

## M-FOLD SYMMETRIC BIUNIVALENT FUNCTIONS DEFINED BY GEGENBAUER POLYNOMIALS: NOVEL SUBCLASSES AND COEFFICIENT BOUNDS

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**ABSTRACT.** In this paper, we use Gegenbauer polynomials to define a novel subclass of holomorphic biunivalent functions which are  $m$ -fold symmetric inside  $\mathbb{U} = \{s : s \in \mathbb{C} \text{ and } |s| < 1\}$ . We compute estimates on initial bounds  $|d_{m+1}|, |d_{2m+1}|$  and the Fekete-Szegő inequalities for functions that are a member of the stated subclasses. Also, several novel results are obtained as special cases of our results.

*Keywords.* Holomorphic function, Biunivalent function,  $m$ -fold symmetric biunivalent function, Gegenbauer polynomials, subordination, Fekete-Szegő inequality.

*2020 Mathematics Subject Classification.* Primary 30C45; Secondary 30C50

### 1. INTRODUCTION AND FUNDAMENTAL NOTIONS

Let  $\mathcal{A}$  represent the class of all holomorphic functions inside the open unit disc  $\mathbb{U} = \{s : s \in \mathbb{C} \text{ and } |s| < 1\}$ , which are of the form:

$$k(s) = s + \sum_{n=2}^{\infty} d_n s^n. \quad (1.1)$$

The function  $k(s)$  of the form (1.1) obviously satisfies the normalization conditions  $k(0) = k'(0) - 1 = 0$ . The subclass of  $\mathcal{A}$  that includes every function that is univalent inside  $\mathbb{U}$  is denoted by  $\mathcal{S}$ .

A disc with a radius of  $1/4$  must be found in the image of  $\mathbb{U}$  under any function  $k(s)$  from  $\mathcal{S}$ , based on the Koebe One-quarter Theorem [8]. Consequently, every univalent function has the inverse  $k^{-1}(r)$  that satisfies

$$k^{-1}(k(s)) = s \ (s \in \mathbb{U}) \ \text{and} \ k(k^{-1}(r)) = r \ \left( |r| < r_0(k); r_0(k) \geq \frac{1}{4} \right),$$

wherever

$$k^{-1}(r) = h(r) = r - d_2 r^2 + (2d_2^2 - d_3) r^3 - (5d_2^3 - 5d_2 d_3 + d_4) r^4 + \dots \quad (1.2)$$

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*Date:* Received: Dec 13, 2025; Revised: Jan 10, 2026; Accepted: Jan 24, 2026.

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If both  $k(s)$  and  $k^{-1}(r)$  are univalent inside  $\mathbb{U}$ , then the function  $k(s) \in \mathcal{A}$  is said to be biunivalent inside  $\mathbb{U}$ . The biunivalent functions class with the form (1.1) is represented by  $\Sigma$ .

Assume that  $k$  and  $f$  are two holomorphic functions inside the open unit disc  $\mathbb{U}$ . If there exists a Schwarz function  $w$  that is holomorphic inside  $\mathbb{U}$  with  $w(0) = 0$  and  $|w(s)| < 1$ , so that  $k(s) = f(w(s))$ , ( $s \in \mathbb{U}$ ), then the function  $k$  is told to be subordinate to  $f$ , denoted  $k \prec f$ . More specifically, if the function  $f$  is univalent in  $\mathbb{U}$ , then  $k \prec f \Leftrightarrow k(0) = f(0)$  and  $k(\mathbb{U}) \subset f(\mathbb{U})$ .

For any function  $k$  in  $\mathcal{S}$ , the univalent function

$$f(s) = (k(s^m))^{\frac{1}{m}} \quad (m \in \mathbb{N}, s \in \mathbb{U}),$$

transforms the open unit disk  $\mathbb{U}$  into an  $m$ -fold symmetrical region (see [13, 15]). The following normalized shape indicates that a function is  $m$ -fold symmetric

$$k(s) = s + \sum_{n=1}^{\infty} d_{mn+1} s^{mn+1} \quad (s \in \mathbb{U}, m \in \mathbb{N}). \quad (1.3)$$

We denote by  $\mathcal{S}_m$  the class of univalent functions which are  $m$ -fold symmetrical inside  $\mathbb{U}$  and normalized based on the series expansion (1.3). In the class  $\mathcal{S}$ , the functions are one-fold symmetrical, meaning that  $m = 1$ .

Every function  $k \in \Sigma$  creates an  $m$ -fold symmetrical biunivalent function for every  $m \in \mathbb{N}$ , according to Srivastava *et al.* [19], which specified that  $m$ -fold symmetrical biunivalent functions are analogous to the idea of  $m$ -fold symmetrical univalent functions. Regarding the normalized form of  $k$  provided based on (1.3), they arrived at the following series expansion for  $k^{-1}$ :

$$\begin{aligned} k^{-1}(r) = h(r) &= r - d_{m+1} r^{m+1} \\ &+ [(m+1) d_{m+1}^2 - d_{2m+1}] r^{2m+1} \\ &- \left[ \frac{1}{2} (2+3m)(1+m) d_{m+1}^3 - (2+3m) d_{m+1} d_{2m+1} + d_{3m+1} \right] r^{3m+1} + \dots \end{aligned} \quad (1.4)$$

We denote by  $\Sigma_m$  the class of  $m$ -fold symmetry biunivalent functions inside  $\mathbb{U}$ . Formula (1.4) for  $m = 1$  is the same as formula (1.2) for the class  $\Sigma$ . Examples of functions belong to the class  $\Sigma_m$  are

$$(-\log(1-s^m))^{\frac{1}{m}}, \left( \frac{s^m}{1-s^m} \right)^{\frac{1}{m}} \text{ and } \left( \frac{1}{2} \log \left( \frac{s^m+1}{1-s^m} \right) \right)^{\frac{1}{m}},$$

along with each of their inverse functions

$$\left( \frac{e^{r^m} - 1}{e^{r^m}} \right)^{\frac{1}{m}}, \left( \frac{r^m}{1+r^m} \right)^{\frac{1}{m}} \text{ and } \left( \frac{e^{2r^m} - 1}{e^{2r^m} + 1} \right)^{\frac{1}{m}}.$$

The importance of orthogonal polynomials in engineering, mathematical physics, probability theory and mathematical statistics has led to a great deal of research on them in the last several years from a variety of angles. From a mathematical

standpoint, orthogonal polynomial often result from ordinary differential equation solutions under certain model-imposed constraints. The classical orthogonal polynomials (Gegenbauer polynomial, Lucas polynomial, Chebyshev polynomial, Jacobi polynomial, Horadam polynomial, Legendre polynomial and Fibonacci polynomial) are the orthogonal polynomials that are most widely utilized in application. Biunivalent functions related with orthogonal polynomials topic of much study recently, including [1, 2, 5, 6, 9, 10, 11, 12, 16, 18]. The definition of a Gegenbauer polynomial generating function for a nonzero real constant  $\alpha$  is

$$H_\alpha(x, s) = \frac{1}{(1 - 2xs + s^2)^\alpha},$$

wherever  $-1 \leq x \leq 1$  and  $s \in \mathbb{U}$ . Since  $H_\alpha$  is holomorphic function inside  $\mathbb{U}$  for fixed  $x$ , it may be expressed as a Taylor series

$$H_\alpha(x, s) = \sum_{n=0}^{\infty} C_n^\alpha(x) s^n,$$

wherever  $C_n^\alpha(x)$  the Gegenbauer polynomial of degree  $n$ .

Clearly, when  $\alpha = 0$ ,  $H_\alpha$  produces nothing. Thus, for  $\alpha = 0$ , we arrive at

$$H_0(x, s) = 1 - \log(1 - 2xs + s^2) = \sum_{n=0}^{\infty} C_n^0(x) s^n.$$

Furthermore, it should be noted that a normalization of  $\alpha$  to be larger than  $-1/2$  is preferred [7, 14, 17]. Additionally, the following recurrence relations may be used to define Gegenbauer polynomials:

$$C_n^\alpha(x) = \frac{1}{n} [2x(n + \alpha - 1)C_{n-1}^\alpha(x) - (n + 2\alpha - 2)C_{n-2}^\alpha(x)],$$

the first few Gegenbauer polynomials are

$$C_0^\alpha(x) = 1, C_1^\alpha(x) = 2\alpha x \text{ and } C_2^\alpha(x) = 2\alpha(1 + \alpha)x^2 - \alpha. \quad (1.5)$$

The polynomials  $C_n^\alpha(x)$  have many special instances: The Chebyshev polynomials are obtained for  $\alpha = 1$  and the Legendre polynomials are obtained for  $\alpha = 1/2$ .

The goal of this work is to submit and study two novel subclasses of m-fold symmetric holomorphic biunivalent functions specified by Gegenbauer polynomials. We get bounds on the initial bounds  $|d_{m+1}|$ ,  $|d_{2m+1}|$  and the Fekete-Szegő inequalities.

## 2. THE SUBCLASS $\mathfrak{B}_{\Sigma_m}(\lambda, x, \alpha)$ AND THE COEFFICIENTS BOUNDS

**Definition 2.1.** A function  $k(s) \in \Sigma_m$  of the shape (1.3) is said to be in the class  $\mathfrak{B}_{\Sigma_m}(\lambda, x, \alpha)$  for  $\lambda \geq 1$ ,  $x \in (1/2, 1]$  and all  $s, r \in \mathbb{U}$  provided the following subordination requirements are met:

$$(1 - \lambda) \frac{k(s)}{s} + \lambda k'(s) \prec H_\alpha(x, s) \quad (2.1)$$

and

$$(1 - \lambda) \frac{h(r)}{r} + \lambda h'(r) \prec H_\alpha(x, r), \quad (2.2)$$

wherever  $h = k^{-1}$  is specified based on (1.4).

The following definition results from choosing  $\lambda = 1$  in Definition 2.1.

**Definition 2.2.** A function  $k(s) \in \Sigma_m$  of the shape (1.3) is said to be in the class  $\mathfrak{B}_{\Sigma_m}(x, \alpha)$  for  $x \in (1/2, 1]$  and all  $s, r \in \mathbb{U}$  provided the following subordination requirements are met:

$$k'(s) \prec H_\alpha(x, s)$$

and

$$h'(r) \prec H_\alpha(x, r),$$

wherever  $h = k^{-1}$  is specified based on (1.4).

*Remark 2.3.*

- (1) The class  $\mathfrak{B}_{\Sigma_m}(\lambda, x, \alpha)$  becomes the class  $\mathfrak{B}_\Sigma(\lambda, x, \alpha)$  submitted and discussed by Amourah *et al.* [4] when  $m = 1$ .
- (2) The class  $\mathfrak{B}_{\Sigma_m}(\lambda, x, \alpha)$  becomes the class  $\mathfrak{B}_\Sigma(x, \alpha)$  submitted and discussed by Amourah *et al.* [4] when  $m = 1$  and  $\lambda = 1$ .

**Theorem 2.4.** Let  $k(s) = s + \sum_{n=1}^{\infty} d_{mn+1} s^{mn+1}$ , belongs to the subclass  $\mathfrak{B}_{\Sigma_m}(\lambda, x, \alpha)$  for  $\lambda \geq 1, x \in (1/2, 1]$  and all  $s, r \in \mathbb{U}$ . Then

$$|d_{m+1}| \leq \frac{2|\alpha|x\sqrt{2x}}{\sqrt{|2(m+2m^2\lambda - m^2\lambda^2)\alpha x^2 + (1+m\lambda)^2(1-2x^2)|}}$$

and

$$|d_{2m+1}| \leq \frac{2|\alpha|x}{2m\lambda+1} + \frac{2(m+1)\alpha^2 x^2}{(m\lambda+1)^2},$$

and for some  $\xi \in \mathbb{R}$ ,

$$|d_{2m+1} - \xi d_{m+1}^2| \leq \begin{cases} \frac{2|\alpha|x}{2m\lambda+1} & \text{if } |m+1-2\xi| \leq \partial_1 \\ \frac{4|m+1-2\xi|\alpha^2 x^3}{|2(m+2m^2\lambda - m^2\lambda^2)\alpha x^2 + (m\lambda+1)^2(1-2x^2)|} & \text{if } |m+1-2\xi| \geq \partial_1, \end{cases}$$

where

$$\partial_1 = \frac{|2(m+2m^2\lambda - m^2\lambda^2)\alpha x^2 + (m\lambda+1)^2(1-2x^2)|}{2(2m\lambda+1)|\alpha|x^2}.$$

*Proof.* Suppose that  $k(s) \in \mathfrak{B}_{\Sigma_m}(\lambda, x, \alpha)$ . Then there are two holomorphic functions  $\omega(s) = p_m s^m + p_{2m} s^{2m} + \dots$  and  $\vartheta(r) = q_m r^m + q_{2m} r^{2m} + \dots$  such that  $\omega(0) = \vartheta(0) = 0, |\omega(s)| < 1, |\vartheta(r)| < 1$  for all  $s, r \in \mathbb{U}$ . From (2.1) and (2.2), we have

$$(1-\lambda) \frac{k(s)}{s} + \lambda k'(s) = 1 + C_1^\alpha(x) \omega(s) + C_2^\alpha(x) [\omega(s)]^2 + C_3^\alpha(x) [\omega(s)]^3 + \dots \quad (2.3)$$

and

$$(1-\lambda) \frac{h(r)}{r} + \lambda h'(r) = 1 + C_1^\alpha(x) \vartheta(r) + C_2^\alpha(x) [\vartheta(r)]^2 + C_3^\alpha(x) [\vartheta(r)]^3 + \dots \quad (2.4)$$

From (2.3) and (2.4), we obtain

$$(1 - \lambda) \frac{k(s)}{s} + \lambda k'(s) = 1 + C_1^\alpha(x) p_m s + [C_1^\alpha(x) p_{2m} + C_2^\alpha(x) p_m^2] s^2 + \dots \quad (2.5)$$

and

$$(1 - \lambda) \frac{h(r)}{r} + \lambda h'(r) = 1 + C_1^\alpha(x) q_m r + [C_1^\alpha(x) q_{2m} + C_2^\alpha(x) q_m^2] r^2 + \dots \quad (2.6)$$

It is well known that if

$$|\omega(s)| = |p_m s^m + p_{2m} s^{2m} + \dots| < 1 \text{ and } |\vartheta(r)| = |q_m r^m + q_{2m} r^{2m} + \dots| < 1,$$

then  $|p_{im}| \leq 1$  and  $|q_{im}| \leq 1$  for each  $i \in \mathbb{N}$ .

When we equate the coefficients in ((2.5) and (2.6)), we get

$$(1 + m\lambda) d_{m+1} = C_1^\alpha(x) p_m, \quad (2.7)$$

$$(1 + 2m\lambda) d_{2m+1} = C_1^\alpha(x) p_{2m} + C_2^\alpha(x) p_m^2, \quad (2.8)$$

$$-(1 + m\lambda) d_{m+1} = C_1^\alpha(x) q_m, \quad (2.9)$$

and

$$(2m\lambda + 1) [(m + 1) d_{m+1}^2 - d_{2m+1}] = C_1^\alpha(x) q_{2m} + C_2^\alpha(x) q_m^2. \quad (2.10)$$

Using ((2.7) and (2.9)), we obtain

$$p_m = -q_m \quad (2.11)$$

and

$$2(m\lambda + 1)^2 d_{m+1}^2 = [C_1^\alpha(x)]^2 (p_m^2 + q_m^2). \quad (2.12)$$

When we add ((2.8) and (2.10)), we get

$$(1 + 2m\lambda) (m + 1) d_{m+1}^2 = C_1^\alpha(x) (p_{2m} + q_{2m}) + C_2^\alpha(x) (p_m^2 + q_m^2). \quad (2.13)$$

Using (2.12) in (2.13) and simplifying, we get

$$d_{m+1}^2 = \frac{[C_1^\alpha(x)]^3 (p_{2m} + q_{2m})}{(m + 1) (1 + 2m\lambda) [C_1^\alpha(x)]^2 - 2(1 + m\lambda)^2 C_2^\alpha(x)}. \quad (2.14)$$

Using (1.5), (2.14) becomes

$$|d_{m+1}| \leq \frac{2|\alpha|x\sqrt{2x}}{\sqrt{|2(m + 2m^2\lambda - m^2\lambda^2)\alpha x^2 + (1 + m\lambda)^2(1 - 2x^2)|}}.$$

When (2.10) is subtracted from (2.8), we get

$$(1 + 2m\lambda) [2d_{2m+1} - (m + 1) d_{m+1}^2] = C_1^\alpha(x) (p_{2m} - q_{2m}) + C_2^\alpha(x) (p_m^2 - q_m^2). \quad (2.15)$$

By applying the equations (2.11), (2.12), together with (2.15), it can be concluded that

$$d_{2m+1} = \frac{C_1^\alpha(x) (p_{2m} - q_{2m})}{2(2m\lambda + 1)} + \frac{(m + 1) (p_m^2 + q_m^2) [C_1^\alpha(x)]^2}{4(m\lambda + 1)^2}.$$

Now, using (1.5), we conclude

$$|d_{2m+1}| \leq \frac{2|\alpha|x}{2m\lambda + 1} + \frac{2(m + 1)\alpha^2 x^2}{(m\lambda + 1)^2}.$$

Lastly, using ((2.14) and (2.15)) for some  $\xi \in \mathbb{R}$ , we get

$$\begin{aligned} d_{2m+1} - \xi d_{m+1}^2 &= \frac{C_1^\alpha(x) (p_{2m} - q_{2m})}{2(2m\lambda + 1)} \\ &+ \frac{(m+1 - 2\xi) [C_1^\alpha(x)]^3 (p_{2m} + q_{2m})}{2((2m\lambda + 1)(m+1) [C_1^\alpha(x)]^2 - 2(m\lambda + 1)^2 C_2^\alpha(x))} \\ &= \frac{C_1^\alpha(x)}{2} \left[ \left( Z(\xi) + \frac{1}{(2m\lambda + 1)} \right) p_{2m} + \left( Z(\xi) - \frac{1}{(2m\lambda + 1)} \right) q_{2m} \right], \end{aligned}$$

where

$$Z(\xi) = \frac{(m+1 - 2\xi) [C_1^\alpha(x)]^2}{(2m\lambda + 1)(m+1) [C_1^\alpha(x)]^2 - 2(m\lambda + 1)^2 C_2^\alpha(x)}.$$

Thus, we conclude that

$$|d_{2m+1} - \xi d_{m+1}^2| \leq \begin{cases} \frac{|C_1^\alpha(x)|}{2m\lambda+1} & \text{if } 0 \leq |Z(\xi)| \leq \frac{1}{2m\lambda+1} \\ |C_1^\alpha(x)| |Z(\xi)| & \text{if } |Z(\xi)| \geq \frac{1}{2m\lambda+1} \end{cases}.$$

After some computations, we get

$$|d_{2m+1} - \xi d_{m+1}^2| \leq \begin{cases} \frac{2|\alpha|x}{2m\lambda+1} & \text{if } |m+1 - 2\xi| \leq \partial_1 \\ \frac{4|m+1-2\xi|\alpha^2 x^3}{|2(m+2m^2\lambda - m^2\lambda^2)\alpha x^2 + (m\lambda+1)^2(1-2x^2)|} & \text{if } |m+1 - 2\xi| \geq \partial_1, \end{cases}$$

where

$$\partial_1 = \frac{|2(m+2m^2\lambda - m^2\lambda^2)\alpha x^2 + (m\lambda+1)^2(1-2x^2)|}{2(2m\lambda+1)|\alpha|x^2}.$$

This proves Theorem 2.4. □

We get the following outcome by using  $\lambda = 1$  in Theorem 2.4.

**Corollary 2.5.** *Let  $k(s) = s + \sum_{n=1}^{\infty} d_{mn+1} s^{mn+1}$ , belongs to the subclass  $\mathfrak{B}_{\Sigma_m}(x)$  for  $x \in (1/2, 1]$  and all  $s, r \in \mathbb{U}$ . Then*

$$|d_{m+1}| \leq \frac{2|\alpha|x\sqrt{2x}}{\sqrt{|2m(m+1)\alpha x^2 + (m+1)^2(1-2x^2)|}}$$

and

$$|d_{2m+1}| \leq \frac{2x|\alpha|}{1+2m} + \frac{2(1+m)\alpha^2 x^2}{(1+m)^2},$$

and for some  $\xi \in \mathbb{R}$ ,

$$|d_{2m+1} - \xi d_{m+1}^2| \leq \begin{cases} \frac{2|\alpha|x}{2m+1} & \text{if } |m+1 - 2\xi| \leq \frac{|2m(m+1)\alpha x^2 + (m+1)^2(1-2x^2)|}{2(2m+1)|\alpha|x^2} \\ \frac{4|m+1-2\xi|\alpha^2 x^3}{|2m(m+1)\alpha x^2 + (m+1)^2(1-2x^2)|} & \text{if } |m+1 - 2\xi| \geq \frac{|2m(m+1)\alpha x^2 + (m+1)^2(1-2x^2)|}{2(2m+1)|\alpha|x^2}. \end{cases}$$

*Remark 2.6.* If we take the following options in Theorem 2.4

- (1) For  $m = 1$ , we obtain the results by Amourah *et al.* [4].

(2) For  $m = 1$  and  $\lambda = 1$ , we get the results by Amourah *et al.* [4].

### 3. THE SUBCLASS $\mathcal{SC}_{\Sigma_m}(\lambda, x, \alpha)$ AND THE COEFFICIENTS BOUNDS

**Definition 3.1.** A function  $k(s) \in \Sigma_m$  of the shape (1.3) is said to be in the class  $\mathcal{SC}_{\Sigma_m}(\lambda, x, \alpha)$  for  $0 \leq \lambda \leq 1, x \in (1/2, 1]$  and all  $s, r \in \mathbb{U}$  provided the following subordination requirements are met:

$$(1 - \lambda) \frac{sk'(s)}{k(s)} + \lambda \left( 1 + \frac{sk''(s)}{k'(s)} \right) \prec H_\alpha(x, s) \quad (3.1)$$

and

$$(1 - \lambda) \frac{rh'(r)}{h(r)} + \lambda \left( 1 + \frac{rh''(r)}{h'(r)} \right) \prec H_\alpha(x, s), \quad (3.2)$$

wherever  $h = k^{-1}$  is specified based on (1.4).

The following definition results from choosing  $\lambda = 0$  in Definition 3.1.

**Definition 3.2.** A function  $k(s) \in \Sigma_m$  of the shape (1.3) is said to be in the class  $\mathcal{S}_{\Sigma_m}(x, \alpha)$  for  $x \in (1/2, 1]$  and all  $s, r \in \mathbb{U}$  provided the following subordination requirements are met:

$$\frac{sk'(s)}{k(s)} \prec H_\alpha(x, s)$$

and

$$\frac{rh'(r)}{h(r)} \prec H_\alpha(x, s),$$

wherever  $h = k^{-1}$  is specified based on (1.4).

The following definition results from choosing  $\lambda = 1$  in Definition 3.1.

**Definition 3.3.** A function  $k(s) \in \Sigma_m$  of the shape (1.3) is said to be in the class  $\mathcal{C}_{\Sigma_m}(x, \alpha)$  for  $x \in (1/2, 1]$  and all  $s, r \in \mathbb{U}$  provided the following subordination requirements are met:

$$1 + \frac{sk''(s)}{k'(s)} \prec H_\alpha(x, s)$$

and

$$1 + \frac{rh''(r)}{h'(r)} \prec H_\alpha(x, s),$$

wherever  $h = k^{-1}$  is specified based on (1.4).

*Remark 3.4.*

- (1) The class  $\mathcal{SC}_{\Sigma_m}(\lambda, x, \alpha)$  becomes the class  $B^*(\alpha)$  submitted and discussed by Amourah *et al.* [3] when  $m = 1$  and  $\lambda = 0$ .
- (2) The class  $\mathcal{SC}_{\Sigma_m}(\lambda, x, \alpha)$  becomes the class  $SM_\Sigma^\alpha(x, 1)$  submitted and discussed by Swamy and Atshan [20] when  $m=1$  and  $\lambda = 1$ .
- (3) The class  $\mathcal{SC}_{\Sigma_m}(\lambda, x, \alpha)$  becomes the class  $\mathfrak{M}_\Sigma(\phi, t, \varphi)$  submitted and discussed by Younis *et al.* [21] when  $m = 1$ .

**Theorem 3.5.** Let  $k(s) = s + \sum_{n=1}^{\infty} d_{mn+1} s^{mn+1}$ , belongs to the subclass  $\mathcal{SC}_{\Sigma_m}(\lambda, x, \alpha)$  for  $0 \leq \lambda \leq 1, x \in (1/2, 1]$  and all  $s, r \in \mathbb{U}$ . Then

$$|d_{m+1}| \leq \frac{2|\alpha|x\sqrt{2x}}{m\sqrt{|2(1-m^2\lambda^2)\alpha x^2 + (m\lambda+1)^2(1-2x^2)|}}$$

and

$$|d_{2m+1}| \leq \frac{|\alpha|x}{m(2m\lambda+1)} + \frac{2(m+1)\alpha^2 x^2}{m^2(m\lambda+1)^2},$$

and for some  $\xi \in \mathbb{R}$ ,

$$|d_{2m+1} - \xi d_{m+1}^2| \leq \begin{cases} \frac{|\alpha|x}{m(2m\lambda+1)} & \text{if } |m+1-2\xi| \leq \partial_2 \\ \frac{4|m+1-2\xi|\alpha^2 x^3}{m^2|2(1-m^2\lambda^2)\alpha x^2 + (1+m\lambda)^2(1-2x^2)|} & \text{if } |m+1-2\xi| \geq \partial_2, \end{cases}$$

where

$$\partial_2 = \frac{m|2(1-m^2\lambda^2)\alpha x^2 + (1+m\lambda)^2(1-2x^2)|}{4(2m\lambda+1)|\alpha|x^2}.$$

*Proof.* Suppose that  $k(s) \in \mathcal{SC}_{\Sigma_m}(\lambda, x, \alpha)$ . Then there are holomorphic functions  $\omega(s) = p_m s^m + p_{2m} s^{2m} + \dots$  and  $\vartheta(r) = q_m r^m + q_{2m} r^{2m} + \dots$  such that  $\omega(0) = \vartheta(0) = 0, |\omega(s)| < 1, |\vartheta(r)| < 1$  for all  $s, r \in \mathbb{U}$ . From (3.1) and (3.2), we have

$$(1-\lambda) \frac{sk'(s)}{k(s)} + \lambda \left(1 + \frac{sk''(s)}{k'(s)}\right) = 1 + C_1^\alpha(x)\omega(s) + C_2^\alpha(x)[\omega(s)]^2 + C_3^\alpha(x)[\omega(s)]^3 + \dots \quad (3.3)$$

and

$$(1-\lambda) \frac{rh'(r)}{h(r)} + \lambda \left(1 + \frac{rh''(r)}{h'(r)}\right) = 1 + C_1^\alpha(x)\vartheta(r) + C_2^\alpha(x)[\vartheta(r)]^2 + C_3^\alpha(x)[\vartheta(r)]^3 + \dots \quad (3.4)$$

From (3.3) and (3.4), we obtain

$$(1-\lambda) \frac{sk'(s)}{k(s)} + \lambda \left(1 + \frac{sk''(s)}{k'(s)}\right) = 1 + C_1^\alpha(x)p_m s + [C_1^\alpha(x)p_{2m} + C_2^\alpha(x)p_m^2]s^2 + \dots \quad (3.5)$$

and

$$(1-\lambda) \frac{rh'(r)}{h(r)} + \lambda \left(1 + \frac{rh''(r)}{h'(r)}\right) = 1 + C_1^\alpha(x)q_m r + [C_1^\alpha(x)q_{2m} + C_2^\alpha(x)q_m^2]r^2 + \dots \quad (3.6)$$

It's well knowledge that if

$$|\omega(s)| = |p_m s^m + p_{2m} s^{2m} + \dots| < 1$$

and

$$|\vartheta(r)| = |q_m r^m + q_{2m} r^{2m} + \dots| < 1,$$

then  $|p_{im}| \leq 1$  and  $|q_{im}| \leq 1$  for each  $i \in \mathbb{N}$ .

When we equate the coefficients in ((3.5) and (3.6)), we get

$$m(\lambda m + 1)d_{m+1} = C_1^\alpha(x)p_m, \quad (3.7)$$

$$\begin{aligned} 2m(2\lambda m + 1)d_{2m+1} - m(1 + 2\lambda m + \lambda m^2)d_{m+1}^2 \\ = C_1^\alpha(x)p_{2m} + C_2^\alpha(x)p_m^2, \end{aligned} \quad (3.8)$$

$$-m(\lambda m + 1)d_{m+1} = C_1^\alpha(x)q_m, \quad (3.9)$$

and

$$\begin{aligned} m(1 + 2m + 2\lambda m + 3\lambda m^2)d_{m+1}^2 - 2m(2\lambda m + 1)d_{2m+1} \\ = C_1^\alpha(x)q_{2m} + C_2^\alpha(x)q_m^2. \end{aligned} \quad (3.10)$$

Using ((3.7) and (3.9)), we get

$$p_m = -q_m \quad (3.11)$$

and

$$2m^2(\lambda m + 1)^2 d_{m+1}^2 = [C_1^\alpha(x)]^2 (p_m^2 + q_m^2). \quad (3.12)$$

When we add ((3.8) and (3.10)), we get

$$2m^2(\lambda m + 1)d_{m+1}^2 = C_1^\alpha(x)(p_{2m} + q_{2m}) + C_2^\alpha(x)(p_m^2 + q_m^2). \quad (3.13)$$

Using (3.12) in (3.13) and simplifying, we get

$$d_{m+1}^2 = \frac{[C_1^\alpha(x)]^3 (p_{2m} + q_{2m})}{2m^2(m\lambda + 1)[C_1^\alpha(x)]^2 - 2m^2(m\lambda + 1)^2 C_2^\alpha(x)}. \quad (3.14)$$

Using (1.5), (3.14) becomes

$$|d_{m+1}| \leq \frac{2|\alpha|x\sqrt{2x}}{m\sqrt{|2(1 - m^2\lambda^2)\alpha x^2 + (m\lambda + 1)^2(1 - 2x^2)|}}.$$

When (3.10) is subtracted from (3.8), we get

$$2m(1 + 2\lambda m)[2d_{2m+1} - (m + 1)d_{m+1}^2] = C_1^\alpha(x)(p_{2m} - q_{2m}) + C_2^\alpha(x)(p_m^2 - q_m^2). \quad (3.15)$$

It follows from (3.11), (3.12) and (3.15) that

$$d_{2m+1} = \frac{C_1^\alpha(x)(p_{2m} - q_{2m})}{4m(2m\lambda + 1)} + \frac{(m + 1)[C_1^\alpha(x)]^2(p_m^2 + q_m^2)}{4m^2(m\lambda + 1)^2}.$$

Now, using (1.5), we conclude

$$|d_{2m+1}| \leq \frac{|\alpha|x}{m(2m\lambda + 1)} + \frac{2(m + 1)\alpha^2 x^2}{m^2(m\lambda + 1)^2}.$$

Lastly, using ((3.14) and (3.15)) for some  $\xi \in \mathbb{R}$ , we get

$$\begin{aligned} d_{2m+1} - \xi d_{m+1}^2 &= \frac{C_1^\alpha(x) (p_{2m} - q_{2m})}{4m(2\lambda m + 1)} \\ &+ \frac{(m + 1 - 2\xi) [C_1^\alpha(x)]^3 (p_{2m} + q_{2m})}{4m^2(\lambda m + 1) [C_1^\alpha(x)]^2 - 4m^2(\lambda m + 1)^2 C_2^\alpha(x)} \\ &= \frac{C_1^\alpha(x)}{4m} \left[ \left( Z(\xi) + \frac{1}{(2m\lambda + 1)} \right) p_{2m} + \left( Z(\xi) - \frac{1}{(2m\lambda + 1)} \right) q_{2m} \right], \end{aligned}$$

where

$$Z(\xi) = \frac{(m + 1 - 2\xi) [C_1^\alpha(x)]^2}{m(\lambda m + 1) [C_1^\alpha(x)]^2 - m(\lambda m + 1)^2 C_2^\alpha(x)}.$$

Thus, we conclude that

$$|d_{2m+1} - \xi d_{m+1}^2| \leq \begin{cases} \frac{|C_1^\alpha(x)|}{2m(2m\lambda + 1)} & \text{if } 0 \leq |Z(\xi)| \leq \frac{1}{2m\lambda + 1} \\ \frac{|C_1^\alpha(x)||Z(\xi)|}{2m} & \text{if } |Z(\xi)| \geq \frac{1}{2m\lambda + 1}. \end{cases}$$

After some computations, we get

$$|d_{2m+1} - \xi d_{m+1}^2| \leq \begin{cases} \frac{|\alpha|x}{m(2m\lambda + 1)} & \text{if } |m + 1 - 2\xi| \leq \partial_2 \\ \frac{4|m+1-2\xi|\alpha^2 x^3}{m^2|2(1-m^2\lambda^2)\alpha x^2 + (1+m\lambda)^2(1-2x^2)|} & \text{if } |m + 1 - 2\xi| \geq \partial_2, \end{cases}$$

where

$$\partial_2 = \frac{m|2(1 - m^2\lambda^2)\alpha x^2 + (1 + m\lambda)^2(1 - 2x^2)|}{4(2m\lambda + 1)|\alpha|x^2}.$$

This proves Theorem 3.5.

We get the following outcome by using  $\lambda = 0$  in Theorem 3.5.  $\square$

**Corollary 3.6.** Let  $k(s) = s + \sum_{n=1}^{\infty} d_{mn+1} s^{mn+1}$ , belongs to the subclass  $\mathcal{S}_{\Sigma_m}(x, \alpha)$  for  $x \in (1/2, 1]$  and all  $s, r \in \mathbb{U}$ . Then

$$|d_{m+1}| \leq \frac{2x|\alpha|\sqrt{2x}}{m\sqrt{|1 - 2(1 - \alpha)x^2|}}$$

and

$$|d_{2m+1}| \leq \frac{|\alpha|x}{m} + \frac{2(m+1)\alpha^2 x^2}{m^2},$$

and for some  $\xi \in \mathbb{R}$ ,

$$|d_{2m+1} - \xi d_{m+1}^2| \leq \begin{cases} \frac{|\alpha|x}{m} & \text{if } |m + 1 - 2\xi| \leq \frac{m|1-2(1-\alpha)x^2|}{4|\alpha|x^2} \\ \frac{4|m+1-2\xi|\alpha^2 x^3}{m^2|1-2(1-\alpha)x^2|} & \text{if } |m + 1 - 2\xi| \geq \frac{m|1-2(1-\alpha)x^2|}{4|\alpha|x^2}. \end{cases}$$

We get the following outcome by using  $\lambda = 1$  in Theorem 3.5.

**Corollary 3.7.** Let  $k(s) = s + \sum_{n=1}^{\infty} d_{mn+1} s^{mn+1}$ , belongs to the subclass  $\mathcal{C}_{\Sigma_m}(x, \alpha)$  for  $x \in (1/2, 1]$  and all  $s, r \in \mathbb{U}$ . Then

$$|d_{m+1}| \leq \frac{2x|\alpha|\sqrt{2x}}{m\sqrt{|2(1-m^2)\alpha x^2 + (m+1)^2(1-2x^2)|}}$$

and

$$|d_{2m+1}| \leq \frac{|\alpha|x}{m(2m+1)} + \frac{2(m+1)\alpha^2 x^2}{m^2(m+1)^2},$$

and for some  $\xi \in \mathbb{R}$ ,

$$|d_{2m+1} - \xi d_{m+1}^2| \leq \begin{cases} \frac{|\alpha|x}{m(2m+1)} \\ \text{if } |m+1-2\xi| \leq \frac{m|2(1-m^2)\alpha x^2 + (m+1)^2(1-2x^2)|}{4(2m+1)|\alpha x^2|} \\ \frac{4|m+1-2\xi|\alpha^2 x^3}{m^2|2(1-m^2)\alpha x^2 + (m+1)^2(1-2x^2)|} \\ \text{if } |m+1-2\xi| \geq \frac{m|2(1-m^2)\alpha x^2 + (m+1)^2(1-2x^2)|}{4(2m+1)|\alpha x^2|}. \end{cases}$$

*Remark 3.8.* If we take the following options in Theorem 3.5

- (1) For  $m = 1$  and  $\lambda = 0$ , we obtain the outcomes by Amourah *et al.* [3].
- (2) For  $m = 1$  and  $\lambda = 1$ , we get the results by Swamy and Atshan [20].
- (3) For  $m = 1$ , we get the outcomes by Younis *et al.* [21].

#### 4. CONCLUSION

In this study, the results presented are novel where by using the concept of subordination, we have submitted two novel subclasses of holomorphic and  $m$ -fold symmetric biunivalent function inside  $\mathbb{U} = \{s : s \in \mathcal{C} \text{ and } |s| < 1\}$ , which are characterized using the Gegenbauer polynomials. We get upper bounds for the first two coefficients  $|d_{m+1}|$ ,  $|d_{2m+1}|$  and Fekete-Szegő inequality  $|d_{2m+1} - \xi d_{m+1}^2|$  for the functions in these subclasses. Also, by specializing the parameters in our main results, we have made several novel discoveries.

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