theta_1|<\delta_n}\{|\dot{\varphi}_n(\theta)-\dot{\varphi}_n(\theta_1)|\}\xrightarrow{P}0,\quad \text{as } n\to\infty \tag{H2}$$

Next, it follows from the law of large numbers that

$$\frac{1}{n}\sum_{i=1}^{n}\dot{\Psi}(U_i^c;\theta_1)\xrightarrow{P}\dot{\varphi}(\theta_1),\quad \text{as } n\to\infty,$$

with $U_i^c=\{S_j(X_{ji})\}_{j=1,2}$. Also, by the continuity of $\dot{\Psi}$ in $\mathbf{u}$,and the fact that $\sup|\hat{S}_j(x_j)-S_j(x_j)|\to 0,\quad$ a.s.

$$\frac{1}{n}\sum_{i=1}^{n}\left|\dot{\Psi}(\hat{U}_i^c;\theta_1)-\dot{\Psi}(U_i^c;\theta_1)\right|\xrightarrow{P}0.$$

Hence

$$|\dot{\varphi}_n(\theta_1)-\dot{\varphi}(\theta_1)|\xrightarrow{P}0,$$

and combined with (H2), this gives (H1). Assumptions $[A1]-[A4]$ readily ensure that conditions (MG0) and (MG3) of Theorem A.10.2 in Bickel *et al.* [3] are satisfied. Consequently, by the general theorem for $Z$-estimators (see van der Vaart and Wellner [29], Theorem 3.3.1), it suffices to establish that

$$\sqrt{n}\left(\dot{\varphi}_n-\dot{\varphi}\right)(\theta_1)$$

converges in distribution to the appropriate limit. This result is a direct consequence of Proposition 3.1, which completes the proof of the theorem 3.2.

□

3.1. **Illustrative Example.** For our illustrative purposes, we consider a representative Archimedean copula, widely recognized for its simple mathematical structure and its flexibility in modeling various dependence patterns. In particular, we focus on the Gumbel copula, which is particularly suitable for our study, defined as:

$$C_\alpha(\mathbf{u})=\exp\left\{-\left(\sum_{j=1}^{d}(-\ln u_j)^\alpha\right)^{1/\alpha}\right\},\quad \alpha\ge 1,$$

whose associated generator function takes the form

$$\varphi_\alpha(t)=(-\ln t)^\alpha,\quad \alpha\ge 1.$$

It is well established that, for modeling high-dimensional datasets, employing copulas with multiple parameters enhances the flexibility in capturing complex dependence structures. Such flexibility can be achieved through the use of transformed copulas, which are expressed in the following form:

$$C_{\varphi}(u) = \varphi^{-1}\Big(C\big(\varphi(u_1), \ldots, \varphi(u_d)\big)\Big),$$

where $\varphi$ is a strictly increasing, concave, and continuous function that satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$, (see Nelsen [22], p. 96). If our analysis is restricted to only two parameters, namely $\alpha$ and $\beta$, we then obtain the following formulation:

$$\widetilde{C}_{\alpha,\beta}(u,v) = u_1 + u_2 - 1 + \left(\left(\left((1-u_1)^{-\alpha} - 1\right)^{\beta} + \left((1-u_2)^{-\alpha} - 1\right)^{\beta}\right)^{1/\beta} + 1\right)^{-1/\alpha}$$

with the generator $\varphi_{\alpha,\beta}(t) = (t^{-\alpha} - 1)^{\beta}$, where $\alpha > 0$ and $\beta \geq 1$ (see Idiou et al. [15] ). In the remainder of this paper, we focus solely on the first scenario described in Corollary 2.1, corresponding to the case of right double censoring, and compute the first two $k$-th CCLM accordingly.

$$\begin{cases} \lambda^{(c)}_{k=1}(V \mid C_1, C_2) = \frac{1}{2}\tilde{c} + J(\alpha,\beta) \\ \\ \lambda^{(c)}_{k=2}(V \mid C_1, C_2) = \frac{1}{6}\tilde{c} + I(\alpha,\beta) \end{cases}$$

where $I$ and $J$ are two functions defined for all $\alpha > 0$ and $\beta \geq 1$ by :

$$J(\alpha,\beta) = \frac{1}{\alpha\beta}\left(\frac{\tilde{c}^{\alpha+1}}{\alpha+2} - \frac{1}{2}\tilde{c} - \frac{(\beta-1)(\tilde{c}^{-\alpha}-1)^{\beta}\Gamma(1-\beta)\Gamma\left(\frac{1}{\alpha}(\alpha\beta+2)\right)}{\alpha\,\tilde{c}^{\alpha+1}\Gamma\left(\frac{2}{\alpha}(\alpha+1)\right)}\right), \quad (3.4)$$

$$I(\alpha,\beta) = I_1(\alpha,\beta) + I_2(\alpha,\beta) + I_3(\alpha,\beta). \quad (3.5)$$

Where

$$\begin{cases} I_1(\alpha,\beta) = \frac{\varphi(\tilde{c})}{\alpha^2\beta\tilde{c}^2}\left(2\,B_{\tilde{c}^{\alpha}}\left(\beta + \frac{3}{\alpha}, 2-\beta\right) - \tilde{c}\,B_{\tilde{c}^{\alpha}}\left(\beta + \frac{2}{\alpha}, 2-\beta\right)\right), \\ I_2(\alpha,\beta) = \frac{\varphi(\tilde{c})}{\alpha^3\beta^2\tilde{c}^2}\left(2\,B_{\tilde{c}^{\alpha}}\left(\beta + \frac{3}{\alpha}, 3-\beta\right) - \varphi(\tilde{c})\,B_{\tilde{c}^{\alpha}}\left(2\beta + \frac{3}{\alpha}, 3-2\beta\right)\right), \\ I_3(\alpha,\beta) = \frac{2}{\alpha\beta}\left(\frac{7}{12}\tilde{c} - \frac{1}{3} + \frac{\tilde{c}^{\alpha}}{\alpha+3} - \frac{\tilde{c}^{\alpha+1}}{2\alpha+4} - \frac{\tilde{c}^{\alpha+1}}{\alpha+3}\right) + \frac{1}{\alpha^2\beta^2}\left(\frac{2\tilde{c}^{\alpha+1}}{\alpha+3} - \frac{\tilde{c}}{3} - \frac{\tilde{c}^{2\alpha+1}}{2\alpha+3}\right), \end{cases}$$

here $B_{\tilde{c}^{\alpha}}(x,y)$ be the Beta function, with $\tilde{c} = \tilde{C}(c_1, c_2)$ representing the ordinary copula, and $\Gamma(x)$ the Gamma function. The CCLM estimator $(\hat{\alpha}^c, \hat{\beta}^c)$ is then described as the following system's singular outcome:

$$\begin{cases} \lambda^c_1(\alpha,\beta) = \hat{\lambda}^c_1, \\ \lambda^c_2(\alpha,\beta) = \hat{\lambda}^c_2. \end{cases}$$

## 4. Simulation studies

A simulation analysis based on the Monte Carlo approach under right-censored settings was carried out to assess the performance of the suggested CCLM estimator [23, 24]. A Gumbel copula model was used to create two-dimensional lifespan instances, with each margin having a Pareto distribution.

$$F(t) = 1 - t^{-\lambda}, \quad t \geq 0,$$

and the censoring times were also assumed to follow Pareto distributions with parameters $\lambda_3$ and $\lambda_4$. For the first sample, the proportion of observed data was defined as $p_1 = \frac{\lambda_2}{\lambda_1+\lambda_2}$. By fixing $\lambda_1 = 0.2$ and considering different observed proportions of $95\%, 90\%, 85\%, 80\%$, the corresponding $\lambda_2$ values were calculated. A similar approach was applied to $\lambda_3$ and $\lambda_4$ using $p_2 = \lambda_4/(\lambda_3 + \lambda_4)$. For each sample size $n = 30, 50, 100, 500, 1000$, a 1000 Monte Carlo replicates were generated to assess the estimator's accuracy. Relative bias and root mean square error (RMSE) were used to gauge the estimation's quality. This procedure was repeated for a range of true copula parameter pairs $(\alpha, \beta)$ representing different dependence levels.

The chosen parameter values correspond to weak ($\tau = 0.05$), moderate ($\tau = 0.5$), and strong ($\tau = 0.7$) levels of dependence. For each dependence level, the simulation study was conducted under four different censoring proportions, namely 5%, 10%, 20%, and 25%, as reported in Tables (1, 2, 3). Kendall's $\tau$ was used as the dependence measure, linked to the copula parameters via $\tau_{\alpha,\beta} = 4\,\mathbb{E}\tilde{C}_{\alpha,\beta}(U_1, U_2)] - 1$, where $U_1$ and $U_2$ are distinct, evenly distributed random variables on [0,1].

The simulation results indicate that the CCLM estimator performs well under different dependence levels and censoring percentages. Relative bias and RMSE values were consistently low, even for large samples, demonstrating accurate estimation. Additionally, after censoring ($\tau_2$), the Kendall's ($\tau$) stays near to its theoretical value ($\tau_1$), indicating that the dependence structure is preserved despite censoring. Overall, the proposed CCLM estimator exhibits robust performance and compares favorably with previously used copula estimation methods.

A comparison between the proposed CCLM estimator and the CCM estimator proposed by Idiou et al. is carried out to highlight the gain achieved by the new approach. The comparison is performed for a fixed sample size $n = 500$ and under four right-censoring proportions, namely 5%, 10%, 20%, and 25%. The comparison is conducted across different dependence levels, ranging from weak to strong association, and the results are reported in Tables 4 and 5 for the parameters $\alpha$ and $\beta$, respectively. The performance of both estimators is evaluated using the bias and the root mean square error (RMSE).

Overall, the CCLM estimator consistently shows smaller bias and lower RMSE values than the CM estimator for all censoring proportions and dependence levels. The superiority of the CCLM estimator becomes more pronounced as the censoring rate increases, where the CCM estimator tends to display higher variability. Moreover, the CCLM estimator remains stable across different dependence structures, providing reliable estimation for weak, moderate, and strong dependence.

Table 1. Bias and RMSE of CCLM estimators for transformed Gumbel copulas under varying censoring levels and weak correlation.

| $\tau = 0.05, \quad \alpha = 0.1 \to \beta = 1.0$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5% of censoring | | | | | | | | |
| $n$ | $\hat{\alpha}$ | | $\hat{\beta}$ | | $\tau$ | | $c$ | |
| | Bias | RMSE | Bias | RMSE | $\tau_1$ | $\tau_2$ | $c_1$ | $c_2$ |
| 30 | 0.0001 | 0.0057 | 0.0018 | 0.0582 | 0.0525 | 0.0506 | 0.0313 | 0.0313 |
| 50 | -0.0002 | 0.0057 | -0.0012 | 0.0581 | 0.0464 | 0.0463 | 0.0176 | 0.0176 |
| 100 | 0.0002 | 0.0059 | 0.0008 | 0.0580 | 0.0482 | 0.0452 | 0.0096 | 0.0096 |
| 500 | 0.0001 | 0.0058 | 0.0012 | 0.0580 | 0.0479 | 0.0462 | 0.0019 | 0.0019 |
| 1000 | -0.0001 | 0.0059 | 0.0015 | 0.0588 | 0.0473 | 0.0455 | 0.0009 | 0.0009 |
| 10% of censoring | | | | | | | | |
| 30 | -0.0003 | 0.0059 | 0.0003 | 0.0556 | 0.0442 | 0.0387 | 0.0303 | 0.0303 |
| 50 | -0.0001 | 0.0057 | 0.0028 | 0.0568 | 0.0495 | 0.0469 | 0.0170 | 0.0170 |
| 100 | -0.0001 | 0.0057 | -0.0033 | 0.0585 | 0.0458 | 0.0414 | 0.0092 | 0.0092 |
| 500 | 0.0001 | 0.0056 | -0.0013 | 0.0584 | 0.0488 | 0.0452 | 0.0018 | 0.0018 |
| 1000 | 0.0002 | 0.0057 | 0.0045 | 0.0571 | 0.0478 | 0.0440 | 0.0009 | 0.0009 |
| 20% of censoring | | | | | | | | |
| 30 | 0.0001 | 0.0057 | -0.0003 | 0.0583 | 0.0487 | 0.0383 | 0.0252 | 0.0252 |
| 50 | -0.0002 | 0.0058 | 0.0040 | 0.0580 | 0.0477 | 0.0409 | 0.0156 | 0.0156 |
| 100 | -0.0001 | 0.0059 | 0.0022 | 0.0590 | 0.0480 | 0.0391 | 0.0082 | 0.0082 |
| 500 | 0.0006 | 0.0057 | 0.0002 | 0.0569 | 0.0482 | 0.0393 | 0.0016 | 0.0016 |
| 1000 | 0.0001 | 0.0057 | 0.0020 | 0.0573 | 0.0473 | 0.0389 | 0.0008 | 0.0008 |
| 25% of censoring | | | | | | | | |
| 30 | 0.0001 | 0.0059 | -0.0014 | 0.0572 | 0.0520 | 0.0395 | 0.0250 | 0.0250 |
| 50 | -0.0001 | 0.0059 | 0.0031 | 0.0587 | 0.0479 | 0.0378 | 0.0147 | 0.0147 |
| 100 | -0.0007 | 0.0057 | 0.0037 | 0.0575 | 0.0489 | 0.0393 | 0.0074 | 0.0074 |
| 500 | 0.0003 | 0.0058 | -0.0017 | 0.0568 | 0.0473 | 0.0380 | 0.0015 | 0.0015 |
| 1000 | -0.0001 | 0.0058 | -0.0026 | 0.0576 | 0.0469 | 0.0363 | 0.0007 | 0.0007 |

## 5. Comments and conclusions

In this study, we proposed an analytical semi-parametric approach based on L-moments for the estimation of copula parameters, particularly in the presence of right-censored data. We established the asymptotic normality of the proposed

TABLE 2. Bias and RMSE of CCLM estimators for transformed Gumbel copulas under varying censoring levels and moderated correlation.

| $\tau = 0.5, \quad \alpha = 0.2 \to \beta = 1.82$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5% of censoring | | | | | | | | |
| $n$ | $\hat{\alpha}$ | | $\hat{\beta}$ | | $\tau$ | | $c$ | |
| | Bias | RMSE | Bias | RMSE | $\tau_1$ | $\tau_2$ | $c_1$ | $c_2$ |
| 30 | -0.0006 | 0.0117 | 0.0015 | 0.1049 | 0.5043 | 0.4615 | 0.0315 | 0.0315 |
| 50 | 0.0001 | 0.0114 | -0.0049 | 0.1057 | 0.4973 | 0.4581 | 0.0196 | 0.0196 |
| 100 | 0.0003 | 0.0116 | -0.0011 | 0.1049 | 0.4983 | 0.4592 | 0.0093 | 0.0093 |
| 500 | 0.0004 | 0.0117 | -0.0051 | 0.1043 | 0.4994 | 0.4586 | 0.0019 | 0.0019 |
| 1000 | 0.0002 | 0.0118 | 0.0024 | 0.1054 | 0.5011 | 0.4608 | 0.0009 | 0.0009 |
| 10% of censoring | | | | | | | | |
| 30 | -0.0004 | 0.0115 | -0.0039 | 0.1080 | 0.5042 | 0.4247 | 0.0301 | 0.0301 |
| 50 | -0.0003 | 0.0115 | 0.0028 | 0.1035 | 0.4999 | 0.4210 | 0.0178 | 0.0178 |
| 100 | 0.0001 | 0.0115 | -0.0060 | 0.1040 | 0.5016 | 0.4212 | 0.0093 | 0.0093 |
| 500 | 0.0005 | 0.0112 | -0.0045 | 0.1051 | 0.5003 | 0.4220 | 0.0018 | 0.0018 |
| 1000 | -0.0005 | 0.0116 | 0.0028 | 0.1031 | 0.5005 | 0.4218 | 0.0008 | 0.0008 |
| 20% of censoring | | | | | | | | |
| 30 | 0.0002 | 0.0115 | -0.0024 | 0.1056 | 0.5040 | 0.3541 | 0.0262 | 0.0262 |
| 50 | -0.0005 | 0.0114 | 0.0041 | 0.1058 | 0.5018 | 0.3522 | 0.0150 | 0.0150 |
| 100 | 0.0003 | 0.0113 | 0.0033 | 0.1060 | 0.5035 | 0.3546 | 0.0083 | 0.0083 |
| 500 | -0.0002 | 0.0114 | 0.0034 | 0.1054 | 0.4998 | 0.3516 | 0.0016 | 0.0016 |
| 1000 | -0.0002 | 0.0115 | 0.0034 | 0.1042 | 0.5004 | 0.3525 | 0.0008 | 0.0008 |
| 25% of censoring | | | | | | | | |
| 30 | 0.0001 | 0.0113 | -0.0042 | 0.1038 | 0.5009 | 0.3288 | 0.0246 | 0.0246 |
| 50 | -0.0008 | 0.0115 | -0.0012 | 0.1055 | 0.4988 | 0.3209 | 0.0136 | 0.0136 |
| 100 | -0.0005 | 0.0116 | -0.0021 | 0.1049 | 0.4988 | 0.3189 | 0.0076 | 0.0076 |
| 500 | 0.0002 | 0.0114 | -0.0006 | 0.1057 | 0.5004 | 0.3192 | 0.0015 | 0.0015 |
| 1000 | 0.0002 | 0.0113 | -0.0053 | 0.1025 | 0.5002 | 0.3185 | 0.0007 | 0.0007 |

estimator, ensuring the statistical validity of the inferences. To evaluate its performance, extensive simulations were conducted, confirming both the accuracy and robustness of the estimator, even under missing data scenarios and in cases of high-dimensional moments.

The results were further compared with the CCM estimator introduced by Idiou et al. [15], highlighting the superior performance of our method in terms of

TABLE 3. Bias and RMSE of CCLM estimators for transformed Gumbel copulas under varying censoring levels and strong correlation.

| $\tau = 0.7, \quad \alpha = 0.4 \to \beta = 2.78$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5% of censoring | | | | | | | | |
| $n$ | $\hat{\alpha}$ | | $\hat{\beta}$ | | $\tau$ | | $c$ | |
| | Bias | RMSE | Bias | RMSE | $\tau_1$ | $\tau_2$ | $c_1$ | $c_2$ |
| 30 | 0.0005 | 0.0238 | 0.0010 | 0.1597 | 0.7009 | 0.6423 | 0.0309 | 0.0309 |
| 50 | 0.0003 | 0.0233 | 0.0003 | 0.1613 | 0.6988 | 0.6403 | 0.0185 | 0.0185 |
| 100 | 0.0019 | 0.0228 | 0.0097 | 0.1630 | 0.7019 | 0.6437 | 0.0091 | 0.0091 |
| 500 | 0.0001 | 0.0225 | -0.0035 | 0.1568 | 0.7002 | 0.6415 | 0.0018 | 0.0018 |
| 1000 | 0.0002 | 0.0229 | 0.0054 | 0.1600 | 0.7001 | 0.6415 | 0.0009 | 0.0009 |
| 10% of censoring | | | | | | | | |
| 30 | 0.0003 | 0.0235 | -0.0015 | 0.1604 | 0.7025 | 0.5904 | 0.0296 | 0.0296 |
| 50 | -0.0003 | 0.0231 | 0.0034 | 0.1600 | 0.6980 | 0.5858 | 0.0179 | 0.0179 |
| 100 | 0.0001 | 0.0233 | -0.0040 | 0.1589 | 0.6994 | 0.5860 | 0.0090 | 0.0090 |
| 500 | -0.0001 | 0.0231 | -0.0035 | 0.1634 | 0.7004 | 0.5870 | 0.0017 | 0.0017 |
| 1000 | 0.0014 | 0.0232 | 0.0063 | 0.1611 | 0.7006 | 0.5878 | 0.0009 | 0.0009 |
| 20% of censoring | | | | | | | | |
| 30 | -0.0007 | 0.0227 | 0.0065 | 0.1639 | 0.6969 | 0.4878 | 0.0272 | 0.0272 |
| 50 | 0.0005 | 0.0236 | 0.0017 | 0.1593 | 0.6990 | 0.4884 | 0.0153 | 0.0153 |
| 100 | 0.0009 | 0.0227 | -0.0035 | 0.1606 | 0.7017 | 0.4910 | 0.0082 | 0.0082 |
| 500 | -0.0009 | 0.0226 | 0.0054 | 0.1574 | 0.6997 | 0.4884 | 0.0017 | 0.0017 |
| 1000 | -0.0014 | 0.0227 | -0.0027 | 0.1620 | 0.6993 | 0.4884 | 0.0008 | 0.0008 |
| 25% of censoring | | | | | | | | |
| 30 | 0.0011 | 0.0229 | -0.0004 | 0.1624 | 0.7026 | 0.4481 | 0.0237 | 0.0237 |
| 50 | -0.0011 | 0.0229 | 0.0050 | 0.1588 | 0.7002 | 0.4460 | 0.0142 | 0.0142 |
| 100 | 0.0005 | 0.0226 | -0.0025 | 0.1602 | 0.6983 | 0.4404 | 0.0079 | 0.0079 |
| 500 | -0.0010 | 0.0230 | -0.0002 | 0.1600 | 0.7004 | 0.4449 | 0.0015 | 0.0015 |
| 1000 | 0.0001 | 0.0225 | 0.0004 | 0.1606 | 0.7003 | 0.4448 | 0.0008 | 0.0008 |

bias and RMSE, precision, and computational efficiency (see Tables 4, 5). The proposed approach provides explicit analytical forms, which makes it particularly effective and reliable, avoiding numerical challenges often encountered with classical maximum likelihood or Kendall's tau inversion methods [11].

Overall, this work demonstrates the relevance of copula-based methods for modeling the dependence structure of multivariate data, even in complex settings

Table 4. Comparison between the CCLM estimator and the CM estimator for $\widehat{\alpha}$ in transformed Gumbel copulas under different dependence levels and censoring proportions ($n = 500$).

| Cens. | Method | $\tau = 0.05$ | | $\tau = 0.5$ | | $\tau = 0.7$ | |
|---|---|---|---|---|---|---|---|
| | | Bias | RMSE | Bias | RMSE | Bias | RMSE |
| 5% | CCLM | 0.0001 | 0.0058 | 0.0004 | 0.0117 | 0.0001 | 0.0225 |
| | CM | -0.0544 | 0.0628 | -0.0257 | 0.0299 | -0.0126 | 0.0146 |
| 10% | CCLM | 0.0001 | 0.0056 | 0.0005 | 0.0112 | -0.0001 | 0.0231 |
| | CM | -0.0526 | 0.0617 | -0.0261 | 0.0302 | -0.0127 | 0.0147 |
| 20% | CCLM | 0.0006 | 0.0057 | -0.0002 | 0.0114 | -0.0009 | 0.0226 |
| | CM | -0.0524 | 0.0615 | -0.0259 | 0.0302 | -0.0132 | 0.0151 |
| 25% | CCLM | 0.0003 | 0.0058 | 0.0002 | 0.0114 | -0.0010 | 0.0230 |
| | CM | -0.0519 | 0.0606 | -0.0254 | 0.0296 | -0.0126 | 0.0146 |

Table 5. Comparison between the CCLM estimator and the CM estimator for $\widehat{\beta}$ in transformed Gumbel copulas under different dependence levels and censoring proportions ($n = 500$).

| Cens. | Method | $\tau = 0.05$ | | $\tau = 0.5$ | | $\tau = 0.7$ | |
|---|---|---|---|---|---|---|---|
| | | Bias | RMSE | Bias | RMSE | Bias | RMSE |
| 5% | CCLM | 0.0012 | 0.0580 | -0.0051 | 0.1043 | -0.0035 | 0.1568 |
| | CM | 0.2339 | 0.0115 | -0.3415 | 0.0063 | 0.3973 | 0.0042 |
| 10% | CCLM | -0.0013 | 0.0584 | -0.0045 | 0.1051 | -0.0035 | 0.1634 |
| | CM | 0.2294 | 0.0120 | 0.3312 | 0.0065 | 0.3927 | 0.0042 |
| 20% | CCLM | 0.0002 | 0.0569 | 0.0034 | 0.1054 | 0.0054 | 0.1574 |
| | CM | 0.2049 | 0.0114 | 0.3021 | 0.0063 | 0.3458 | 0.0041 |
| 25% | CCLM | -0.0017 | 0.0568 | -0.0002 | 0.1600 | -0.0002 | 0.1600 |
| | CM | 0.1782 | -0.0118 | 0.2982 | 0.0064 | 0.0040 | 0.0040 |

with incomplete observations. The efficiency and robustness of our estimator open avenues for applications in survival analysis, finance, and other fields where multivariate dependence is crucial. Future work will focus on extending this methodology to higher-dimensional copulas and more complex copula models, as well as applying it to real datasets to further validate its practical applicability.

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