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ON SOME TOPOLOGICAL STRUCTURES OF TOPOLOGICAL MONOIDS

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Abstract. Let G be a topological monoid, meaning that it is both a monoid and a topological space with continuous multiplication. This paper focuses on points in G that do not possess compact neighborhoods. In the context of topological groups, if the identity element, denoted e, has a compact neighborhood, then the space is locally compact. However, in the context of topological monoids, we construct an example where the identity element has a compact neighborhood while the space is not locally compact. Elements x and y in G are called mutually inverse if their products xy and yx equal e. We first investigate the lack of compact neighborhoods in topological monoids and confirm that results applicable to topological groups also hold true for topological monoids in the case of mutually inverse elements. Next, we introduce the concept of a strictly mutually inverse pair (y, x), where $yx = e$ but xy does not necessarily equal e. In this case, we demonstrate that specific relationships between x, y, and their neighborhoods are relevant when considering left and right inverses within this context.

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1. Introduction

Topology and algebra are two branches of mathematics that are deeply interconnected. Algebraic structures are utilized to study and classify topological shapes. In algebraic topology, algebraic invariants such as homotopy groups, homology groups, and cohomology groups are employed to analyze topological spaces [5, 9]. These invariants remain unchanged under continuous deformations [17, 15]. This framework allows us to categorize topological spaces based on their algebraic invariants. The interplay between topology and algebra also establishes connections to other domains, including algebraic geometry [1] and mathematical physics, particularly through the study of topological quantum field theories [16, 4].

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Conversely, topological algebra investigates algebraic structures such as rings, groups, and modules, which are endowed with a topology, ensuring that all algebraic operations are continuous. This integration means we can leverage topological concepts such as compactness, connectedness, convergence, and completeness to analyze algebraic behavior. Simultaneously, algebraic structures can elucidate certain topological properties. This framework allows for an exploration of the compatibility conditions between algebraic operations and topology, as noted in [2]. Recent work in representation theory [13] delves into the structural analyses of topological rings and continuity conditions, examining free objects in topological varieties [14] and the exploration of topological classes within algebraic logic [3]. Further advances in the geometry of topological modules highlight the evolving interface between algebra and topology [6].

Topological and algebraic structures interact coherently via topological groups, particularly through the continuity of multiplication and inversion maps. The existence of Haar measure provides a robust analytic framework for these groups, especially in the setting of local compactness [7]. For abelian locally compact groups, the classical structural theorem classifies each group based on the extension of an Euclidean component by a compact component, elucidating both connectedness and decomposition scenarios [12]. A well-known result in topological groups states that any connected locally compact space can be expressed as a countable union of compact spaces [8]. Moreover, it has been shown that every almost connected locally compact space can be represented via inverse limits of Lie groups [11].

In this paper, we investigate a specific type of topological semigroups that we term topological monoids. The key distinction in this context is that in monoids, we do not assume the existence of two-sided inverses for each element. It is established that in a topological group, if the identity element possesses a compact neighborhood, then the space itself is locally compact. However, in the case of a topological monoid, we construct an example wherein the identity element has a compact neighborhood, yet the space does not exhibit local compactness. Additionally, we present several results throughout this paper. We define a compact neighborhood of a point x as an open neighborhood U_x such that the closure $\overline{U_x}$ is compact within the considered space. For our purposes, a space is termed locally compact if every point within the topological space has a compact neighborhood.

2. Topological Monoid and Local Compactness

Definition 2.1. [10] Let G be a set equipped with a binary operation $*$. The structure $(G, *)$ is called a monoid if the following conditions are satisfied:

(1) Closure: The set G is closed under the operation $*$.
(2) Associativity: For all $a, b, c \in G$, the operation satisfies the associative property: $(a * b) * c = a * (b * c)$.
(3) Identity Element: There exists an element $e \in G$ known as the identity element, such that for all $a \in G$, the following holds: $e * a = a * e = a$.

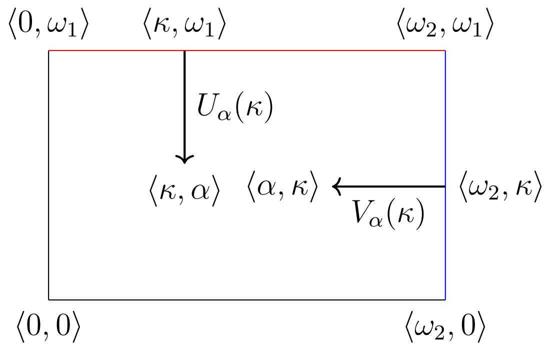
These conditions together ensure that $(G, *)$ forms a monoid.

Definition 2.2. [2] A topological monoid G consists of a topological space G equipped with an associative operation $*: G \times G \to G$, along with an element e such that $e * a = a * e = a$ for all $a \in G$. Additionally, we require the map

$$m : G \times G \to G : x \times y \mapsto x * y$$

to be continuous.

Example 2.3. Modified *Dieudonné Plank* :
Let $X = ([0, \omega_2] \times [0, \omega_1]) \setminus \langle \omega_2, \omega_1 \rangle$, where ω_1 denotes the first uncountable ordinal and ω_2 is the successor ordinal of ω_1. The topology τ on X is generated by considering each point of the space $([0, \omega_2) \times [0, \omega_1))$ as an open set, in addition to the sets defined as $U_\alpha(\kappa) = \{\langle \kappa, \gamma \rangle : \alpha < \gamma \leq \omega_1\}$ and $V_\alpha(\kappa) = \{\langle \gamma, \kappa \rangle : \alpha < \gamma \leq \omega_2\}$.



This is a non-locally compact Tychonoff space. To demonstrate that this space is not locally compact, consider the point $c = \langle \omega_2, 0 \rangle$ and take an open neighborhood of c in the form $V_\alpha(0)$. By definition, α is an infinite ordinal, and each set of the form $\{\langle \alpha, 0 \rangle\}$ is both open and closed for $\alpha < \omega_2$.

Now, select an infinite ordinal $\eta > \alpha$. We can express the set $V_\alpha(\omega_2)$ as $F = \bigcup_\xi \{\langle \xi, 0 \rangle\} \cup V_\eta(\omega_2)$, where $\alpha < \xi \leq \eta$. Consequently, the closure $\overline{F}$ cannot be compact in X, which implies that X is not locally compact. Define a monoid structure on X as follows: $\langle x_1, y_1 \rangle * \langle x_2, y_2 \rangle = \langle max(x_1, x_2), max(y_1, y_2) \rangle$. This indeed defines a monoid structure. To see this, take any two elements in X, say $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle$. Then,

$$\langle max(x_1, x_2), max(y_1, y_2) \rangle = \begin{cases} \langle x_1, y_1 \rangle, & \text{if } x_1 \geq x_2 \text{ and } y_1 \geq y_2 \\ \langle x_1, y_2 \rangle, & \text{if } x_1 \geq x_2 \text{ and } y_2 \geq y_1 \\ \langle x_2, y_1 \rangle, & \text{if } x_2 \geq x_1 \text{ and } y_1 \geq y_2 \\ \langle x_2, y_2 \rangle, & \text{if } x_2 \geq x_1 \text{ and } y_2 \geq y_1 \end{cases}$$

This shows that the operation is closed under multiplication. Take any element $\langle x, y \rangle \in X$, then $\langle max(0, x), max(0, y) \rangle = \langle max(x, 0), max(y, 0) \rangle = \langle x, y \rangle$. Hence, $\langle 0, 0 \rangle$ is the identity e. Also, it is easy to see that the operation is associative, e.g., let $x_1 \geq x_2$, $x_2 \geq x_3, y_2 \geq y_1, y_2 \geq y_3$. Then,

$$(\langle x_1, y_1 \rangle * \langle x_2, y_2 \rangle) * \langle x_3, y_3 \rangle = \langle x_1, y_2 \rangle * \langle x_3, y_3 \rangle = \langle x_1, y_2 \rangle$$

and

$$\langle x_1, y_1 \rangle * (\langle x_2, y_2 \rangle * \langle x_3, y_3 \rangle) = \langle x_1, y_1 \rangle * \langle x_2, y_2 \rangle = \langle x_1, y_2 \rangle.$$

All other cases can be done similarly.
Now we aim to check the continuity of $m : X \times X \to X$. Let $U \subseteq X$ be an open set. This implies that U can be expressed as a union singletons off blue and red segments, sets of the form $V_\alpha(\kappa)$ and sets of the form $U_\alpha(\kappa)$:

(1) Let $V = \{\langle a, b\rangle\} \subseteq U$ off both blue and red segments. Then,
$$m^{-1}(V) = (\bigcup_{\substack{i\leq a\\ j\leq b}} \langle i, j\rangle \times \langle a, b\rangle) \cup (\bigcup_{\substack{i\leq a\\ j\leq b}} \langle a, b\rangle \times \langle i, j\rangle),$$
which is open with respect to the product topology;

(2) Let $V \subseteq U$ be of the form $V_\alpha(\kappa)$. Then, $m^{-1}(V)$ is the union of the following:
   (a) $(\bigcup_{i,j}\{\langle i, j\rangle\} \times V) \cup (\bigcup_{i,j} V \times \{\langle i, j\rangle\})$, where each $\langle i, j\rangle$ is off both the blue and red segments and it must be below V. Hence, it is open.
   (b) $(\bigcup_i V \times R_i) \cup (\bigcup_i R_i \times V)$, where each R_i is of the form $V_\alpha(\kappa)$ and below V. Hence, it is open.

(3) Let $V \subseteq U$ be of the form $U_\alpha(\kappa)$. Then, $m^{-1}(V)$ is the union of the following:
   (a) $(\bigcup_{i,j}\{\langle i, j\rangle\} \times V) \cup (\bigcup_{i,j} V \times \{\langle i, j\rangle\})$, where each $\langle i, j\rangle$ is off both the blue and red segments and it must be to the left of V. Hence, it is open.
   (b) $(\bigcup_i V \times R_i) \cup (\bigcup_i R_i \times V)$, where each R_i is of the form $U_\alpha(\kappa)$ and to the left V. Hence, it is open.

Consequently, the preimage of any open set under m is open, verifying that m is indeed continuous.

3. Mutually Inverse Case

This section draws on arguments from the theory of topological groups. The primary goal is to confirm that the results remain valid in the context of topological monoids, even when the requirement for every point to have a two-sided inverse is removed. We will examine a specific pair of elements within a monoid G, focusing on the interesting case where these elements are two-sided inverses of each other. Understanding the properties and implications of two-sided inverses is essential, as they play a significant role in the overall structure of the monoid. Through this exploration, we aim to emphasize the importance of two-sided inverses in algebraic structures and their applications in various mathematical contexts. This discussion will serve as an introduction and preparation for the next section.

Lemma 3.1. *Let G be a monoid, and let $x \in G$ have an inverse. This means there exists $y \in G$ such that $yx = e = xy$. In this case, the maps $L_x : G \to G$ defined by $g \mapsto xg$ and $R_y : G \to G$ defined by $g \mapsto gy$ are bijective. Similarly, the maps L_y and R_x are bijective.*

Proof. Let's start with L_x and L_y. Suppose that $L_x(a) = L_x(b)$, then this implies $xa = xb$ and multiplying both sides with y from the left gives $a = b$. Similarly, let $L_y(a) = L_y(b)$, then $ya = yb$ and multiplying both sides with x from the left gives $a = b$. Hence, L_x and L_y are one-to-one. Now, let $g \in G$, then $L_x(yg) = L_y(xg) = g$ and this implies both L_x and L_y are onto.
Let's now show that R_x and R_y are bijective. Let $R_x(a) = R_x(b)$, then $ax = bx$ and multiplying both sides with y from the right gives $a = b$. Similarly, suppose that $R_y(a) = R_y(b)$. Then, $ay = by$ and this implies $ayx = byx$, i.e, $a = b$. Therefore, both R_x and R_y are one-to-one. Now, to show R_x and R_y are onto, let $g \in G$, then $R_x(gy) = R_y(gx) = g$. Henec, both R_x and R_y are onto. □

Definition 3.2. For a monoid G, we define an element x as an invertible element if there exists $y \in G$ such that $xy = yx = e$. In this case, we refer to the pair (x, y) as mutually inverse elements.

Lemma 3.3. *Let G be a topological monoid. If (x, y) is a pair of mutually inverses. Then, the maps $L_x : G \to G$ defined by $g \mapsto xg$ and the map $R_y : G \to G$ defined by $g \mapsto gy$ are homeomorphisms. Similarly, the maps L_y and R_x are homeomorphisms.*

Proof. We show that L_x and R_y are homeomorphisms. The other two maps are done in a similar manner. From assumption, the multiplication map $m : G \times G \to G : x \times y \mapsto x * y$ is continuous as G being a topological monoid. Lemma 3 implies that L_x is bijective. We can express L_x as $m|_{\{x\}\times G}$, i.e., $L_x(g) = m|_{\{x\}\times G}(x \times g) = xg$. Since m is continuous, then its restriction is also continuous. Hence,L_x is continuous. The inverse of L_x is given $L_x^{-1} : G \to G : g \mapsto yg$. L_x^{-1} is also continuous since it can be expressed as $m|_{\{y\}\times G}$. Therefore, L_x is a homeomorphism. Now, for the same reason R_y is bijective and it can be written as $m|_{G\times\{y\}}$. This allow us to say that R_y is continuous. Its inverse $R_y^{-1} : G \to G : g \mapsto gx$ is also continuous as it is the same as $m|_{G\times\{x\}}$. Therefore, R_y is a homeomorphism. □

Remark 3.4. Following the notation given in [18], in a topological space G we mean by $G^\bullet$ the set of all points in G which do not have compact neighbourhood.

Proposition 3.5. *Let G be a topological monoid, and let $G^\bullet$ be the set of all points that lack compact neighborhoods. Let (x, y) be mutually inverses. Then, $x \in G^\bullet$ if and only if $e \in G^\bullet$.*

Proof. ($\Longrightarrow$) Suppose that $e \in G \setminus G^\bullet$. This implies that there exits a compact neighborhood U_e of e, i.e., an open set U_e containing e such that $\overline{U_e}$ is compact in G. We know that L_x is a homeomorphism, then $L_x(U_e) \subseteq L_x(\overline{U_e}) = \overline{L_x(\overline{U_e})}$. Hence, $V = L_x(U_e)$ is a compact neighborhood of x, which is a contradiction. Therefore, $e \in G^\bullet$.

($\Longleftarrow$) Suppose that x has a compact neighborhood U_x. Then, $L_y(U_x)$ is a neighborhood of e. We also have that $L_y(U_x) \subseteq L_y(\overline{U_x}) = \overline{L_y(\overline{U_x})}$ as L_y is a homeomorphism. Hence, $W = L_y(U_x)$ is a compact neighborhood of e, which is a contradiction. Therefore, $x \in G^\bullet$. □

Proposition 3.6. *Let G be a topological monoid, and let $G^\bullet$ be the set of all points that lack compact neighborhoods. Let (x, y) be mutually inverses. Then, $x \in G^\bullet$ if and only if $y \in G^\bullet$.*

Proof. From Proposition 3.5 we conclude that $x \in G \setminus G^\bullet$ if and only if $e \in G \setminus G^\bullet$. If U_e is a compact neighborhood of e, then both R_y and L_y are compact neighborhoods of y. That completes the proof. □

Theorem 3.7. *Let G be a topological monoid, H be the set of all invertible elements and let $G^\bullet$ denote the set of all points in G that do not have compact neighborhoods. The following statements are equivalent:*

1. *For a pair mutually inverses (x, y), $x \in G^\bullet$.*
2. *For a pair mutually inverses (x, y), $y \in G^\bullet$.*
3. *The identity element $e \in G$ belongs to $G^\bullet$.*
4. *$H \subseteq G^\bullet$*

Proof. The implication (1) $\implies$ (2) follows from Proposition 3.6, as it holds that $x \in G^\bullet$ if and only if $y \in G^\bullet$.

The implication (2) $\implies$ (3) comes from combining Propositions 3.5 and 3.6. It is important to note that $y \in G^\bullet$ if and only if $x \in G^\bullet$, which is equivalent to $e \in G^\bullet$.

For the implication (3) $\implies$ (4), consider any pair of mutually inverse elements (x, y). Propositions 3.5 and 3.6 indicate that $e \in G^\bullet$ if and only if $x, y \in G^\bullet$. Since x and y are arbitrary, we conclude that $H \subseteq G^\bullet$.

Finally, the implication (4) $\implies$ (1) is evident. □

4. Strictly Mutually Inverse Case

In this section, we do not assume that a pair of mutual inverses (y, x) in a monoid G commute with each other; specifically, $yx = e$, but xy does not necessarily equal e. In this context, x is referred to as the right inverse of y, while y is the left inverse of x. We will call the pair (y, x) strictly mutually inverses. Let $L = \{(y_i, x_i)\}$ denote the set of all pairs of strictly mutually inverses in the monoid G. We assume that each y_i commutes with all elements in G, except for x_j when $j = i$.

Lemma 4.1. *Let G be a monoid, and let (y, x) be a pair of strictly mutually inverses. In this case, the maps $L_y : G \to G$ defined by $g \mapsto yg$ and $R_x : G \to G$ defined by $g \mapsto gx$ are bijective.*

Proof. We first demonstrate that L_y is injective. Suppose $L_y(a) = L_y(b)$. This implies that $ya = yb$. Now, multiplying both sides by x from the right gives us $yax = ybx$. Since we know that y commutes with all elements except x, we can simplify this: $yax = ayx = a$ and $ybx = byx = b$. Therefore, we conclude that $a = b$.

Next, we prove that R_x is injective. Suppose $R_x(a) = R_x(b)$. This implies that $ax = bx$. By multiplying both sides by y from the left, we obtain $yax = ybx$. From our earlier conclusion, we have $yax = ayx = a$ and $ybx = byx = b$. Thus, we again find that $a = b$.

Let $g \in G$. We can express g as either $L_y(xg)$ or $R_x(gy)$, which shows that both L_y and R_x are onto. This completes the proof. □

Lemma 4.2. *Let G be a topological monoid, and let (y, x) be a pair of strictly mutually inverses. In this case, the maps $L_y : G \to G$ defined by $g \mapsto yg$ and $R_x : G \to G$ defined by $g \mapsto gx$ are homeomorphisms.*

Proof. Let us demonstrate that L_y is a homeomorphism. By assumption, the multiplication map $m : G \times G \to G$ defined by $(x, y) \mapsto x * y$ is continuous, since G is a topological monoid. According to Lemma 4.1, L_y is bijective. We can express L_y as the restriction of m to $\{y\} \times G$; that is, $L_y(g) = m|_{\{y\}\times G}(y, g) = y * g$. Because m is continuous, its restriction is also continuous. Hence, L_y is continuous.

The inverse of L_y is given by $L_y^{-1} : G \to G$ defined as $g \mapsto gx$, which can be expressed as the right multiplication map R_x. Since R_x can be expressed as $m|_{G\times\{x\}}$, it follows that L_y^{-1} is also continuous. Therefore, L_y is a homeomorphism.

Similarly, R_x can be written as $m|_{G\times\{x\}}$. This is equivalent to saying that R_x is continuous. Its inverse, $R_x^{-1} = L_y$, is also continuous. Consequently, R_x is a homeomorphism. □

Example 4.3. Let $\mathbb{R}$ be equipped with the usual topology, and consider the interval $[0, 1] \subset \mathbb{R}$ with the relative topology. Define the space of functions $S = [0, 1]^{[0,1]}$, which consists of all continuous maps $f : [0, 1] \to [0, 1]$. This space S is equipped with the compact-open topology.

We can define a monoid structure on S by using function composition as the operation. It is well-known that the multiplication map $m : S \times S \to S$ is continuous.

Now, let us take two specific functions: $f_1(x) = \frac{x}{2}$ for $x \in [0, 1]$, and $f_2(x) = \min(2x, 1)$ for $x \in [0, 1]$. The composition $f_2 \circ f_1$ yields:

$$f_2 \circ f_1(x) = x,$$

which shows that f_2 is a left inverse of f_1. It's important to note that $f_1 \circ f_2 \neq f_2 \circ f_1$. Indeed, we have:

$$f_1 \circ f_2 = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2} & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Next, consider the map $L_{f_2} : S \to S$ defined by $L_{f_2}(h) = f_2 \circ h$, where f_2 is as defined before. In this case, L_{f_2} is not a homeomorphism. To illustrate this, let us take two functions $h_1 = 0.6$ and $h_2 = 0.8$. Here, we see that $h_1 \neq h_2$, but:

$$L_{f_2}(h_1) = 1 = L_{f_2}(h_2).$$

This shows that L_{f_2} is not one-to-one.

Proposition 4.4. *Let G be a topological monoid, and let $G^\bullet$ be the set of all points that lack compact neighborhoods. Let (y, x) be a pair of strictly mutually inverses. Then, $y \in G^\bullet$ if and only if $e \in G^\bullet$.*

Proof. ($\Longrightarrow$) Suppose $e \in G \setminus G^\bullet$ and $y \in G^\bullet$. This implies that there exists a compact neighborhood U_e of e; specifically, this is an open set U_e containing e such that $\overline{U_e}$ is compact in G. Since L_y is a homeomorphism, we have $L_y(U_e) \subseteq L_y(\overline{U_e}) = \overline{L_y(\overline{U_e})}$. Hence, let $V = L_y(U_e)$, which is a compact neighborhood of y. This leads to a contradiction because we assumed $y \in G^\bullet$. Therefore, we conclude that $e \notin G^\bullet$.

($\Longleftarrow$) Suppose that y has a compact neighborhood U_y and $e \in G^\bullet$. Then, $R_x(U_y)$ is a neighborhood of e. Since R_x is also a homeomorphism, we have $R_x(U_y) \subseteq R_x(\overline{U_y}) = \overline{R_x(\overline{U_y})}$. Thus, let $W = R_x(U_y)$, which is a compact neighborhood of e. This again leads to a contradiction, showing that our assumption must be false. Therefore, it follows that $y \in G^\bullet$. $\square$

Proposition 4.5. *Let G be a topological monoid, and let $G^\bullet$ be the set of all points that lack compact neighborhoods. Let (y, x) be a pair of strictly mutually inverses. Then, $y \in G^\bullet$ if and only if $x \in G^\bullet$.*

Proof. From Proposition 4.4, we conclude that $y \in G \setminus G^\bullet$ if and only if $e \in G \setminus G^\bullet$. If U_e is a compact neighborhood of e, then $R_x(U_e)$ is a compact neighborhood of x. That completes the proof. $\square$

Remark 4.6. Let G be a topological monoid. We denote G_l as the set of all elements in G that have left inverses. Similarly, G_r will refer to the set of all elements in G that have right inverses.

Theorem 4.7. *Let G be a topological monoid, and let $G^\bullet$ denote the set of all points in G that do not have compact neighborhoods. The following statements are equivalent:*

(1) *There exists an element $y \in G_r$ such that $x \in G^\bullet$.*
(2) *There exists an element $x \in G_l$ such that $y \in G^\bullet$.*
(3) *The identity element $e \in G$ belongs to $G^\bullet$.*
(4) $G_r \subseteq G^\bullet$
(5) $G_l \subseteq G^\bullet$

Proof. (1) $\Longrightarrow$ (2): This follows from Proposition 4.5, as $y \in G^\bullet$ if and only if $x \in G^\bullet$.

(2) $\Longrightarrow$ (3): This is established by combining Propositions 4.4 and 4.5. Note that $x \in G^\bullet$ if and only if $y \in G^\bullet$, which is also equivalent to $e \in G^\bullet$.

(3) $\Longrightarrow$ (4): Let (x, y) be any pair of strictly mutually inverse elements. According to Proposition 4.4, $e \in G^\bullet$ if and only if $y \in G^\bullet$. Since y was chosen arbitrarily, we conclude that $G_r \subseteq G^\bullet$.

(4) $\Longrightarrow$ (5): Again, let (x, y) be any pair of strictly mutually inverse elements. By Proposition 4.5, we find that $y \in G^\bullet$ if and only if $x \in G^\bullet$. Since both x and y were arbitrary, it follows that $G_l \subseteq G^\bullet$.

(5) $\Longrightarrow$ (1): This implication is evident. $\square$

Corollary 4.8. *If $x \in G_l \subseteq G^\bullet$ or $x \in G_r \subseteq G^\bullet$, then the union $G_l \cup G_r$ consists of non-isolated points in G if G is a T_1 space.*

5. Conclusion

It is well-known that in any topological group G, if the identity element $e \in G$ has a compact neighborhood, then all other elements in G will also have compact neighborhoods. This observation raises further questions regarding local compactness. The first question is: Are all topological monoids locally compact if the identity element has a compact neighborhood? We arrive at a negative answer to this question and construct a counterexample in which the identity element has a compact neighborhood, but the space is not locally compact. We then examine two cases and derive several results. The first case involves the set of all invertible elements, while the second case considers the set of all elements that have right inverses and the set of points that have left inverses.

References

1. Adhikari, M. R. (2016). *Basic algebraic topology and its applications*. Springer. https://doi.org/10.1007/978-3-319-26938-0.
2. Arhangel'skii, A. V. (2023). Some concepts of topology and questions in topological algebra. *Quaestiones Mathematicae, 46*(S1), 41-51. https://doi.org/10.2989/16073606.2023.2247726.
3. Barbieri, G., Di Nola, A., & Lenzi, G. (2025). Topological classes of MV-algebras. *Studia Logica*. https://doi.org/10.1007/s11225-025-10188-8.
4. Beck, J., Losev, A., & Mnev, P. (2024). Combinatorial 2D higher topological quantum field theory from a local cyclic A_∞ algebra. *Letters in Mathematical Physics, 114*, Article 125. https://doi.org/10.1007/s11005-024-01874-0.
5. El Houari, L., Belaouidel, H., & El Houari, A. (2025). Study of a Lane–Emden type equation under Navier boundary conditions and topological insight using persistent homology. *Gulf Journal of Mathematics, 21*(1), 288–301. https://doi.org/10.56947/gjom.v21i1.3318.
6. Garcia-Pacheco, F. J. (2024). Topological modules geometry. *Journal of Mathematical Sciences, 280*, 117-130. https://doi.org/10.1007/s10958-023-06806-2.
7. Haar, A. (1933). Der Maßbegriff in der Theorie der kontinuierlichen Gruppen. *Annals of Mathematics, 34*, 147-169.
8. Hofmann, K. H., & Morris, S. A. (2013). *The structure of compact groups*. De Gruyter.
9. Kaddoura, I. (2014). Construction of semi-free S^3 actions on homotopy spheres with untwisted fixed point set.*Gulf Journal of Mathematics, 2*(3), 19-25. https://doi.org/10.56947/gjom.v2i3.225.
10. Lang, S. (2002). *Algebra*. Springer. https://doi.org/10.1007/978-1-4612-0868-6.
11. Montgomery, D., & Zippin, L. (1955). *Topological transformation groups*. Wiley.
12. Morris, S. A. (1977). *Pontryagin duality and the structure of locally compact abelian groups*. Cambridge University Press.
13. Positselski, L., & Stovicek, J. (2024). Topologically semiperfect topological rings. *Algebras and Representation Theory, 27*, 245-278. https://doi.org/10.1007/s10468-023-10217-x.
14. Sipacheva, O. V., & Solonkov, A. A. (2024). Free topological Mal'tsev algebras. *arXiv*. https://arxiv.org/abs/2412.10379.
15. Spanier, E. H. (1966). *Algebraic topology*. McGraw-Hill. https://doi.org/10.1007/978-1-4684-9322-1.
16. Torzewska, F. (2024). Topological quantum field theories and homotopy cobordisms. *Applied Categorical Structures, 32*, Article 22. https://doi.org/10.1007/s10485-024-09776-x.
17. Vick, J. W. (1994). *Homology theory: An introduction to algebraic topology*. Springer. https://doi.org/10.1007/978-1-4612-0881-5.
18. Zailai, M. (2023). Locally compact spaces with defects. *European Journal of Pure and Applied Mathematics, 16*(2), 1228-1235. https://doi.org/10.29020/nybg.ejpam.v16i2.4767.

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