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BOUNDS FOR THE ZEROS OF QUATERNIONIC POLYNOMIALS

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Abstract. This paper develops new bounds for the zeros of polynomials with quaternionic coefficients. We establish eigenvalue localization theorems for quaternionic matrices and derive estimates for their spectral radii. Several explicit bounds are presented for the zeros of both simple and unilateral quaternionic polynomials, including generalizations of classical polynomial bounds. The theoretical framework employs companion matrix constructions and norm-based techniques. We demonstrate applications to stability analysis of discrete-time quaternionic systems, providing methods for estimating stability margins. Numerical examples validate the effectiveness of the proposed bounds and demonstrate their improvement over existing results in the literature.

Keywords. Quaternionic matrices, Right eigenvalues, Quaternionic polynomials, Right spectral radius, Companion quaternionic matrix, Quaternionic discrete systems, Stability
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1. Introduction

Quaternions, first introduced by Sir William Rowan Hamilton in 1843, constitute a noncommutative division algebra that extends the complex number system. Due to their unique algebraic properties, quaternions have found significant applications in diverse scientific and engineering domains, including quantum mechanics [1], computer graphics and vision [8], signal and image processing [14, 15, 30], and geometric algebra [18, 20]. Their particular utility in representing three-dimensional and four-dimensional rotations makes them especially valuable in computer vision and aerospace engineering [20]. Additional applications of quaternions can be found in differential equations [9, 10], numerical linear algebra [12], eigenvalue problems [11] and spectral analysis [19].

The noncommutative nature of quaternion multiplication necessitates multiple definitions of polynomials over quaternions. Unlike the real or complex case, coefficients can appear on either side of the indeterminate, producing distinct

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algebraic objects [16, 24, 28, 29]. In this paper, we focus on two primary forms:

$$p_l(z) := q_m z^m + q_{m-1} z^{m-1} + \cdots + q_1 z + q_0, \tag{1.1}$$

$$p_r(z) := z^m q_m + z^{m-1} q_{m-1} + \cdots + z q_1 + q_0, \tag{1.2}$$

where $q_j, z \in \mathbb{H}$ for $j = 0, 1, \ldots, m$. These are commonly termed *simple quaternionic polynomials* ([16]). A detailed discussion of their properties is provided in section 4.

The problem of finding zeros for quaternionic polynomials was first investigated by Niven [23], with subsequent algorithmic developments by Serôdio et al. [29]. Subsequent research on root-finding and structural properties appears in [13, 16, 24, 27]. Despite these contributions, relatively little literature addresses the systematic *bounding* of such zeros. The initial bounds were established by Janovská & Opfer [16], Opfer [24], while Kalantari [17] derived norm bounds for the zeros of unilateral quaternionic polynomials. Accurate localization of polynomial zeros is crucial for stability analysis of quaternionic discrete-time systems and for estimating stability margins.

This paper proposes new bounds for the zeros of quaternionic polynomials using companion matrix constructions and spectral radius estimates. Our specific contributions are as follows:

- We present norm-based bounds for right eigenvalues of quaternionic matrices and establish their connection to polynomial zero localization.
- We derive several explicit bounds—including generalizations of the Cauchy, Montel, and Carmichael–Mason bounds—for simple quaternionic polynomials.
- We extend these results to unilateral quaternionic polynomials and demonstrate that our bounds are often sharper than existing ones in the literature.
- We illustrate the practical utility of these bounds in estimating stability margins for quaternionic discrete-time systems.
- We provide numerical examples to validate the theoretical developments and demonstrate comparative performance.

The remainder of this paper is organized as follows. Section 2 introduces notation and preliminary concepts. Section 3 discusses eigenvalue estimates for quaternionic matrices. Section 4 presents bounds for the zeros of simple quaternionic polynomials. Section 5 extends these bounds to unilateral polynomials. Section 6 provides an application to system stability analysis. Finally, section 7 illustrates the results with numerical examples.

2. Notation and Preliminaries

Throughout this paper, we adopt the following notation. Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. The set of real quaternions is defined as:

$$\mathbb{H} := \big\{ q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} : a_0, a_1, a_2, a_3 \in \mathbb{R} \big\},$$

with multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are imaginary units that commute with real numbers. The set $\mathbb{H}$ forms a skew field. For $q \in \mathbb{H}$, we denote its conjugate and modulus by $\overline{q}$ and $|q|$, respectively, defined as:

$$\overline{q} := a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}, \quad |q| := \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}.$$

Moreover, there exists a unique ordered pair $(p_0, p_1) \in \mathbb{C}^2$ such that $q = p_0 + p_1\mathbf{j}$.

Let $\mathbb{H}^n$ be a right vector space over $\mathbb{H}$. For $x, y \in \mathbb{H}^n$, we denote the inner product by $\langle x, y \rangle = y^*x$, where $x, y \in \mathbb{H}^n$. The corresponding norm on $\mathbb{H}^n$ is $\|x\| = \sqrt{\langle x, x \rangle}$.

We denote by $\mathbb{R}^{n\times n}$, $\mathbb{C}^{n\times n}$, and $\mathbb{H}^{n\times n}$ the sets of $n \times n$ real, complex, and quaternionic matrices, respectively. For $A \in \mathbb{H}^{n\times n}$, we denote its transpose and conjugate-transpose by A^T and $A^* = \overline{A}^T \in \mathbb{H}^{n\times n}$, respectively.

Furthermore, we define the following matrix norms:

- Frobenius norm: $\|A\|_F := \left(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2\right)^{1/2} = (\text{trace}(A^*A))^{1/2}$
- Spectral (operator) norm: $\|A\|_2 := \max\{\|Ax\| : x \in \mathbb{H}^n, \|x\| = 1\}$
- Maximum column sum norm: $\|A\|_1 := \max\limits_{1\le j\le n} \sum\limits_{i=1}^n |a_{ij}|$
- Maximum row sum norm: $\|A\|_\infty := \max\limits_{1\le i\le n} \sum\limits_{j=1}^n |a_{ij}|$

The sets of right, standard right, and left eigenvalues of $A \in \mathbb{H}^{n\times n}$ are denoted by $\Lambda_r(A)$, $\Lambda_s(A)$, and $\Lambda_l(A)$, respectively. Additionally, the set of complex eigenvalues of $A \in \mathbb{C}^{n\times n}$ is denoted by $\Lambda_c(A)$.

Definition 2.1. Two quaternions p and q are similar if there exists $\rho(\neq 0) \in \mathbb{H}$ such that $p = \rho^{-1}q\rho$. Note that p is similar to q if and only if $\Re(p) = \Re(q)$ and $|\Im(p)| = |\Im(q)|$. The equivalence class of q is denoted by

$$[q] := \{\rho^{-1}q\rho : 0 \neq \rho \in \mathbb{H}\}.$$

Due to the non-commutative nature of quaternion multiplication, there are two distinct types of eigenvalues for quaternionic matrices: right eigenvalues and left eigenvalues. We now define these concepts formally.

Definition 2.2. Given $A \in \mathbb{H}^{n\times n}$, $\lambda \in \mathbb{H}$ is called a *right eigenvalue* of A if $Ax = x\lambda$ for some nonzero $x \in \mathbb{H}^n$. Moreover, if $\rho \in \mathbb{H}$, $\rho \neq 0$, then $\rho^{-1}\lambda\rho$ is also a right eigenvalue of the quaternionic matrix A; that is, all elements of the equivalence class $[\lambda]$ are right eigenvalues. The set of right eigenvalues of A is defined as

$$\Lambda_r(A) := \{\lambda \in \mathbb{H} : Ax = x\lambda \text{ for some nonzero } x \in \mathbb{H}^n\}.$$

From Definition 2.2, it follows that if λ is a complex right eigenvalue of a quaternionic matrix A, then A has infinitely many eigenvalues up to equivalence class.

We define the mapping $\Psi : \mathbb{H}^{n\times n} \to \mathbb{C}^{2n\times 2n}$ by

$$\Psi_A = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix},$$

where $A_i \in \mathbb{C}^{n\times n}$ for $i = 1, 2$. The matrix Ψ_A is called the adjoint complex matrix of A.

Definition 2.3 ([7]; [22])**.** Let $A \in \mathbb{H}^{n\times n}$. The adjoint complex matrix Ψ_A has $2n$ complex eigenvalues, among which n have nonnegative imaginary parts. These are called the standard right eigenvalues of A.

Definition 2.4. Given $A \in \mathbb{H}^{n\times n}$, $\lambda \in \mathbb{H}$ is called a left eigenvalue of A if $Ax = \lambda x$ for some nonzero $x \in \mathbb{H}^n$. The set of left eigenvalues of A is defined as

$$\Lambda_l(A) := \{\lambda \in \mathbb{H} : Ax = \lambda x \text{ for some nonzero } x \in \mathbb{H}^n\}.$$

We will frequently use the following established results in developing our theory.

Lemma 2.5 ([32], Lemma 2.3)**.** *Let $A, B \in \mathbb{H}^{n\times n}$. If A is similar to B, then they have the same right eigenvalues.*

Lemma 2.6 ([29], Proposition 2)**.** *If $p_l(z)$ is a simple quaternionic polynomial of degree m, then its zeros belong to at most m equivalence classes of quaternions.*

Lemma 2.7 ([29], Corollary 2)**.** *If a quaternionic polynomial has two distinct zeros belonging to the same equivalence class, then all quaternions in that class are also zeros of the polynomial.*

Theorem 2.8 ([24], Theorem 4.2)**.** *Let $p(x)$ be a simple monic polynomial over $\mathbb{H}$ of degree m with $q_0 \neq 0$. Then all zeros of $p(x)$ are contained in the ball*

$$S := \{z \in \mathbb{H} : |z| \le R\}, \quad \text{where } R := \max\left\{1, \sum_{j=0}^{m-1} |q_j|\right\}.$$

Theorem 2.9 ([16], Lemma 4.1)**.** *The two polynomials $\tilde{p}_m(x) := \sum_{j=0}^{m} x^j q_j$ and $p_m(x) := \sum_{j=0}^{m} \overline{q_j} x^j$ have the same real and spherical zeros. For non-real isolated zeros, we have*

$$p_m(x) = 0 \iff \tilde{p}_m(\overline{x}) = 0.$$

3. Localization theorem for left and right eigenvalue of quaternionic matrices

We will frequently use the following theorem to find the bounds for the zeros of quaternionic polynomial.

Proposition 3.1. *Let $A = (a_{ij}) \in \mathbb{H}^{n\times n}$. Then the left and right spectral radii of A are bounded by its maximum row sum norm, i.e.,*

$$\rho_l(A),\ \rho_r(A) \le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| = \|A\|_\infty.$$

Proof. Suppose λ is a right eigenvalue of A with nonzero eigenvector $x = [x_1, \dots, x_n]^T$. Let $|x_t| = \max_i |x_i| > 0$. From the t-th equation of $Ax = x\lambda$, we have

$$\sum_{j=1}^{n} a_{tj} x_j = x_t \lambda.$$

Taking absolute values and applying the triangle inequality yields

$$|x_t|\,|\lambda| \leq \sum_{j=1}^{n} |a_{tj}|\,|x_j| \leq |x_t| \sum_{j=1}^{n} |a_{tj}|.$$

Thus

$$|\lambda| \leq \sum_{j=1}^{n} |a_{tj}| \leq \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| = \|A\|_\infty.$$

Since this holds for every right eigenvalue λ, we obtain $\rho_r(A) \leq \|A\|_\infty$. The proof for left eigenvalues is analogous. □

To find bounds of spectral radius, we build up the machinery for proof of our results. We begin some basic concepts that will be required for development of our theory.

Definition 3.2. The left and right spectral radius $\rho_r(A)$ of a matrix $A \in \mathbb{H}^{n\times n}$ are

$$\rho_l(A) = \max\big\{|\lambda| : \lambda \in \Lambda_l(A)\big\} \quad \text{and} \quad \rho_r(A) = \max\big\{|\lambda| : \lambda \in \Lambda_r(A)\big\}.$$

Theorem 3.3 (Theorem 3, [31])**.** *Let $A \in \mathbb{H}^{n\times n}$. Then*

$$\rho_l(A),\, \rho_r(A) \leq \max_{\|x\|_2=1} \|Ax\|_2 = \|A\|_2.$$

Theorem 3.4. *Let $A \in \mathbb{H}^{n\times n}$ be a Hermitian matrix. Then,*

$$\text{trace}(A) = \sum_{i=1}^{n} \lambda_i(A),$$

where λ_i are the right eigenvalues of A.

Proof. Suppose A is a quaternionic Hermitian matrix. By Schur's triangularization theorem for quaternionic matrices [Theorem 2, [7]], there exist $U, T \in \mathbb{H}^{n\times n}$ such that U is unitary, T is upper triangular, and $A = UTU^*$. Since A is Hermitian, i.e., $A = A^*$, it follows that $T^* = T$, so T is diagonal with real diagonal entries. Thus $A = UDU^*$.

Applying the mapping Ψ and taking the trace, we have

$$\text{trace}\big(\Psi_A\big) = \text{trace}\big(\Psi_{UDU^*}\big) = \text{trace}\big(\Psi_U \Psi_D \Psi_{U^*}\big) = \text{trace}\big(\Psi_D\big).$$

If $A \in \mathbb{H}^{n\times n}$ has real diagonal entries, then $\text{trace}(\Psi_A) = 2\,\text{trace}(A)$, yielding the desired result. □

Since upper bounds for the right spectral radius of a quaternionic matrix play an important role in finding bounds for the zeros of quaternionic polynomials, we now present some useful results for the development of our theory.

Theorem 3.5 (Lemma 3.5, [4]). *Let $A \in \mathbb{H}^{n\times n}$. Then*
$$\|A\|_2 \leq \|A\|_F.$$

Theorem 3.6. *If $\|\cdot\|_q$ $(q = 1, \infty, F)$ is a quaternionic matrix norm and $A \in \mathbb{H}^{n\times n}$, then*
$$\rho_l(A),\ \rho_r(A) \leq \|A\|_q.$$

Proof. Let λ be a left or right eigenvalue of A with quaternionic eigenvector x. If $\|\cdot\|_q$ is any quaternionic matrix norm as stated, then
$$\begin{aligned}\|\lambda x\|_q = \|x\lambda\|_q = \|Ax\|_q \leq \|A\|_q\|x\|_q,\\ |\lambda|\|x\|_q \leq \|A\|_q\|x\|_q.\end{aligned}$$
Thus we obtain
$$\rho_l(A),\ \rho_r(A) \leq \|A\|_q.$$
□

Theorem 3.7 (Corollary 3.9, [4]). *Let $A \in \mathbb{H}^{n\times n}$. Then*
$$\rho_r(A) \leq \left(\|A\|_F^2 - \left[\max_{1\leq i\leq n} |R_i - C_i|\right]^2\right)^{1/2},$$
where
$$R_i(A) = \left(\sum_{j\neq i}^{n} |a_{ij}|\right)^{1/2} \quad \text{and} \quad C_i(A) = \left(\sum_{j\neq i}^{n} |a_{ji}|\right)^{1/2}.$$

4. Bounds for the zeros of simple quaternionic polynomials

We begin this section by recalling the definitions of simple quaternionic polynomials. Consider polynomials 1.1 and 1.2. If $q_m = 1$, these are called monic simple polynomials. For non-monic polynomials, we may factor out the leading coefficient, e.g., $q_l(z) = q_m p_l(z)$ for the left case, where $p_l(z)$ is monic.
The companion matrices associated with monic $p_l(z)$ and $p_r(z)$ are respectively:
$$C_{p_l} = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -q_0 & -q_1 & \dots & -q_{m-1} \end{bmatrix}, C_{p_r} = \begin{bmatrix} 0 & \dots & 0 & -q_0 \\ 1 & 0 & & -q_1 \\ & \ddots & & \vdots \\ 0 & & 1 & -q_{m-1} \end{bmatrix}.$$
The following known results link the zeros of $p_l(z)$ and $p_r(z)$ to eigenvalues of their companion matrices.

Proposition 4.1 (Proposition 1, [29]). *If λ is a left eigenvalue of C_{p_l}, then λ is a zero of $p_l(z)$.*

Corollary 4.2 (Corollary 1, [29]). *If λ is a left eigenvalue of C_{p_l}, then it is also a right eigenvalue.*

Example 4.3. Consider the simple polynomial $p_l(z) = z^2 + \mathbf{j}z + 2$, then its companion matrix
$$C_{p_l} = \begin{bmatrix} 0 & 1 \\ -2 & -\mathbf{j} \end{bmatrix}.$$
$\mathbf{i}$ is a right eigenvalue of C_{p_l}. However, $\mathbf{i}$ is not a left eigenvalue.

Similarly, we obtain analogous results for C_{p_r}. Moreover, the left eigenvalues of C_{p_r} coincide with the zeros of $p_r(z)$; see [29] for details. From 2.6, we conclude that all zeros of $p_l(z)$ are right eigenvalues of C_{p_l}. However, the converse is not necessarily true.

Example 4.4. Consider the simple quaternionic polynomial

$$\begin{aligned} p_l(z) = z^6 + (\mathbf{i}+3\mathbf{k})z^5 + (3+\mathbf{j})z^4 + (5\mathbf{i}+15\mathbf{k})z^3 \\ + (-4+5\mathbf{j})z^2 + (6\mathbf{i}+18\mathbf{k})z - 12 + 6\mathbf{j}, \end{aligned}$$

$\mathbf{i}\sqrt{5}$ is a right eigenvalue of C_{p_l}. But $\mathbf{i}\sqrt{5}$ is not a zero of $p_l(z)$. The converse of 4.2 is not true in general (see 4.3). Moreover, all zeros of $p_l(z)$ are right eigenvalues of C_{p_l}, but the converse need not hold (4.4).

Basic norm bounds for zeros The following theorem provides several classical norm bounds for the zeros of monic simple quaternionic polynomials, derived from matrix norm inequalities applied to the companion matrix.

Theorem 4.5. *Let $p_l(z)$ be a monic simple polynomial over $\mathbb{H}$ of degree m. Then every zero $\tilde{z}$ of $p_l(z)$ satisfies $|\tilde{z}| \le \alpha$, where α can be taken as any of the following:*

(1) $\alpha_1 = \max\big\{|q_0|,\ 1+|q_1|,\ \ldots,\ 1+|q_{m-1}|\big\}$,
(2) $\alpha_2 = 1 + \max\big\{|q_0|, |q_1|, \ldots, |q_{m-1}|\big\}$ *(Generalized Cauchy bound),*
(3) $\alpha_3 = 1 + \sum_{j=0}^{m-1} |q_j|$ *(Generalized Montel bound),*
(4) $\alpha_4 = \Big(1 + \sum_{j=0}^{m-1} |q_j|^2\Big)^{1/2}$ *(Generalized Carmichael–Mason bound).*

Proof. Since every zero $\tilde{z}$ of $p_l(z)$ is a right eigenvalue of C_{p_l}, we have $|\tilde{z}| \le \rho_r(C_{p_l})$. By Theorem 3.6, $\rho_r(C_{p_l}) \le \|C_{p_l}\|_q$ for any matrix norm $\|\cdot\|_q$. For α_1 and α_2, we use the maximum column sum norm $\|\cdot\|_1$:

$$\|C_{p_l}\|_1 = \max\big\{|q_0|,\ 1+|q_1|,\ \ldots,\ 1+|q_{m-1}|\big\}.$$

Taking the maximum over this set gives α_1, while adding 1 to the maximum coefficient magnitude yields α_2. For α_3, we use the maximum row sum norm $\|\cdot\|_\infty$:

$$\|C_{p_l}\|_\infty = \max\Big\{1,\ \sum_{j=0}^{m-1} |q_j|\Big\} \le 1 + \sum_{j=0}^{m-1} |q_j|.$$

For α_4, we use the spectral norm $\|\cdot\|_2$. Write $C_{p_l} = X + Y$, where

$$X = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \\ -q_0 & -q_1 & \ldots & -q_{m-1} \end{bmatrix}.$$

Then $XY^* = YX^* = 0$, $\|XX^*\|_2 = 1$, and $\|YY^*\|_2 = \sum_{j=0}^{m-1} |q_j|^2$. By the triangle inequality,

$$\|C_{p_l}\|_2^2 = \|(X+Y)(X+Y)^*\|_2 \le \|XX^*\|_2 + \|YY^*\|_2 = 1 + \sum_{j=0}^{m-1} |q_j|^2.$$

Thus

$$\|C_{p_l}\|_2 \le \alpha_4.$$

□

Remark 4.6. Analogous bounds hold for $p_r(z)$ via the transformation $\overline{p_r(\overline{z})}$, using Theorem 2.9.

Higher-order bounds via powers of the companion matrix

The following bounds are derived by considering powers of the companion matrix and their Frobenius norms. While algebraically straightforward, they can sometimes yield sharper estimates than the basic norm bounds.

Let $C_{p_l}^r = [d_{jk}^{(r)}]$ for $r \ge 2$, and define

$$b_j = q_{m-1}q_j - q_{j-1}, \qquad c_j = -q_{m-1}b_j + q_{m-2}q_j - q_{j-2},$$

with $q_{-1} = q_{-2} = 0$.

Theorem 4.7. *Let $p_l(z)$ be monic of degree m and $C_{p_l}^r = [d_{jk}^{(r)}]$ for $r \ge 2$. Then every zero $\tilde{z}$ of $p_l(z)$ satisfies*

$$|\tilde{z}| \le \beta, \quad \textit{where} \quad \beta = \max_{1\le j\le m} \left(\sqrt[r]{\sum_{k=1}^{m} |d_{jk}^{(r)}|} \right).$$

Proof. If λ is a right eigenvalue of C_{p_l}, then λ^r is a right eigenvalue of $C_{p_l}^r$, because $C_{p_l}x = x\lambda$ implies $C_{p_l}^r x = x\lambda^r$. Applying Proposition 3.1 (the row-sum bound) to $C_{p_l}^r$ yields

$$|\lambda^r| \le \max_j \sum_k |d_{jk}^{(r)}|,$$

hence $|\lambda| \le \beta$. □

Theorem 4.8. *For $P = C_{p_l}^2$, we have*

$$\rho_r(C_{p_l}) \le \left[1 + \sum_{j=0}^{m-1} \left(|q_j|^2 + |b_j|^2\right)\right]^{1/4}.$$

Proof. Decompose $P = X + Y + Z$, where

$$X = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \\ b_0 & b_1 & \dots & b_{m-1} \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \\ -q_0 & -q_1 & \dots & -q_{m-1} \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

These matrices satisfy $XY^* = XZ^* = YX^* = YZ^* = ZX^* = ZY^* = 0$. Their squared Frobenius norms are:

$$\|XX^*\|_2 = \sum_{j=0}^{m-1} |b_j|^2, \quad \|YY^*\|_2 = \sum_{j=0}^{m-1} |q_j|^2, \quad \|ZZ^*\|_2 = 1.$$

Thus,

$$\|P\|_2^2 \le \|XX^*\|_2 + \|YY^*\|_2 + \|ZZ^*\|_2 = 1 + \sum_{j=0}^{m-1} (|q_j|^2 + |b_j|^2).$$

Since $\rho_r(C_{p_l})^2 \le \rho_r(C_{p_l}^2) \le \|C_{p_l}^2\|_2$, the result follows. $\square$

Corollary 4.9. *Every zero of $p_l(z)$ satisfies*

$$|z| \le \left[1 + \sum_{j=0}^{m-1} \left(|q_j|^2 + |b_j|^2\right)\right]^{1/4}.$$

A similar bound using $C_{p_l}^3$ can be derived analogously, yielding a sixth-root estimate.

Comparison with Opfer's bound

We compare the Generalized Cauchy bound with Opfer's bound (Theorem 2.8).

Lemma 4.10. *If $|q_j| \ge 1$ for at least one j, then*

$$\alpha_2 \le R, \quad \textit{where} \quad R = \max\left\{1, \sum_{j=0}^{m-1} |q_j|\right\}.$$

Proof. Let $|q_p| \ge 1$. Then

$$\alpha_2 = 1 + \max_j |q_j| \le |q_p| + \max_j |q_j| \le \sum_{j=0}^{m-1} |q_j| \le R.$$

$\square$

Thus, under the stated condition, the Generalized Cauchy bound is sharper than Opfer's.

Remarks on the derivation: The bounds presented in this section rely on elementary matrix norm properties and the algebraic fact that right eigenvalues satisfy $\lambda^r \in \sigma_r(C^r)$ if $\lambda \in \sigma_r(C)$. While the derivations are algebraically consistent, they are conceptually straightforward and essentially amount to computing Frobenius norms of powers of the companion matrix. Their utility lies in providing explicit, computable estimates for zero locations, which can be tighter than earlier bounds in certain parameter regimes.

5. The bounds for the zeros of unilateral quaternionic polynomials

We assume that the variable z commutes with the quaternionic coefficients in 1.1 and 1.2. Then we have $p_l(z) = p_r(z)$, and this quaternionic polynomial is called a *unilateral quaternionic polynomial* [17]. Consider the unilateral quaternionic polynomial

$$P(z) = q_m z^m + q_{m-1} z^{m-1} + \ldots + q_1 z + q_0 = \sum_{j=0}^{m} q_j z^j, \quad q_j \in \mathbb{H}. \tag{5.1}$$

The quaternion conjugate of $P(z)$ is defined as

$$\overline{P}(z) = \overline{q_m} z^m + \overline{q_{m-1}} z^{m-1} + \ldots + \overline{q_1} z + \overline{q_0} = \sum_{j=0}^{m} \overline{q}_j z^j, \quad q_j \in \mathbb{H}. \tag{5.2}$$

Define

$$F(z) = P(z)\overline{P}(z) = \left(\sum_{j=0}^{m} q_j z^j\right)\left(\sum_{j=0}^{m} \overline{q}_j z^j\right) = \sum_{i=0}^{m}\sum_{k=0}^{m} q_i q_k z^{i+k}. \tag{5.3}$$

Theorem 5.1 ([17], Theorem 7.1)**.** *Let $P(z)$ be a quaternion polynomial and $F(z) = P(z)\overline{P}(z)$. For each root of $P(z)$ there is a root of $F(z)$ having the same norm. Conversely, for each root of $F(z)$ there is a root of $P(z)$ having the same norm.*

For properties of such polynomials over general division rings, see [21].

Remark 5.2. Let $F(z)$ be a quaternionic polynomial as defined in 5.3. Then all zeros of $F(z)$ satisfy $|z| \le \eta$, where η can be taken as any of the following bounds:

(1) $\eta := \max\{|q_0|, 1+|q_1|, 1+|q_2|, \ldots, 1+|q_{m-1}|\}$
(2) $\eta := 1 + \max\{|q_0|, |q_1|, \ldots, |q_{m-1}|\}$ (Generalized Cauchy's bound)
(3) $\eta := 1 + |q_0| + |q_1| + \ldots + |q_{m-1}|$ (Generalized Montel's bound)
(4) $\eta := (1 + |q_0|^2 + \ldots + |q_{m-1}|^2)^{1/2}$ (Generalized Carmichael & Mason's bound)

Justification: Follows directly from 4.5 applied to $P(z)$ and the norm-preserving correspondence between zeros of $P(z)$ and $F(z)$ (5.1).

Remark 5.3. Let $F(z)$ be a unilateral quaternionic polynomial with $|q_m| > |q_j|$ for $j = 0, 1, \ldots, m-1$. Then all zeros of $F(z)$ are contained in the ball

$$\{q \in \mathbb{H} : |q| \le 2\}.$$

Remark 5.4. Let $F(z)$ be a unilateral quaternionic polynomial and $C_P^r = [d_{ij}^{(r)}]$ for some $r \ge 2$. Then all zeros of $F(z)$ satisfy $|z| \le \beta$, where

$$\beta := \max_{1\le j\le m}\left\{\sqrt[r]{\sum_{k=1}^{m} |d_{jk}^{(r)}|}\right\}.$$

Justification: Follows from Theorem 4.7

Corollary 5.5. *Let $F(z)$ be a unilateral quaternionic polynomial of degree m. Then all zeros of $F(z)$ satisfy*

$$|z| \le \left[1 + \sum_{j=0}^{m-1}\left(|q_j|^2 + |b_j|^2\right)\right]^{1/4},$$

where $b_j = q_{m-1}q_j - q_{j-1}$ *with* $q_{-1} = 0$.

Corollary 5.6. *Let* $F(z)$ *be a unilateral quaternionic polynomial of degree* m. *Then all zeros of* $F(z)$ *satisfy*

$$|z| \leq \left[1 + \sum_{j=0}^{m-1} \left(|q_j|^2 + |b_j|^2 + |c_j|^2\right)\right]^{1/6},$$

where $b_j = q_{m-1}q_j - q_{j-1}$ *and* $c_j = -q_{m-1}b_j + q_{m-2}q_j - q_{j-2}$*, with* $q_{-1} = q_{-2} = 0$.

6. Application

The quaternionic linear state space systems $\dot{x}(t) = Ax(t)$ with $A \in \mathbb{H}^{n\times n}$ is stable if $\Lambda_r(A) \subset \mathbb{H}^- := \{q \in \mathbb{H} : \Re(q) < 0\}$, see [25] for details. The quaternionic discrete systems $x(t+1) = Ax(t)$ with $A \in \mathbb{H}^{n,n}$ is stable if $\Lambda_r(A) \subset S_{\mathbb{Z}} := \{q \in \mathbb{H} : |q| < 1\}$, see [25] for details. We can extend the stability margin of stable complex discrete systems [6] to the quaternionic case [2, 3, 25, 5]. The stability margin of stable quaternionic discrete systems $x(t+1) = Ax(t)$ with $A \in \mathbb{H}^{n\times n}$ is defined as

$$M_q := 1 - \max_{1\leq i\leq n} |\lambda_i(A)|,$$

where $\lambda_i(A) \in \Lambda_s(A)$. Moreover, $|\lambda_i| = |\gamma_i^{-1}\lambda_i\gamma_i|$ for any nonzero $\gamma_i \in \mathbb{H}$.

The $\max\limits_{1\leq i\leq n} |\lambda_i(A)|$ is uniquely determined by A up to similarity, that is, replacing $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $\gamma_1^{-1}\lambda_1\gamma_1, \gamma_2^{-1}\lambda_2\gamma_2, \ldots, \gamma_n^{-1}\lambda_n\gamma_n$, where $\gamma_i \in \mathbb{H} \setminus \{0\}, i = 1, 2, \ldots, n$.

We now illustrate an application. Consider the quaternionic discrete system

$$w(t+3) + 0.02\mathbf{j}\, w(t+2) + 0.2\mathbf{i}\, w(t+1) + 0.03\mathbf{k} = 0, \tag{6.1}$$

whose companion matrix is

$$C := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.03\mathbf{k} & -0.2\mathbf{i} & -0.02\mathbf{j} \end{bmatrix}.$$

The set of standard right eigenvalues of C is

$$\Lambda_s(C) = \{-0.3257 + 0.3085\mathbf{i},\ 0.3257 + 0.3085\mathbf{i},\ 0.1491\mathbf{i}\}.$$

All these eigenvalues satisfy $|z| < 1$, so the system is stable. The true spectral radius (maximum modulus) is

$$\rho_s(C) = \max |\lambda_i(C)| \approx 0.449,$$

giving the actual stability margin

$$M_q = 1 - \rho_s(C) \approx 0.551.$$

Applying 5.5 to C^4, we obtain the bound $\beta \approx 0.5082$, yielding a *lower bound* on the stability margin:

$$M_q \geq 1 - \beta \approx 0.4918.$$

Thus, the system is not only stable but also guaranteed to have a stability margin of at least 0.4918, indicating robust stability. Note that $1-\beta$ is a conservative guarantee, not the margin itself; the true margin M_q may be larger (here $0.551 > 0.4918$).

$$C^4 := \begin{bmatrix} 6\times 10^{-4}\mathbf{i} & -0.0340\mathbf{k} & -4\times 10^{-4} - 0.2000\mathbf{i} \\ 1.2\times 10^{-6}\mathbf{k} - 0.0060\mathbf{j} & 6.8\times 10^{-4}\mathbf{i} - 0.0400 & 8\times 10^{-5}\mathbf{j} - 0.0300\mathbf{k} \\ -2.4\times 10^{-6}\mathbf{i} - 9\times 10^{-4} & 1.72\times 10^{-5}\mathbf{k} & -0.0400 + 8\times 10^{-5}\mathbf{i} \end{bmatrix}.$$

7. Numerical Examples

Example 7.1. Let us consider a simple polynomial over $\mathbb{H}$:

$$p_l(z) = z^6 + \mathbf{j}z^5 + \mathbf{i}z^4 - z^2 - \mathbf{j}z - \mathbf{i}.$$

The zeros of $p_l(z)$ are shown in Table 1.

Table 1. Zeros of $p_l(z)$ from Example 7.1

$z_1 =$	$0.5(1-\mathbf{i}-\mathbf{j}-\mathbf{k})$
$z_2 =$	$0.5(-1+\mathbf{i}-\mathbf{j}-\mathbf{k})$
$z_3 =$	$[\mathbf{i}]$
$z_4 =$	1
$z_5 =$	-1

Example 7.2. Consider the polynomial over $\mathbb{H}$:

$$p_l(z) = z^{10} + (1+2\mathbf{i}-4\mathbf{j})z^9 - (3.1\mathbf{i}+\mathbf{k})z^8 + (2.5\mathbf{j}+2.1\mathbf{k})z^7 + (3-\mathbf{i})z^6 - 1.7z^5 - (\mathbf{i}+\mathbf{j})z^4 - 7.2z^3 - \mathbf{j}z + 2.9(\mathbf{j}-\mathbf{k}) - 4.$$

The zeros of $p_l(z)$ are shown in Table 2.

Table 2. Zeros of $p_l(z)$ from Example 7.2

$z_1 =$	$-1.26112 - 1.92564\mathbf{i} + 4.10523\mathbf{j} - 0.55813\mathbf{k}$
$z_2 =$	$0.93019 + 0.27871\mathbf{i} - 0.51178\mathbf{j} - 0.45249\mathbf{k}$
$z_3 =$	$-1.07301 + 0.46409\mathbf{i} - 0.09237\mathbf{j} - 0.14655\mathbf{k}$
$z_4 =$	$1.1405 + 0.08057\mathbf{i} + 0.07784\mathbf{j} - 0.04811\mathbf{k}$
$z_5 =$	$-0.65287 - 0.01858\mathbf{i} + 0.88395\mathbf{j} - 0.27907\mathbf{k}$
$z_6 =$	$0.21713 - 0.24588\mathbf{i} + 1.02275\mathbf{j} + 0.16627\mathbf{k}$
$z_7 =$	$-0.38874 + 0.08867\mathbf{i} - 0.46591\mathbf{j} - 0.83920\mathbf{k}$
$z_8 =$	$0.28474 - 0.45581\mathbf{i} - 0.33143\mathbf{j} + 0.5855\mathbf{k}$
$z_9 =$	$0.60157 + 0.21245\mathbf{i} + 0.44266\mathbf{j} - 0.28997\mathbf{k}$
$z_{10} =$	$-0.79994 - 0.03749\mathbf{i} + 0.16559\mathbf{j} - 0.06488\mathbf{k}$

Example 7.3. Suppose the simple polynomial over $\mathbb{H}$:

$$p_r(z) = z^6 - z^5(\mathbf{i}+3\mathbf{k}) + z^4(3-\mathbf{j}) - z^3(5\mathbf{i}+15\mathbf{k}) - z^2(4+5\mathbf{j}) - z(6\mathbf{i}+18\mathbf{k}) - 6\mathbf{j} - 12.$$

The zeros of $p_r(z)$ are shown in Table 3.

TABLE 3. Zeros of $p_r(z)$ from Example 7.3

$z_1 =$	$-\mathbf{i} - 2\mathbf{k}$
$z_2 =$	$[\mathbf{i}\sqrt{3}]$
$z_3 =$	$[\mathbf{i}\sqrt{2}]$
$z_4 =$	$-0.6\mathbf{i} - 0.8\mathbf{k}$

Example 7.4. Let us consider another simple polynomial over $\mathbb{H}$:

$$p_r(z) = -z^3\mathbf{i} - z^2\mathbf{j} - z\mathbf{k} + 1.$$

The zeros of $p_r(z)$ are shown in Table 4.

TABLE 4. Zeros of $p_r(z)$ from Example 7.4

$z_1 =$	$\mathbf{k}$
$z_2 =$	$0.5(\sqrt{2} + \mathbf{i} + \mathbf{k})$
$z_3 =$	$0.5(-\sqrt{2} + \mathbf{i} + \mathbf{k})$

Example 7.5. Take the simple polynomial over $\mathbb{H}$:

$$p_r(z) = z^6 - z^5\mathbf{j} - z^4\mathbf{i} - z^2 + z\mathbf{j} + \mathbf{i}.$$

The zeros of $p_r(z)$ are shown in Table 5.

TABLE 5. Zeros of $p_r(z)$ from Example 7.5

$z_1 =$	1
$z_2 =$	-1
$z_3 =$ $z_4 =$	$0.5(1 - \mathbf{i} - \mathbf{j} - \mathbf{k})$ $0.5(-1 + \mathbf{i} - \mathbf{j} - \mathbf{k})$
$z_5 =$	$[\mathbf{i}]$

Example 7.6. Let

$$p(z) = p_l(z) = p_r(z) = z^4 + 2z^3 + 5z^2 + 8z + 12.$$

be the simple polynomial over $\mathbb{H}$.

The zeros of $p(z)$ are shown in Table 6.

TABLE 6. Zeros of $p(z)$ from Example 7.6

$z_1 =$	$[0.359481 + 1.91107\mathbf{i}]$
$z_2 =$	$[-1.35948 + 1.15118\mathbf{i}]$

Remark 7.7. From Tables 1–6, it is clear that all our results regarding bounds for the zeros of quaternionic polynomials are verified.

Now, we present Table 7 for bounds of zeros of quaternionic polynomials.

Table 7. Bounds for zeros of quaternionic polynomials from examples

Example	Cauchy	Montel	Carmichael & Mason	G. Opfer	Corollary 5.5	Corollary 5.6
7.1	2	6	2.4495	5	1.8988	1.6188
7.2	8.2	32.3085	11.9925	31.3085	7.3	6.323
7.3	19.9737	61.9291	29.1890	60.9291	9.6374	7.1088
7.4	2	4	2	3	1.732	1.6475
7.5	2	6	2.4495	5	1.8988	1.6188
7.6	13	28	15.4272	27	5.375	3.5878

Remark 7.8. From Table 7, we conclude that the bounds in Corollary 5.5 and Corollary 5.6 are better than G. Opfer's bounds. Moreover, the "Carmichael and Mason" and Cauchy's bounds are also better than G. Opfer's bounds.

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