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MAXIMUM PRINCIPLE FOR DELAYED VOLTERRA JUMPS SYSTEMS USING ANTICIPATING ADJOINTS

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Abstract. We establish a complete stochastic maximum principle for optimal control of stochastic Volterra integral equations with delay, Lévy processes, and partial observation. Anticipated backward stochastic Volterra integral equations characterize the adjoint processes. We rigorously establish existence and uniqueness of solutions. Employing Hida-Malliavin calculus, we derive necessary and sufficient optimality conditions. Applicability is demonstrated through explicit solutions to linear problems with memory and delay effects.

Keywords. stochastic Volterra integral equations, Lévy processes, optimal control, maximum principle, Hida-Malliavin calculus, anticipated backward equations.
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1. Introduction

The maximum principle stochastic control problems related to memory systems named SVIE have been the center of attention for many researchers, considering its wide applicability in different fields such as biology and finance. Several papers studied variations of these problems where the system takes many forms of stochastic Volterra equations. Some authors used the theory of Hida-Malliavin calculus in their demonstrations, we refer to [1, 15, 16, 17, 22] for more information.

The solvability of the optimal control problem for Volterra stochastic integral equations driven by Brownian motion and an independent compensated Poisson jump process is established in [3]. The sufficient and necessary optimality conditions for forward-backward Volterra stochastic integral equations with jumps have been established in [2].
Most of the work on stochastic control of delayed equations is in infinite-dimensional time. For this kind of problem in the Brownian case, we refer to [4, 6, 8, 9, 10, 11, 14]. In application, a delayed system can be reduced to finite-dimensional systems

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because a finite-dimensional variable can represent the information from the system. For the extension of [13] to the case when a Lévy noise drives the equation, we refer to [5]. For the stochastic control problems using the Girsanov transformation, we refer to [7]. The presence of a delayed noise component invalidates this technique. In contrast, [19] addresses the non-Markovian problem formulation through a stochastic maximum principle (Pontryagin-Bismut-Bensoussan type), abandoning the dynamic programming framework. Numerical approaches for solving linear stochastic Volterra integral equations have also been developed; we refer to [20] for Walsh function approximation techniques and operational matrix methods.

This paper addresses the delay stochastic optimal control problem, focusing on systems modeled by Volterra integral equations with jumps. We establish necessary and sufficient conditions for optimal control within this framework, utilizing a Malliavin calculus. The associated adjoint processes conform to an advanced backward stochastic Volterra integral, adding depth to our control strategy. To strengthen the validity of our findings, we provide evidence of the existence and uniqueness of the Delay Stochastic Volterra Integral Equation (DSVIE). To demonstrate its practical significance, we present an example that involves the optimal control of Volterra type with delay, highlighting the flexibility of our suggested approach.

The rest of the paper is organized as follows. Section 2 formulates the controlled delayed stochastic Volterra equation with jumps and partial observation, and states the optimal control problem. Section 3 establishes existence and uniqueness of a strong solution to the forward state equation under classical Lipschitz and linear-growth conditions. Section 4 contains the core contribution of the paper: a complete stochastic maximum principle (both sufficient and necessary versions) for this non-Markovian, partially observed control problem; the corresponding adjoint processes are characterized by a new class of anticipated backward stochastic Volterra integral equations that fully reflect the mixed point-and-distributed delay structure. Section 5 illustrates the theory by solving a problem of optimal consumption from a cash flow modeled by linear DSVIE. Section 6 concludes the paper and suggests several promising directions for future research.

2. State Dynamics and Control Objective

This section formulates the optimal control problem under consideration. The controlled state process is governed by a Volterra stochastic equation that incorporates explicit delay and exponential kernel, and jump perturbations. Our objective in this work is to study under partial information the maximum principle of DSVIE in the form of:

$$\begin{aligned} X(t) = \chi(t) &+ \int_0^t \alpha(t,s,X(s),Y(s),A(s),v(s))ds \\ &+ \int_0^t \sigma(t,s,X(s),Y(s),A(s),v(s))dW(s) \\ &+ \int_0^t \int_{\mathbb{R}} \theta(t,s,X(s),Y(s),A(s),v(s),z)\tilde{N}(ds,dz); \quad t \in [0,T], \end{aligned} \tag{2.1}$$

$$X(t) = \chi(t) \ , \quad t \in [-\delta, 0] \, , \tag{2.2}$$

such that:

(1) $Y(t) = X(t-\delta)$, $\qquad A(t) = \int_{t-\delta}^t e^{-\rho(t-r)}X(r)dr.$
(2) The function χ is supposed to be deterministic , continuously and differentiable.
(3) The coefficients are defined as follows:

$$\begin{aligned} \alpha &: [0,T]^2 \times \mathbb{R}^3 \times \mathbb{V} \times \Omega \to \mathbb{R}, \\ \sigma &: [0,T]^2 \times \mathbb{R}^3 \times \mathbb{V} \times \Omega \to \mathbb{R}, \\ \theta &: [0,T]^2 \times \mathbb{R}^3 \times \mathbb{V} \times \mathbb{R}_0 \times \Omega \to \mathbb{R}. \end{aligned}$$

(4) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$ be a filtred probability space .
(5) $\mathbb{G} = \{\mathcal{G}_t\}_{t\ge 0}$, where $\mathcal{G}_t \subseteq \mathcal{F}_t$,$t \in [0,T]$ represent information available to the controller.
(6) $W(t) = W(t,\omega)$ is Brownian motion .
(7) $\tilde{N}(dt,dz) := N(dt,dz) - \nu(dz)dt$, with $N(dt,dz)$, is a poisson random measure and ν is the Lévy measure.

We associate the system (2.1)-(2.2) *the cost functional*:

$$J(v) = \mathbb{E}\left[\int_0^T h(s,X(s),Y(s),A(s),v(s))ds + g(X(T))\right], \tag{2.3}$$

such as:

$$h : [0,T] \times \mathbb{R}^3 \times \mathbb{V} \times \Omega \to \mathbb{R},$$

and

$$g : \mathbb{R} \times \Omega \to \mathbb{R},$$

where the set $\mathbb{V} \subset \mathbb{R}$ is convex.
Our aim of optimisation control problem is to obtain $v^* \in \mathcal{V}$ ($\mathcal{V}$ is the family of admissible controls $\mathcal{G}_t$-predictable) satisfies the following equation:

$$sup_{v\in\mathcal{V}} J(v) = J(v^*). \tag{2.4}$$

The following spaces equipped with their norms will be used throughout the rest of this paper:

- $\mathcal{S}^2$ is the set of $\mathbb{R}$-valued $\mathcal{F}$-adapted càdlàg processes $(X(t))_{t\in[0,T]}$ with the following norm:

$$\|X\|^2_{\mathcal{S}^2} := \mathbb{E}[\sup_{t\in[0,T]} |X(t)|^2] \ < \ \infty \ .$$

- We define $\mathbb{L}^2$ as the space of $\mathbb{R}$-valued $\mathcal{F}$-adapted processes $\{Q(t,s)\}_{(t,s)\in[0,T]^2}$ such that:

$$\|Q\|^2_{\mathbb{L}^2} := \mathbb{E}[\int_0^T \int_0^T |Q(t,s)|^2 ds dt] < \ \infty \ .$$

- $\mathbb{L}^2_\nu$ is the set of Borel functions $K : \mathbb{R}_0 \to \mathbb{R}$ and the norm:

$$\| \ K \ \|^2_{\mathbb{L}^2_\nu} := \int_0^T K(\zeta)^2 \nu(d\zeta) < \infty,$$

such that $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$.
- $\mathbb{H}^2_\nu$ is the set of $\mathcal{F}$-adapted predictable processes $R : [0,T]^2 \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$, where:
(1) $\mathbb{E}[\int_0^T \int_0^T \int_0^T |R(t,s,\zeta)|^2 \nu(d\zeta) ds dt] < \infty$.
(2) $\|R\|^2_{\mathbb{H}^2_\nu} := \mathbb{E}\left[\int_0^T \int_0^T \int_0^T |R(t,s,\zeta)|^2 \nu(d\zeta) ds dt\right]$.

3. Uniqueness and Existence of the State Equation

This section is devoted to proving existence and uniqueness results for delay SVIEs under Brownian motion and independent compensated Poisson jumps, under suitable hypotheses. Prior work [21] studied the more general framework where delay SVIEs are driven by right-continuous semi-martingales.
Let us consider the following SVIE with jumps:

$$\begin{aligned} X(t) = \chi(t) &+ \int_0^t \alpha(t,s,X(s),Y(s),A(s),v(s))\, ds \\ &+ \int_0^t \sigma(t,s,X(s),Y(s),A(s),v(s))\, dW(s) \\ &+ \int_0^t \int_{\mathbb{R}_0} \theta(t,s,X(s),Y(s),A(s),v(s),z)\, \tilde{N}(ds,dz); t \in [0,T] \, . \end{aligned} \tag{3.1}$$

We introduce the following hypotheses:

Assumption 3.1. (i) *$\chi(t)$ is a given $\mathcal{F}$_adapted cádlág process.*
(ii) *$\alpha(t,s,x,y,a,v)$, $\sigma(t,s,x,y,a,v)$ and $\theta(t,s,x,y,a,v,z)$ are $\mathcal{F}_s$ predictable for $s \le t$, for each (t,x,y,a,v) and (t,x,y,a,v,z) respectively.*

(iii) *There exists a constant $C > 0$ such that: for all $0 \le s \le t \le T, v \in V$ satisfy:*

$$|\alpha(t,s,0,0,0,v)| + |\sigma(t,s,0,0,0,v)| + \left(\int_{\mathbb{R}_0} |\theta(t,s,0,0,0,v,z)|^2 \nu(dz)\right)^{\frac{1}{2}} \le C.$$

(iv) *(Lipschitz continuity) for all $x, x', y, y', a, a' \in \mathbb{R}$, we have:*

$$\begin{aligned} |\alpha(t,s,x,y,a,v) &- \alpha(t,s,x',y',a',v)| + |\sigma(t,s,x,y,a,v) - \sigma(t,s,x',y',a',v)| \\ &+ \left(\int_{\mathbb{R}_0} |\theta(t,s,x,y,a,v,z) - \theta(t,s,x',y',a',v,z)|^2 \nu dz\right)^{\frac{1}{2}} \\ &\le C\left[|x-x'| + |y-y'| + |a-a'|\right]. \end{aligned}$$

(v) $|\alpha(t,s,x,y,a,v)| + |\sigma(t,s,x,y,a,v)| + \left(\int_{\mathbb{R}_0} |\theta(t,s,x,y,a,v,z)|^2 \nu dz\right)^{\frac{1}{2}} \le C\left[1 + |x| + |y| + |a|\right].$

Theorem 3.2. *Given Assumptions 3.1, the delayed SVIE (3.1) admits a unique solution.*

Proof. • **Existence:** Let $v \in V$ and construct X^n recursively for $n \in \mathbb{N}$ as follows:

$$\begin{cases} X^0(t) &= \chi(t) \\ X^{n+1}(t) &= \chi(t) + \int_0^t \alpha(t,s,X^n(s),Y^n(s),A^n(s),v(s))\,ds \\ & \quad + \int_0^t \sigma(t,s,X^n(s),Y^n(s),A^n(s),v(s))\,dW(s) \\ & \quad + \int_0^t \int_{\mathbb{R}_0} \theta(t,s,X^n(s),Y^n(s),A^n(s),v(s),z)\,\tilde{N}(ds,dz)\,; t \in [0,T]. \end{cases} \tag{3.2}$$

Define $\overline{X}^n := X^{n+1} - X^n$; . Utilizing the Cauchy-Schwarz inequality and assuming (iv), we will have: for all $t \in [0,T]$, and $n \ge 1$:

$$\begin{aligned} \mathbb{E}\left[\left|\overline{X}^n(s)\right|^2\right] &\le 3E\left[t\int_0^t \left|\alpha^n(t,s) - \alpha^{n-1}(t,s)\right|^2 ds + \int_0^t \left|\sigma^n(t,s) - \sigma^{n-1}(t,s)\right|^2 ds \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}_0} \left|\theta^n(t,s,z) - \theta^{n-1}(t,s,z)\right|^2 \nu(dz)\,ds\right] \\ &\le 3C^2 \mathbb{E}\left[t\int_0^t \left\{\left|\left|\overline{X}^{n-1}(s)\right| + \left|\overline{Y}^{n-1}(s)\right| + \left|\overline{A}^{n-1}(s)\right|\right|^2 \right.\right. \\ &\quad \left.\left. + 2\left|\left|\overline{X}^{n-1}(s)\right| + \left|\overline{Y}^{n-1}(s)\right| + \left|\overline{A}^{n-1}(s)\right|\right|^2\right\} ds\right], \end{aligned}$$

via the simplified form $\alpha^n(t,s) = \alpha(t,s,X^n(s),Y^n(s),A^n(s),v(s))$. Put $k = 3C^2(T+2)$. Then:

$$E\left[\left|\overline{X}^n(s)\right|^2\right] \le kE\left[\int_0^t \left|\left|\bar{X}^{n-1}(s)\right| + \left|\overline{Y}^{n-1}(s)\right| + \left|\overline{A}^{n-1}(s)\right|\right|^2 ds\right], \tag{3.3}$$

we have:

$$\left|\overline{Y}^{n-1}(s)\right| := \left|Y^{n}(s) - Y^{n-1}(s)\right|,$$

so:

$$\begin{aligned}
\mathbb{E}\int_0^t \left|\overline{Y}^{n-1}(s)\right|^2 ds &= \mathbb{E}\int_0^t \left|X^{n}(s-\delta) - X^{n-1}(s-\delta)\right|^2 ds \\
&= \mathbb{E}\int_{-\delta}^{t-\delta} \left|X^{n}(v) - X^{n-1}(v)\right|^2 dv \\
&\leq \mathbb{E}\int_0^t \left|X^{n}(v) - X^{n-1}(v)\right|^2 dv \\
&= \mathbb{E}\int_0^t \left|\bar{X}^{n-1}(s)\right|^2 ds, \qquad (3.4)
\end{aligned}$$

and

$$\left|\overline{A}^{n-1}(s)\right| := \left|A^{n}(s) - A^{n-1}(s)\right|.$$

Applying the Cauchy-Schwarz inequality and reversing the integration order yields:

$$\begin{aligned}
\mathbb{E}\int_0^t \left|\overline{A}^{n-1}(s)\right|^2 ds &= \mathbb{E}\int_0^t \left|\int_{s-\delta}^{s} e^{-\rho(s-r)}\left(X^{n}(r) - X^{n-1}(r)\right) dr\right|^2 ds \\
&\leq \mathbb{E}\int_0^t \left[\int_{s-\delta}^{s} e^{-2\rho(s-r)} dr\right]\left[\int_{s-\delta}^{s} \left|X^{n}(r) - X^{n-1}(r)\right|^2 dr\right] ds \\
&\leq \lambda\mathbb{E}\int_0^t \left\{\int_{s-\delta}^{s} \left|X^{n}(r) - X^{n-1}(r)\right|^2 dr\right\} ds \\
&\leq \lambda\mathbb{E}\int_{-\delta}^t \left\{\int_{r}^{r+\delta} \left|X^{n}(r) - X^{n-1}(r)\right|^2 ds\right\} dr \\
&\leq \lambda\mathbb{E}\int_{-\delta}^t \left|X^{n}(r) - X^{n-1}(r)\right|^2 \left\{\int_{r}^{r+\delta} ds\right\} dr \\
&\leq \delta\lambda\mathbb{E}\int_0^t \left|X^{n}(r) - X^{n-1}(r)\right|^2 dr, \qquad (3.5)
\end{aligned}$$

where:

$$\lambda = \frac{1}{2\rho}\left(1 - e^{-2\rho\delta}\right).$$

By combining equations (3.5) and (3.4), we derive:

$$\begin{aligned}
&\mathbb{E}\int_0^t \left[\left|\overline{X}^{n-1}(s)\right| + \left|\overline{Y}^{n-1}(s)\right| + \left|\overline{A}^{n-1}(s)\right|\right]^2 ds \\
&\leq 3\left[\mathbb{E}\int_0^t \left|\overline{X}^{n-1}(s)\right|^2 ds + \mathbb{E}\int_0^t \left|\overline{Y}^{n-1}(s)\right|^2 ds + \mathbb{E}\int_0^t \left|\overline{A}^{n-1}(s)\right|^2 ds\right] \\
&\leq 3\left[\mathbb{E}\int_0^t \left|\overline{X}^{n-1}(s)\right|^2 ds + \mathbb{E}\int_0^t \left|\overline{X}^{n-1}(s)\right|^2 ds + \mathbb{E}\int_0^t \delta\lambda\left|\overline{X}^{n-1}(s)\right|^2 ds\right] \\
&\leq 3\left[\mathbb{E}\int_0^t (2+\delta\lambda)\left|\overline{X}^{n-1}(s)\right|^2 ds\right] \\
&\leq 3(2+\delta\lambda)\left[\mathbb{E}\int_0^t \left|\overline{X}^{n-1}(s)\right|^2 ds\right],
\end{aligned}$$

and combining it with (3.3) gives us:

$$\mathbb{E}\left[\left|\overline{X}^n\right|^2\right] \leq 3k(2+\delta\lambda)\left[\mathbb{E}\int_0^t \left|\overline{X}^{n-1}(s)\right|^2 ds\right]. \tag{3.6}$$

Now, employing the assumption of linear growth (v), we similarly deduce as explained earlier:

$$\begin{aligned}
\mathbb{E}\left[\left|X^1(t) - X^0(t)\right|^2\right] &= \mathbb{E}\Bigg[\Bigg|\int_0^t \alpha(t,s,X^0(s),Y^0(s),A^0(s),v(s))ds \\
&\quad + \int_0^t \sigma(t,s,X^0(s),Y^0(s),A^0(s),v(s))dW(s) \\
&\quad + \int_0^t\int_{\mathbb{R}_0} \theta(t,s,X^0(s),Y^0(s),A^0(s),v(s),z)\tilde{N}(ds,dz)\Bigg|^2\Bigg] \\
&\leq 6C^2(2+\lambda\delta)E\int_0^t \left(1+|\chi(s)|^2\right)ds \\
&\leq 2kt \sup_{t\in[0,T]} E\left[1+|\chi(s)|^2\right],
\end{aligned}$$

and combine it with (3.6), results in:

$$\mathbb{E}\left[\left|X^{n+1}(t) - X^n(t)\right|^2\right] \leq \frac{2K''(K't)^{n+1}}{(n+1)!},$$

where:

$$K' := 3k(2+\delta\lambda),\ K'' = \sup_{t\in[0,T]} \mathbb{E}\left[1+\chi^2(t)\right] < \infty.$$

For $m > n > 0$, it follows that:

$$\mathbb{E}\left[\int_0^T |X^m(t) - X^n(t)|^2 \, dt\right] \leq \sum_{k=n}^{m-1} \frac{2K'' \int_0^T (K't)^{k+1} \, dt}{(k+1)!}$$

$$= \sum_{k=n}^{m-1} \frac{2K'' K'^{k+1} T^{k+2}}{(k+2)!} \to 0, \text{ as } m, n \to \infty.$$

Therefore, the sequence $\{X^n(t)\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{L}^2(\Lambda \times P)$. Passing to the limit in the Picard iterates as $n \to \infty$ yields a solution to the equation.

- **Uniqueness:** To prove uniqueness, consider the difference between any two solutions and apply an estimate analogous to the one used for convergence.

$\square$

4. Stochastic maximum principle

Having established that the forward system is well-posed in Section 3, this section focuses on the derivation of optimality conditions. Specifically, we develop a stochastic maximum principle to identify the necessary and sufficient requirements for the problem posed in Section 2.

The approach relies on an appropriately defined Hamiltonian functional and a system of anticipating adjoint processes, which take the form of backward stochastic Volterra integral equations (ABSVIEs). As well as the partial observation structure through conditional expectations.

In the remaider of this paper, we assume the following conditions:

(1) For all $s \leq t$, the functions b, σ, θ, and f are adapted to the filtration $\mathcal{F}_s$, and are all twice continuously differentiable with respect to t, x, y, and a. Additionally, these functions are continuously differentiable with respect to u for each s.
(2) The function g is $\mathcal{F}_T$-measurable and continuously differentiable in x.
(3) All partial derivatives are assumed to be bounded.

4.1. Hamiltonian and adjoint processes.

We introduce the Hamiltonian function associated with the problem:

$$\mathcal{H}(t, x, y, a, v, p, q, r(\cdot)) := H_0(t, x, y, a, v, p, q, r(\cdot)) + H_1(t, x, y, a, v, p, q, r(.)), \tag{4.1}$$

with:

$$H_0 : [0, T] \times \mathbb{R}^3 \times \mathbb{V} \times \mathbb{R}^2 \times {\mathbb{L}_v}^2 \to \mathbb{R},$$

is defined by:

$$H_0(t, x, y, a, v, p, q, r(.)) := h(t, x, y, a, v) + \alpha(t, t, x, y, a, v)p(t) + \sigma(t, t, x, y, a, v)q(t, t) + \int_{\mathbb{R}_0} \theta(t, t, x, y, a, v)r(t, t, z)\nu(dz), \tag{4.2}$$

and

$$H_1 : [0,T] \times \mathbb{R}^3 \times \mathbb{V} \times \mathbb{R}^2 \times {\mathbb{L}_v}^2 \to \mathbb{R},$$

is defined by

$$\begin{aligned} H_1(t,x,y,a,v,p,q,r\,(.)) :=& \int_t^T \frac{\partial \alpha}{\partial s}(s,t,x,y,a,v)p(s)ds \\ &+ \int_t^T \frac{\partial \sigma}{\partial s}(s,t,x,y,a,v)q\,(s,t)\,ds \\ &+ \int_t^T \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial s}(s,t,x,y,a,v,z)r(s,t,z)\nu(dz)ds. \end{aligned}$$

We denote by x, y, a, p, q, r the values of the processes $X(\cdot), Y(\cdot), A(\cdot), p(\cdot), q(\cdot), r(\cdot)$ respectively.

The adjoint processes $p(t), q(t,s), r(t,s,\cdot)$ satisfy the following BSVIE:

$$p(t) := \frac{\partial}{\partial x} g(X(T)) + \int_t^T \mathbb{E}(\mu(s)|\mathcal{F}s)ds - \int_t^T q(t,s)dW(s) - \int_t^T \int_{\mathbb{R}_0} r(t,s,z)\tilde{N}\,(ds,dz)\,, 0 \le t \le T, \tag{4.3}$$

$$\mu(t) := -\frac{\partial \mathcal{H}}{\partial x}(t) - \frac{\partial \mathcal{H}}{\partial y}(t+\delta)1_{[o,T-\delta]}(t) - e^{\rho t}\left(\int_t^{t+\delta} \frac{\partial \mathcal{H}}{\partial a}\,(s)\,e^{-\rho s}1_{[o,T]}(s)ds\right). \tag{4.4}$$

With

$$\begin{aligned} \mu(t) &= \mu_0(t) + \mu_1(t) \\ \mu_0(t) &= -\frac{\partial H_0}{\partial x}(t) - \frac{\partial H_0}{\partial y}(t+\delta)1_{[0,T-\delta]}(t) - e^{\rho t}\left(\int_t^{t+\delta} \frac{\partial H_0}{\partial a}(s)e^{-\rho s}1_{[0,T]}(s)ds\right) \\ \mu_1(t) &= -\frac{\partial H_1}{\partial x}(t) - \frac{\partial H_1}{\partial y}(t+\delta)1_{[0,T-\delta]}(t) - e^{\rho t}\left(\int_t^{t+\delta} \frac{\partial H_1}{\partial a}(s)e^{-\rho s}1_{[0,T]}(s)ds\right) \end{aligned} \tag{4.5}$$

employing the shorthand notation:

$$\frac{\partial \mathcal{H}}{\partial x}(t) = \left[\frac{\partial \mathcal{H}}{\partial x}(t, X(t), Y\,(t)\,, A\,(t)\,, v\,(t)\,, p(t), q(t,t), r(t,t,\cdot)\right]_{x=X(t)}, \tag{4.6}$$

and similarly for $\frac{\partial \mathcal{H}}{\partial y}(t)$, $\frac{\partial \mathcal{H}}{\partial a}(t)$.

Henceforth, we introduce the following additional assumption:

Assumption 4.1. Suppose that $t \mapsto q(t,s)$ and $t \mapsto r(t,s,\cdot)$ are continuously differentiable for all s, z, ω. Furthermore,

$$\mathbb{E}\left[\int_0^T \int_0^T \left(\tfrac{\partial q}{\partial t}(t,s)\right)^2 dsdt + \int_0^T \int_0^T \int_{\mathbb{R}_0} \left(\tfrac{\partial r}{\partial t}(t,s,z)\right)^2 \nu\,(dz)\,dsdt\right] < +\infty.$$

From equation (2.1), we derive the equivalent representation holding for all $(t,s)\in[0,T]^2$*:*

$$\begin{aligned} dX(t) = \chi'(t)dt &+ \alpha\left(t,t,X(t),Y(t),A(t),v(t)\right)dt \\ &+ \left(\int_0^t \frac{\partial\alpha}{\partial t}\left(t,s,X(s),Y(s),A(s),v(s)\right)ds\right)dt \\ &+ \sigma\left(t,t,X(t),Y(t),A(t),v(t)\right)dW(t) \\ &+ \left(\int_0^t \frac{\partial\sigma}{\partial t}\left(t,s,X(s),Y(s),A(s),v(s)\right)dW(s)\right)dt \\ &+ \int_{\mathbb{R}_0}\theta(t,t,X(t),Y(t),A(t),v(t),z)\tilde{N}(dt,dz) \\ &+ \left(\int_0^t\int_{\mathbb{R}_0}\frac{\partial\theta}{\partial t}(t,s,X(s),Y(s),A(s),v(s),z)\tilde{N}(ds,dz)\right)dt. \end{aligned} \tag{4.7}$$

Under Assumption 4.1, we obtain the adjoint system:

$$\begin{cases} dp(t) &= [\mathbb{E}(\mu(t)|\mathcal{F}_t) - \int_t^T \frac{\partial q}{\partial t}(t,s)dW(s) - \int_t^T\int_{\mathbb{R}_0}\frac{\partial r}{\partial t}(t,s,z)\tilde{N}(ds,dz)]dt \\ &\quad + q(t,t)dW(t) + \int_{\mathbb{R}_0} r(t,t,z)\tilde{N}(dt,dz), \\ p(T) &= \frac{\partial g}{\partial x}(X(T)). \end{cases} \tag{4.8}$$

Remark 4.2. In a specific subset of linear BSVIEs with jumps, as elaborated in Section 5, Assumption 4.1 is confirmed. Further details can be found in [12].

4.2. **Sufficient maximum principle.** We establish and prove a sufficient optimality criterion via a maximum principle:

Theorem 4.3. *(Sufficient Optimality via Maximum Principle) Let* $\hat{v}\in\mathcal{V}$ *with corresponding state and adjoint processes* $\hat{X}(t),\hat{Y}(t),\hat{A}(t),\hat{p}(t),\hat{q}(t,s),\hat{r}(t,s,z)$ *solving equations (2.1)–(2.2) and (4.8) respectively. Suppose:*

- *(Concavity Requirement) Suppose that:*

$$x\mapsto g(x)$$

and the Hamiltonian

$$(x,y,a,v)\mapsto\mathcal{H}(t,x,y,a,v,\hat{p},\hat{q},\hat{r}(\cdot))$$

are both concave in (x,y,a,v) *for each* $t\in[0,T]$*.*

- *(The maximum condition)*

$$\begin{aligned} &\max_{w\in\mathcal{V}}\mathbb{E}\left[\mathcal{H}(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),w,\hat{p}(t),\hat{q}(t,t),\hat{r}(t,t,\cdot))\mid\mathcal{G}_t\right] \\ &=\mathbb{E}\left[\mathcal{H}(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\hat{v}(t),\hat{p}(t),\hat{q}(t,t),\hat{r}(t,t,\cdot))\mid\mathcal{G}_t\right]\ \forall t\ \mathrm{P}-a.s. \end{aligned} \tag{4.9}$$

Therefore, $\hat{v}$ *serves as an optimal control for our given problem.*

Proof. Consider an increasing sequence of stopping times converging to T. This localization technique ensures that dW and $\tilde{N}$-integrals in the subsequent expressions are martingales, hence have vanishing expectation. Reference [22, Lemma 3.1] provides the full argument.

Let $v \in \mathcal{V}$ be given. Our objective is to demonstrate $J(v) \leq J(\hat{v})$. Using the explicit form of the cost functional:

$$J(v) - J(\hat{v}) = \Pi_1 + \Pi_2, \tag{4.10}$$

where

$$\Pi_1 = \mathbb{E}\left[\int_0^T \left\{h(t) - \hat{h}(t)\right\} dt\right], \quad \Pi_2 = \mathbb{E}\left[g(X(T)) - g(\hat{X}(T))\right], \tag{4.11}$$

where $h(t) = h(t, X(t), Y(t), A(t), v(t))$, and $\hat{h}(t) = h(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{v}(t))$.

Using the following simplified notation:

$$\begin{aligned}
&\hat{\alpha}(t,t) = \alpha(t,t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\hat{v}(t)), \alpha(t,.) = \alpha(t,.,X(.),Y(.),A(.),v(.)),\\
&\hat{\sigma}(t,t) = \sigma(t,t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\hat{v}(t)), \sigma(t,.) = \sigma(t,.,X(.),Y(.),A(.),v(.)),\\
&\hat{\theta}(t,t,z) = \theta(t,t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\hat{v}(t),z), \theta(t,.,z) = \theta(t,.,X(t),Y(t),A(t),v(t),z),\\
&\hat{H}_0(t) = H_0(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\hat{v}(t),\hat{p}(t),\hat{q}(t,t),\hat{r}(t,t,.)),\\
&H_0(t) = H_0(t,X(t),Y(t),A(t),v(t),p(t),q(t,t),r(t,t,.)),\\
&\hat{H}_1(t) = H_1(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\hat{v}(t),\hat{p}(t),\hat{q}(t,t),\hat{r}(t,t,.)),\\
&H_1(t) = H_1(t,X(t),Y(t),A(t),v(t),p(t),q(t,t),r(t,t,.)).
\end{aligned}$$

From the definition of the Hamiltonian given in (4.1), we obtain:

$$\begin{aligned}
\Pi_1 = \mathbb{E}\Bigg[\int_0^T \Big\{&H_0(t) - \hat{H}_0(t) - \hat{p}(t)\left(\alpha(t,t) - \hat{\alpha}(t,t)\right) - \hat{q}(t,t)\left(\sigma(t,t) - \hat{\sigma}(t,t)\right)\\
&- \int_{\mathbb{R}_0} \hat{r}(t,t,z)\left(\theta(t,t,z) - \hat{\theta}(t,t,z)\right)\nu(dz)\Big\} dt\Bigg]\\
\leq \mathbb{E}\Bigg[\int_0^T \Big\{&\frac{\partial \hat{H}_0}{\partial x}(t)\left(X(t) - \hat{X}(t)\right) + \frac{\partial \hat{H}_0}{\partial y}(t)\left(Y(t) - \hat{Y}(t)\right)\\
&+ \frac{\partial \hat{H}_0}{\partial a}(t)\left(A(t) - \hat{A}(t)\right) + \frac{\partial \hat{H}_0}{\partial v}(t)\left(v(t) - \hat{v}(t)\right) - \left(\alpha(t,t) - \hat{\alpha}(t,t)\right)\hat{p}(t)\\
&- \left(\sigma(t,t) - \hat{\sigma}(t,t)\right)\hat{q}(t,t) - \int_{\mathbb{R}_0}\left(\theta(t,t,z) - \hat{\theta}(t,t,z)\right)\hat{r}(t,t,z)\nu dz\Big\} dt\Bigg].
\end{aligned} \tag{4.12}$$

Applying Itô's formula combined with the concavity property yields:

$$
\begin{aligned}
\Pi_2 &\leq E\left[g'(\hat{X}(T))\left(X(T)-\hat{X}(T)\right)\right] = E\left[\hat{p}(T)\left(X(T)-\hat{X}(T)\right)\right] \\
&= E\left[\int_0^T \hat{p}(t)\left(dX(t)-d\hat{X}(t)\right) + \int_0^T \left(X(t)-\hat{X}(t)\right)d\hat{p}(t) + \int_0^T d<\hat{p}(t), X(t)-\hat{X}(t)>\right] \\
&= E\left[\int_0^T \left\{\hat{p}(t)\left(\alpha(t,t)-\hat{\alpha}(t,t)\right) + \left(X(t)-\hat{X}(t)\right)\hat{\mu}(t) + \left(\sigma(t,t)-\hat{\sigma}(t,t)\right)\hat{q}(t,t)\right.\right. \\
&+ \int_{\mathbb{R}_0}\left(\theta(t,t,z)-\hat{\theta}(t,t,z)\right)\hat{r}(t,t,z)\nu dz + \hat{p}(t)\int_0^t \frac{\partial\left(\alpha(t,s)-\hat{\alpha}(t,s)\right)}{\partial t}ds \\
&\left.\left.+\hat{p}(t)\int_0^t \frac{\partial\left(\sigma(t,s)-\hat{\sigma}(t,s)\right)}{\partial t}dW(s) + \hat{p}(t)\int_0^t\int_{\mathbb{R}_0}\frac{\partial\left(\theta(t,s,z)-\hat{\theta}(t,s,z)\right)}{\partial t}\tilde{N}(ds,dz)\right\}dt\right].
\end{aligned}
\tag{4.13}
$$

Combining (4.12) and (4.13), we obtain:

$$
\begin{aligned}
\Pi_1+\Pi_2 \leq & E\left[\int_0^T\left\{\frac{\partial\hat{H}_0}{\partial x}(X(t)-\hat{X}(t)) + \frac{\partial\hat{H}_0}{\partial y}(Y(t)-\hat{Y}(t)) + \frac{\partial\hat{H}_0}{\partial a}(A(t)-\hat{A}(t))\right.\right. \\
&+ \frac{\partial\hat{H}_0}{\partial v}(v(t)-\hat{v}(t)) + \hat{\mu}(t)(X(t)-\hat{X}(t)) + \hat{p}(t)\int_0^t\frac{\partial\left(\alpha(t,s)-\hat{\alpha}(t,s)\right)}{\partial t}ds \\
&\left.\left.+\hat{p}(t)\int_0^t\frac{\partial\left(\sigma(t,s)-\hat{\sigma}(t,s)\right)}{\partial t}dW(s) + \hat{p}(t)\int_0^t\int_{\mathbb{R}_0}\frac{\partial\left(\theta(t,s,z)-\hat{\theta}(t,s,z)\right)}{\partial t}\tilde{N}(ds,dz)\right\}dt\right].
\end{aligned}
\tag{4.14}
$$

Via the change of variables $t=r+\delta$ and Fubini's theorem:

$$
\begin{aligned}
\int_0^T\frac{\partial\hat{H}_0}{\partial a}(s)\left(A(s)-\hat{A}(s)\right)ds &= \int_0^T\frac{\partial\hat{H}_0}{\partial a}(s)\int_{s-\delta}^s e^{-\rho(s-r)}\left(X(r)-\hat{X}(r)\right)drds \\
&= \int_\delta^T\left(\int_r^{r+\delta}\frac{\partial\hat{H}_0}{\partial a}(s)e^{-\rho s}1_{[0,T]}(s)ds\right)e^{\rho r}\left(X(r)-\hat{X}(r)\right)dr \\
&= \int_\delta^{T+\delta}\left(\int_{t-\delta}^t\frac{\partial\hat{H}_0}{\partial a}(s)e^{-\rho s}1_{[0,T]}(s)ds\right)e^{\rho(t-\delta)}\left(X(t-\delta)-\hat{X}(t-\delta)\right)dt.
\end{aligned}
\tag{4.15}
$$

Uniting this expression with (4.14), we obtain:

$$\begin{aligned}
\Pi_1+\Pi_2 \le E\Bigg[\int_0^T \Bigg\{ &\frac{\partial \hat{H}_0}{\partial x}(t)(X(t)-\hat{X}(t)) + \frac{\partial \hat{H}_0}{\partial y}(t)(Y(t)-\hat{Y}(t)) + \mu_0(t)(X(t)-\hat{X}(t)) \\
&+\frac{\partial \hat{H}_0}{\partial a}(t)(A(t)-\hat{A}(t)) + \frac{\partial \hat{H}_0}{\partial v}(t)(v(t)-\hat{v}(t))\Bigg\} dt \\
&+\int_0^T \hat{\mu_1}(t)(X(t)-\hat{X}(t))dt + \int_0^T \Bigg\{\hat{p}(t)\int_0^t \frac{\partial\left(\alpha(t,s)-\hat{\alpha}(t,s)\right)}{\partial t}ds \\
&+\hat{p}(t)\int_0^t \frac{\partial\left(\sigma(t,s)-\hat{\sigma}(t,s)\right)}{\partial t}dW(s) \\
&+\hat{p}(t)\int_0^t\int_{\mathbb{R}_0} \frac{\partial\left(\theta(t,s,z)-\hat{\theta}(t,s,z)\right)}{\partial t}\tilde{N}(ds,dz)\Bigg\} dt\Bigg]
\end{aligned}$$

$$\begin{aligned}
&= E\Bigg[\int_\delta^{T+\delta} \Bigg\{\frac{\partial \hat{H}_0}{\partial x}(t-\delta) + \frac{\partial \hat{H}_0}{\partial y}(t)1_{[0,T]}(t) + \mu_0(t-\delta)\Bigg\}(Y(t)-\hat{Y}(t))dt \\
&+\int_0^T \frac{\partial \hat{H}_0}{\partial a}(t)(A(t)-\hat{A}(t))dt + \int_0^T \frac{\partial \hat{H}_0}{\partial v}(t)(v(t)-\hat{v}(t))dt \\
&+\int_0^T \hat{\mu_1}(t)(X(t)-\hat{X}(t))dt + \int_0^T \Bigg\{\hat{p}(t)\int_0^t \frac{\partial\left(\alpha(t,s)-\hat{\alpha}(t,s)\right)}{\partial t}ds \\
&+\hat{p}(t)\int_0^t \frac{\partial\left(\sigma(t,s)-\hat{\sigma}(t,s)\right)}{\partial t}dW(s) \\
&+\hat{p}(t)\int_0^t\int_{\mathbb{R}_0} \frac{\partial\left(\theta(t,s,z)-\hat{\theta}(t,s,z)\right)}{\partial t}\tilde{N}(ds,dz)\Bigg\} dt\Bigg]
\end{aligned}$$

$$\begin{aligned}
&= E\Bigg[\int_0^T \frac{\partial \hat{H}_0}{\partial v}(t)(v(t)-\hat{v}(t))dt + \int_0^T -\frac{\partial \hat{H}_1}{\partial x}(t)(X(t)-\hat{X}(t))dt \\
&+\int_\delta^{T+\delta} -\frac{\partial \hat{H}_1}{\partial y}(t)1_{[0,T]}(t)(Y(t)-\hat{Y}(t))dt + \int_0^T -\frac{\partial \hat{H}_1}{\partial a}(t)(A(t)-\hat{A}(t))dt \\
&+\int_0^T \Bigg\{\hat{p}(t)\int_0^t \frac{\partial\left(\alpha(t,s)-\hat{\alpha}(t,s)\right)}{\partial t}ds + \hat{p}(t)\int_0^t \frac{\partial\left(\sigma(t,s)-\hat{\sigma}(t,s)\right)}{\partial t}dW(s) \\
&+\hat{p}(t)\int_0^t\int_{\mathbb{R}_0} \frac{\partial\left(\theta(t,s,z)-\hat{\theta}(t,s,z)\right)}{\partial t}\tilde{N}(ds,dz)\Bigg\} dt\Bigg]
\end{aligned}$$

$$= E\Bigg[\int_0^T \frac{\partial \hat{H}_0}{\partial v}(t)(v(t)-\hat{v}(t))dt$$
$$+\int_0^T \left\{-\frac{\partial \hat{H}_1}{\partial x}(t)(X(t)-\hat{X}(t)) - \frac{\partial \hat{H}_1}{\partial y}(t)(Y(t)-\hat{Y}(t)) - \frac{\partial \hat{H}_1}{\partial a}(t)(A(t)-\hat{A}(t))\right\}dt$$
$$+\int_0^T \Bigg\{\hat{p}(t)\int_0^t \frac{\partial\left(\alpha(t,s)-\hat{\alpha}(t,s)\right)}{\partial t}ds + \hat{p}(t)\int_0^t \frac{\partial\left(\sigma(t,s)-\hat{\sigma}(t,s)\right)}{\partial t}dW(s)$$
$$+\hat{p}(t)\int_0^t\int_{\mathbb{R}_0} \frac{\partial\left(\theta(t,s,z)-\hat{\theta}(t,s,z)\right)}{\partial t}\tilde{N}(ds,dz)\Bigg\}dt\Bigg].$$

By employing the Fubini's theorem, we obtain:

$$\int_0^T \hat{p}(t)(\int_0^T \tfrac{\partial\tilde{\alpha}}{\partial t}(t,s)ds)dt = \int_0^T(\int_0^T \hat{p}(t)\tfrac{\partial\tilde{\alpha}}{\partial t}(t,s)dt)ds = \int_0^T(\int_0^T \hat{p}(s)\tfrac{\partial\tilde{\alpha}}{\partial s}(s,t)ds)dt. \tag{4.16}$$

Using the Hida-Malliavin calculus, together with [3, Equality (2.3)] and (4.16), we obtain:

$$\begin{aligned}\mathbb{E}[\int_0^T \hat{p}(t)(\int_0^T \tfrac{\partial\tilde{\sigma}}{\partial t}(t,s)dW(s))dt] &= \int_0^T \mathbb{E}[\int_0^T \hat{p}(t)\tfrac{\partial\tilde{\sigma}}{\partial t}(t,s)dW(s)]dt\\ &= \int_0^T \mathbb{E}[\int_0^T \mathbb{E}[D_s\hat{p}(t)|\mathcal{F}_s]\tfrac{\partial\tilde{\sigma}}{\partial t}(t,s)ds]dt.\end{aligned}$$

Fubini's theorem yields:

$$\begin{aligned}\mathbb{E}[\int_0^T \hat{p}(t)(\int_0^T \tfrac{\partial\tilde{\sigma}}{\partial t}(t,s)dW(s))dt] &= \int_0^T \mathbb{E}[\int_0^T \mathbb{E}[D_s\hat{p}(t)|\mathcal{F}_s]\tfrac{\partial\tilde{\sigma}}{\partial t}(t,s)dt]ds\\ &= \mathbb{E}[\int_0^T\int_0^T \mathbb{E}[D_t\hat{p}(s)|\mathcal{F}_t]\tfrac{\partial\tilde{\sigma}}{\partial s}(s,t)dsdt],\end{aligned}$$

Using the Hida-Malliavin calculus, and applying [3, Equality (2.6)], we obtain:

$$\mathbb{E}[\int_0^T \hat{p}(t)(\int_0^T \tfrac{\partial\tilde{\sigma}}{\partial t}(t,s)dW(s))dt] = \mathbb{E}[\int_0^T\int_0^T \hat{q}(s,t)\tfrac{\partial\tilde{\sigma}}{\partial s}(s,t)dsdt]. \tag{4.17}$$

Applying analogous considerations to the Brownian setting for jumps, including Fubini's theorem and [3, Proposition 2.4, and Theorem 2.5], we get:

$$\begin{aligned}
\mathbb{E}\left[\int_0^T\left(\int_0^t\int_{\mathbb{R}_0}\hat{p}(t)\tfrac{\partial\tilde{\theta}}{\partial t}(t,s,z)\tilde{N}(ds,dz)\right)dt\right] &= \int_0^T\mathbb{E}\left[\int_0^t\int_{\mathbb{R}_0}\hat{p}(t)\tfrac{\partial\tilde{\theta}}{\partial t}(t,s,z)\tilde{N}(ds,dz)\right]dt\\
&= \int_0^T\mathbb{E}\left[\int_0^t\int_{\mathbb{R}_0}\mathbb{E}[D_{s,z}\hat{p}(t)|\mathcal{F}_s]\tfrac{\partial\tilde{\theta}}{\partial t}(t,s,z)\nu(dz)ds\right]dt\\
&= \int_0^T\mathbb{E}\left[\int_s^T\int_{\mathbb{R}_0}\mathbb{E}[D_{s,z}\hat{p}(t)|\mathcal{F}_s]\tfrac{\partial\tilde{\theta}}{\partial t}(t,s,z)\nu(dz)dt\right]ds\\
&= \mathbb{E}\left[\int_0^T\int_t^T\int_{\mathbb{R}_0}\mathbb{E}[D_{s,z}\hat{p}(t)|\mathcal{F}_s]\tfrac{\partial\tilde{\theta}}{\partial t}(s,t,z)\nu(dz)dsdt\right]\\
&= \mathbb{E}\left[\int_0^T\int_t^T\int_{\mathbb{R}_0}\hat{r}(s,t,z)\tfrac{\partial\tilde{\theta}}{\partial s}(s,t,z)\nu(dz)dsdt\right].
\end{aligned}\tag{4.18}$$

Substituting (4.16), (4.17), (4.18), and (4.1) into (4.10) yields:

$$\begin{aligned}
&J(v)-J(\hat{v})\le E\left[\int_0^T\frac{\partial\hat{H}_0}{\partial v}(t)(v(t)-\hat{v}(t))dt\right.\\
&+\int_0^T\left\{-\frac{\partial\hat{H}_1}{\partial x}(t)(X(t)-\hat{X}(t))-\frac{\partial\hat{H}_1}{\partial y}(t)(Y(t)-\hat{Y}(t))-\frac{\partial\hat{H}_1}{\partial a}(t)(A(t)-\hat{A}(t))\right\}dt\\
&+\int_0^T\left(H_1(t)-\hat{H}_1(t)\right)dt.
\end{aligned}$$

Due to the concavity of H_1, we observe:

$$\begin{aligned}
H_1(t)-\hat{H}_1(t)&\le\frac{\partial\hat{H}_1}{\partial x}(t)\left(X(t)-\hat{X}(t)\right)+\frac{\partial\hat{H}_1}{\partial y}(t)\left(Y(t)-\hat{Y}(t)\right)\\
&+\frac{\partial\hat{H}_1}{\partial a}(t)\left(A(t)-\hat{A}(t)\right)+\frac{\partial\hat{H}_1}{\partial v}(t)\left(v(t)-\hat{v}(t)\right).
\end{aligned}$$

Therefore, given that $v=\hat{v}$ is adapted to $\mathbb{G}$ and maximizes the conditional Hamiltonian:

$$\begin{aligned}
J(v)-J(\hat{v})&\le\mathbb{E}[\int_0^T\tfrac{\partial\mathcal{H}}{\partial v}(t)(v(t)-\hat{v}(t))dt]\\
&=\mathbb{E}[\int_0^T\mathbb{E}[\tfrac{\partial\mathcal{H}}{\partial v}(t)|\mathcal{G}_t](v(t)-\hat{v}(t))dt]\le 0,
\end{aligned}\tag{4.19}$$

implying that $\hat{v}$ serves as an optimal control. □

4.3. **Necessary Conditions for Optimality.** Suppose v is an optimal control in the set $\mathcal{V}$ and β belongs to the same set. If the function $l\mapsto J(v+l\beta)$ is well-defined and differentiable in a neighborhood of 0, then:

$$\frac{d}{dl}J\left(v+l\beta\right)|_{l=0}=0.\tag{4.20}$$

Subject to appropriate assumptions on the coefficients, we aim to demonstrate that:

$$\frac{d}{dl}J\left(v+l\beta\right)|_{l=0}=0, \tag{4.21}$$

is equivalent to

$$\mathbb{E}\left[\frac{\partial\mathcal{H}}{\partial v}(t)\mid\mathcal{G}_t\right]=0.\ P-a.s.\ \forall t\in[0,T]. \tag{4.22}$$

Here are the details: For any given $t\in[0,T]$, consider $b=b(t)$ as a bounded $\mathcal{G}_t$-measurable random variable. Let $h\in[T-t,T]$, and introduce the following definition:

$$\beta\left(s\right):=b.1_{[t,t+h]}\left(s\right),\ \ s\in\left[0,T\right]. \tag{4.23}$$

Under the premise that:

$$v+l\beta\in\mathcal{V}, \tag{4.24}$$

for all such β and all $v\in\mathcal{V}$, and all nonzero h are sufficiently small. Then the derivative processes are defined by

$$\begin{array}{lcl}\varphi(t) & = & \frac{\mathrm{d}}{\mathrm{d}l}X^{v+l\beta}(t)_{|l=0}, and\ \ \forall t\in[-\delta,0]\,\varphi(t)=0,.\\ \varphi(t-\delta) & = & \frac{\mathrm{d}}{\mathrm{d}l}Y^{v+l\beta}(t)_{|l=0}.\\ \int_{t-\delta}^{t}e^{-\rho(t-r)}\varphi(r)dr & = & \frac{\mathrm{d}}{\mathrm{d}l}A^{v+l\beta}(t)_{|l=0}.\end{array}$$

Then we see that:

$$\begin{aligned}\varphi(t)=&\int_0^t\left\{\frac{\partial\alpha(t,s)}{\partial x}\varphi(s)+\frac{\partial\alpha(t,s)}{\partial y}\varphi(s-\delta)\right.\\ &\left.+\frac{\partial\alpha(t,s)}{\partial a}\int_{s-\delta}^{s}e^{-\rho(s-r)}\varphi(r)dr+\frac{\partial\alpha(t,s)}{\partial v}\beta(s)\right\}ds\\ &+\int_0^t\left\{\frac{\partial\sigma(t,s)}{\partial x}\varphi(s)+\frac{\partial\sigma(t,s)}{\partial y}\varphi(s-\delta)\right.\\ &\left.+\frac{\partial\sigma(t,s)}{\partial a}\int_{s-\delta}^{s}e^{-\rho(s-r)}\varphi(r)dr+\frac{\partial\sigma(t,s)}{\partial v}\beta(s)\right\}dW(s)\\ &+\int_0^t\int_{\mathbb{R}_0}\left\{\frac{\partial\theta(t,s,z)}{\partial x}\varphi(s)+\frac{\partial\theta(t,s,z)}{\partial y}\varphi(s-\delta)\right.\\ &\left.+\frac{\partial\theta(t,s,z)}{\partial a}\int_{s-\delta}^{s}e^{-\rho(s-r)}\varphi(r)dr+\frac{\partial\theta(t,s,z)}{\partial v}\beta(s)\right\}\tilde{N}(ds,dz).\end{aligned} \tag{4.25}$$

Hence

$$d\varphi(t)=\Big\{\frac{\partial\alpha(t,t)}{\partial x}\varphi(t)+\frac{\partial\alpha(t,t)}{\partial y}\varphi(t-\delta)+\frac{\partial\alpha(t,t)}{\partial a}\int_{t-\delta}^{t}e^{-\rho(t-r)}\varphi(r)dr+\frac{\partial\alpha(t,t)}{\partial v}\beta(t)\Big\}dt$$

$$+\Big\{\frac{\partial\sigma(t,t)}{\partial x}\varphi(t)+\frac{\partial\sigma(t,t)}{\partial y}\varphi(t-\delta)+\frac{\partial\sigma(t,t)}{\partial a}\int_{t-\delta}^{t}e^{-\rho(t-r)}\varphi(r)dr+\frac{\partial\sigma(t,t)}{\partial v}\beta(t)\Big\}dW(t)$$

$$+\int_{\mathbb{R}_0}\Big\{\frac{\partial\theta(t,t,z)}{\partial x}\varphi(t)+\frac{\partial\theta(t,t,z)}{\partial y}\varphi(t-\delta)+\frac{\partial\theta(t,t,z)}{\partial a}\int_{t-\delta}^{t}e^{-\rho(t-r)}\varphi(r)dr+\frac{\partial\theta(t,t,z)}{\partial v}\beta(t)\Big\}\tilde{N}(ds,dz)$$

$$+\Big[\int_0^t \Big\{\frac{\partial^2\alpha(t,s)}{\partial t\partial x}\varphi(s)+\frac{\partial^2\alpha(t,s)}{\partial t\partial y}\varphi(s-\delta)+\frac{\partial^2\alpha(t,s)}{\partial t\partial a}\int_{s-\delta}^s e^{-\rho(s-r)}\varphi(r)dr+\frac{\partial^2\alpha(t,s)}{\partial t\partial v}\beta(s)\Big\}ds\Big]dt$$
$$+\Big[\int_0^t \Big\{\frac{\partial^2\sigma(t,s)}{\partial t\partial x}\varphi(s)+\frac{\partial^2\sigma(t,s)}{\partial t\partial y}\varphi(s-\delta)+\frac{\partial^2\sigma(t,s)}{\partial t\partial a}\int_{s-\delta}^s e^{-\rho(s-r)}\varphi(r)dr$$
$$+\frac{\partial^2\sigma(t,s)}{\partial t\partial u}\beta(s)\Big\}dW(s)\Big]dt+\Big[\int_0^t\int_{\mathbb{R}_0}\Big\{\frac{\partial^2\theta(t,s,z)}{\partial t\partial x}\varphi(s)+\frac{\partial^2\theta(t,s,z)}{\partial t\partial y}\varphi(s-\delta)$$
$$+\frac{\partial^2\theta(t,s,z)}{\partial t\partial a}\int_{s-\delta}^s e^{-\rho(s-r)}\varphi(r)dr+\frac{\partial^2\theta(t,s,z)}{\partial t\partial v}\beta(s)\Big\}\tilde N(ds,dz)\Big]dt.$$

Theorem 4.4 (Necessary maximum principle)**.** *Consider $\hat v\in\mathcal{V}$ is such that, for all β and the corresponding solutions $\hat X(t)$ to equations (2.1)-(2.2) and $(\hat p(t),\hat q(t,t),\hat r(t,t,\cdot))$ to (4.8) exist. Then the equivalence of the following statements holds:*

(i)

$$\frac{d}{dl}J\left(\hat v+l\beta\right)|_{l=0}=0,$$

(ii)

$$\mathbb{E}\left[\frac{\partial\mathcal{H}}{\partial v}(t)\mid\mathcal{G}_t\right]_{v=\hat v(t)}=0.$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}l}J(\hat v+l\beta)_{|l=0}=0$$
$$=\frac{\mathrm{d}}{\mathrm{d}l}E\left[\int_0^T f(t,X^{\hat v+l\beta}(t),Y^{\hat v+l\beta}(t),A^{\hat v+l\beta}(t),\hat v(t)+l\beta)dt+g(X^{\hat v+l\beta}(T))\right]$$
$$=E\Bigg[\int_0^T\left\{\frac{\partial f(t)}{\partial x}\varphi(t)+\frac{\partial f(t)}{\partial y}\varphi(t-\delta)+\frac{\partial f(t)}{\partial a}\int_{t-\delta}^t e^{-\rho(t-r)}\varphi(r)dr+\frac{\partial f(t)}{\partial v}\beta(t)\right\}dt$$
$$+g'(X(t))\varphi(T)\Bigg]$$
$$=E\Bigg[\int_0^T\left\{\frac{\partial H_0(t)}{\partial x}-\frac{\partial b(t,t)}{\partial x}p(t)-\frac{\partial\sigma(t,t)}{\partial x}q(t,t)-\int_{\mathbb{R}_0}\frac{\partial\theta(t,t,z)}{\partial x}r(t,t,z)\nu dz\right\}\varphi(t)dt$$
$$+\int_0^T\left\{\frac{\partial H_0(t)}{\partial y}-\frac{\partial b(t,t)}{\partial y}p(t)-\frac{\partial\sigma(t,t)}{\partial y}q(t,t)-\int_{\mathbb{R}_0}\frac{\partial\theta(t,t,z)}{\partial y}r(t,t,z)\nu dz\right\}\varphi(t-\delta)dt$$
$$+\int_0^T\left\{\frac{\partial H_0(t)}{\partial a}-\frac{\partial b(t,t)}{\partial a}p(t)-\frac{\partial\sigma(t,t)}{\partial a}q(t,t)-\int_{\mathbb{R}_0}\frac{\partial\theta(t,t,z)}{\partial a}r(t,t,z)\nu dz\right\}$$
$$\left(\int_{t-\delta}^t e^{-\rho(t-r)}\varphi(r)dr\right)dt$$
$$+\int_0^T\left\{\frac{\partial H_0(t)}{\partial v}-\frac{\partial b(t,t)}{\partial v}p(t)-\frac{\partial\sigma(t,t)}{\partial v}q(t,t)-\int_{\mathbb{R}_0}\frac{\partial\theta(t,t,z)}{\partial v}r(t,t,z)\nu dz\right\}\beta(t)dt$$
$$+g'(X(T))\varphi(T)\Bigg].$$

By the Itô formula:

$$
\begin{aligned}
E\left[g'(X(T))\varphi(T)\right] = E\Bigg[&\int_0^T \left\{p(t)\left(\frac{\partial\alpha(t,t)}{\partial x}\varphi(t) + \frac{\partial\alpha(t,t)}{\partial y}\varphi(t-\delta)\right.\right. \\
&+ \left.\left.\frac{\partial\alpha(t,t)}{\partial a}\int_{t-\delta}^{t} e^{-\rho(t-r)}\varphi(r)dr + \frac{\partial\alpha(t,t)}{\partial v}\beta(t)\right)\right\} dt \\
&+ \int_0^T p(t)\left(\int_0^t \left\{\frac{\partial^2\alpha(t,s)}{\partial t\partial x}\varphi(s) + \frac{\partial^2\alpha(t,s)}{\partial t\partial y}\varphi(s-\delta)\right.\right. \\
&+ \left.\left.\frac{\partial^2\alpha(t,s)}{\partial t\partial a}\int_{s-\delta}^{s} e^{-\rho(s-r)}\varphi(r)dr + \frac{\partial^2\alpha(t,s)}{\partial t\partial v}\beta(s)\right\} ds\right) dt \\
&+ \int_0^T p(t)\left(\int_0^t \left\{\frac{\partial^2\sigma(t,s)}{\partial t\partial x}\varphi(s) + \frac{\partial^2\sigma(t,s)}{\partial t\partial y}\varphi(s-\delta)\right.\right. \\
&+ \left.\left.\frac{\partial^2\sigma(t,s)}{\partial t\partial a}\int_{s-\delta}^{s} e^{-\rho(s-r)}\varphi(r)dr + \frac{\partial^2\sigma(t,s)}{\partial t\partial v}\beta(s)\right\} dW(s)\right) dt \\
&+ \int_0^T p(t)\left(\int_0^t \int_{\mathbb{R}_0} \left\{\frac{\partial^2\theta(t,s,z)}{\partial t\partial x}\varphi(s) + \frac{\partial^2\theta(t,s,z)}{\partial t\partial y}\varphi(s-\delta)\right.\right. \\
&+ \left.\left.\frac{\partial^2\theta(t,s,z)}{\partial t\partial a}\int_{s-\delta}^{s} e^{-\rho(s-r)}\varphi(r)dr + \frac{\partial^2\theta(t,s,z)}{\partial t\partial v}\beta(s)\right\} \tilde{N}(ds,dz)\right) dt \\
&+ \int_0^T \varphi(t)E\left(\mu(t)/\mathcal{F}_t\right) dt + \int_0^T \left\{q(t,t)\left(\frac{\partial\sigma(t,t)}{\partial x}\varphi(t) + \frac{\partial\sigma(t,t)}{\partial y}\varphi(t-\delta)\right.\right. \\
&+ \left.\left.\frac{\partial\sigma(t,t)}{\partial a}\int_{t-\delta}^{t} e^{-\rho(t-r)}\varphi(r)dr + \frac{\partial\sigma(t,t)}{\partial v}\beta(t)\right)\right\} dt \\
&+ \int_0^T \left\{\int_{\mathbb{R}_0} r(t,t,z)\left(\frac{\partial\theta(t,t,z)}{\partial x}\varphi(t) + \frac{\partial\theta(t,t,z)}{\partial y}\varphi(t-\delta)\right.\right. \\
&+ \left.\left.\frac{\partial\theta(t,t,z)}{\partial a}\int_{t-\delta}^{t} e^{-\rho(t-r)}\varphi(r)dr + \frac{\partial\theta(t,t,z)}{\partial v}\beta(t)\right)\nu\, dz\right\} dt\Bigg].
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}l}J(\hat{v}+l\beta)\Big|_{l=0} = \mathbb{E}\Big[\int_0^T \Big\{ &\frac{\partial H_0}{\partial x}(t)\varphi(t) + \frac{\partial H_0}{\partial y}(t)\varphi(t-\delta) + \frac{\partial H_0}{\partial a}(t)\int_{t-\delta}^t e^{-\rho(t-r)}\varphi(r)\,dr \\
&+ \frac{\partial H_0}{\partial v}(t)\beta(t)\Big\}\,dt + \int_0^T \varphi(t)(\mu_0(t)+\mu_1(t))\,dt \\
&+ \int_0^T p(t)\Big\{ \int_0^t \frac{\partial^2\alpha(t,s)}{\partial t\partial x}\varphi(s)\,ds + \int_0^t \frac{\partial^2\sigma(t,s)}{\partial t\partial x}\varphi(s)\,dW(s) \\
&+ \int_0^t\int_{\mathbb{R}_0} \frac{\partial^2\theta(t,s,z)}{\partial t\partial x}\varphi(s)\,\tilde{N}(ds,dz)\Big\}\,dt + \int_0^T p(t)\Big\{ \int_0^t \frac{\partial^2\alpha(t,s)}{\partial t\partial y}\varphi(s-\delta)\,ds \\
&+ \int_0^t \frac{\partial^2\sigma(t,s)}{\partial t\partial y}\varphi(s-\delta)\,dW(s) + \int_0^t\int_{\mathbb{R}_0} \frac{\partial^2\theta(t,s,z)}{\partial t\partial y}\varphi(s-\delta)\,\tilde{N}(ds,dz)\Big\}\,dt \\
&+ \int_0^T p(t)\Big\{ \int_0^t \frac{\partial^2\alpha(t,s)}{\partial t\partial v}\beta(s)\,ds + \int_0^t \frac{\partial^2\sigma(t,s)}{\partial t\partial v}\beta(s)\,dW(s) \\
&+ \int_0^t\int_{\mathbb{R}_0} \frac{\partial^2\theta(t,s,z)}{\partial t\partial v}\beta(s)\,\tilde{N}(ds,dz)\Big\}\,dt \\
&+ \int_0^T p(t)\Big\{ \int_0^t \Big(\frac{\partial^2\alpha(t,s)}{\partial t\partial a}\int_{s-\delta}^s e^{-\rho(s-r)}\varphi(r)\,dr\Big)ds \\
&+ \int_0^t \Big(\frac{\partial^2\sigma(t,s)}{\partial t\partial a}\int_{s-\delta}^s e^{-\rho(s-r)}\varphi(r)\,dr\Big)dW(s) \\
&+ \int_0^t\int_{\mathbb{R}_0} \Big(\frac{\partial^2\theta(t,s,z)}{\partial t\partial a}\int_{s-\delta}^s e^{-\rho(s-r)}\varphi(r)\,dr\Big)\tilde{N}(ds,dz)\Big\}\,dt\Big].
\end{aligned}
$$

By utilizing (4.16) and (4.17), applying the Fubini theorem, and considering the definition of β, we derive:

$$
\frac{d}{dl}J\left(\hat{v}+l\beta\right)|_{l=0} = \mathbb{E}\left[\int_0^T \frac{\partial\mathcal{H}}{\partial v}\left(s\right)\beta\left(s\right)ds\right] = \mathbb{E}\left[\int_t^{t+h}\frac{\partial\mathcal{H}}{\partial v}\left(s\right)dsb\right].
$$

In conclusion:

$$
\frac{d}{dl}J\left(\hat{v}+l\beta\right)|_{l=0} = 0,
$$

if and only if:

$$
\mathbb{E}\left[\frac{\partial\mathcal{H}}{\partial v}(t)\mid\mathcal{G}_t\right] = 0.
$$

□

5. Application: Optimal control of Volterra cash flow with delay

In this example we chose a particuliar linear stochastic Volterra integral equations with delay (5.1), where the state $X(t)$ of the system countains a memory term given by the ds- integral.

Let $X(t) = X^v(t)$ denotes the cash flow at a rate of consumption $v(t)$ $\mathbb{G}$-adapted wich satisfies the following DSVIE:

$$\begin{aligned} X(t) = \xi + \int_0^t (\alpha(t,s)X(s-\delta) - v(s))\, ds + \int_0^t \beta(t)X(s)dW(s) \\ + \int_0^t \int_{\mathbb{R}_0} \pi(t,\zeta)X(s)\tilde{N}(ds,d\zeta), t \in [0,T], \\ X(t) = \xi, t \in [-\delta, 0], X(t-\delta) = Y(t), \end{aligned} \tag{5.1}$$

where we suppose that ξ is a constant(deterministic) and α, $\beta : [0,T]^2 \to \mathbb{R}$ and $\pi : [0,T]^2 \times \mathbb{R}_0 \to \mathbb{R}$ are deterministic functions with α, β and π bounded, we also assume that $\alpha(t,s)$ is continuously differentiable with respect to t for each s. For simplicity we assume that these functions are bounded, and we assume that there exists $\varepsilon > 0$ such that $\pi(s,\zeta) \geq -1 + \varepsilon$ for all s, ζ and the initial value $\xi \in \mathbb{R}$.

Remark 5.1. In this example we chose a particular linear stochastic Volterra integral equations with delay (5.1) because the delay is only in term represented by the ds-integral then the corresponding adjoint equation is reduced to a particular linear ABSVIE (5.6) which can be solved explicitly as in [12].

Problem 5.2. Find $\hat{v} \in \mathcal{V}$, ,wich maximises the function J such that:

$$\sup_v J(v) = J(\hat{v}), \tag{5.2}$$

where

$$J(v) = \mathbb{E}[\theta X(T) + \int_0^T \log(v(t))dt]. \tag{5.3}$$

With $\theta = \theta(\omega)$ being $\mathcal{F}_T$-measurable, the Hamiltonian functional simplifies to:

$$\begin{aligned} \mathcal{H}(t,x,y,v,\hat{p},\hat{q},\hat{r}(\cdot)) = \log(v) + \alpha(t,t)y(t)p(t) - vp(t) + \beta(t)x(t)q(t,t) \\ + \int_{\mathbb{R}_0} \pi(t,\zeta)\, x(t) r(t,t,\zeta)\nu(d\zeta) + \int_t^T \tfrac{\partial\alpha}{\partial s}(s,t)y(s)p(s)ds. \end{aligned} \tag{5.4}$$

Assume an optimal control $\hat{v} \in \mathcal{V}$ exists for (5.3) with associated state and adjoint processes $\hat{X}, \hat{p}, \hat{q}, \hat{r}$. The optimality condition then yields, for each t:

$$\mathbb{E}[\tfrac{\partial}{\partial v}\mathcal{H}(t,\hat{X}(t),\hat{Y}(t),v,\hat{p}(t),\hat{q}(t,s),\hat{r}(t,s,\cdot))|\mathcal{G}_t]_{v=\hat{v}(t)} = 0,$$

i.e.,

$$\mathbb{E}[\frac{1}{\hat{v}(t)} - \hat{p}(t)|\mathcal{G}_t] = 0.$$

Since $\hat{v}(t)$ is $\mathbb{G}$-adapted, the following relationship holds:

$$\hat{v}(t) = \frac{1}{\mathbb{E}[\hat{p}(t)|\mathcal{G}_t]}. \tag{5.5}$$

For the optimal strategy $\hat{v}(t)$, the corresponding adjoint system reduces to a linear ABSVIE of the form:

$$\begin{aligned} \hat{p}(t) &= \theta + \int_t^T \Big[E\left[\alpha(t,s+\delta)\hat{p}(s+\delta)1_{[0,T-\delta](s)}|\mathcal{F}_s\right] + \beta(s)\hat{q}(t,s) + \int_{\mathbb{R}_0} \pi(s,\zeta)\,\hat{r}(t,s,\zeta)\nu(d\zeta) \Big] ds \\ &\quad - \int_t^T \hat{q}(t,s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \hat{r}(t,s,\zeta)\tilde{N}(dt,d\zeta); \quad t \in [0,T]. \end{aligned} \tag{5.6}$$

Via a change of variables:

$$\begin{aligned}\hat{p}(t) &= \theta + \int_t^T \left[E\left[\alpha(t,s)\hat{p}(s)1_{[t+\delta,T]}|\mathcal{F}_{s-\delta}\right] + \beta(s)\hat{q}(t,s) + \int_{\mathbb{R}_0} \pi(s,\zeta)\,\hat{r}(t,s,\zeta)\nu(d\zeta)\right] ds \\ &- \int_t^T \hat{q}(t,s)dW(s) - \int_t^T \int_{\mathbb{R}_0} \hat{r}(t,s,\zeta)\tilde{N}(dt,d\zeta); \quad t \in [0,T].\end{aligned}$$

Following the methodology in [12, Theorem 1.2], we construct a new probability measure $\mathbb{Q}$ defined by:

$$d\mathbb{Q} \ := M(T)dP \text{ on } \mathcal{F}_T,$$

where the Radon-Nikodym derivative $M(t)$ satisfies:

$$\begin{cases} dM(t) &= M(t^-)[(\beta(t)dW(t) + \int_{\mathbb{R}_0} \pi(s,\zeta)\,\tilde{N}(dt,d\zeta)]; \quad t \in [0,T], \\ M(0) &= 1, \end{cases}$$

This admits the explicit solution:

$$\begin{aligned} M(t) := \exp(&\int_0^t \beta(s)dW(s) - \tfrac{1}{2}\int_0^t \beta^2(s)ds + \int_0^t \int_{\mathbb{R}_0} \ln(1+\pi(s,\zeta))\tilde{N}(ds,d\zeta) \\ &+ \int_0^t \int_{\mathbb{R}_0} \{\ln(1+\pi(s,\zeta)) - \pi(s,\zeta)\}\nu(d\zeta)ds); \quad t \in [0,T]. \end{aligned}$$

Under the measure $\mathbb{Q}$, the process

$$W_{\mathbb{Q}}(t) := W(t) - \int_0^t \beta(s)ds\,, \quad t \in [0,T] \tag{5.7}$$

becomes a Brownian motion, and the random measure

$$\tilde{N}_{\mathbb{Q}}(dt,d\zeta) := \tilde{N}(dt,d\zeta) - \pi(t,\zeta)\nu(d\zeta)dt, \tag{5.8}$$

is the $\mathbb{Q}$-compensated Poisson random measure of $N(\cdot,\cdot)$. Specifically, for all predictable processes $\gamma(t,\zeta)$ satisfying

$$\int_0^T \int_{\mathbb{R}_0} \pi^2(t,\zeta)\gamma^2(t,\zeta)\nu(d\zeta)dt < \infty,$$

the process

$$\tilde{N}_\gamma(t) := \int_0^t \int_{\mathbb{R}_0} \gamma(s,\zeta)\tilde{N}_{\mathbb{Q}}(ds,d\zeta)$$

is a local $\mathbb{Q}$-martingale. For all $0 \le t \le l \le T$, define the iterated kernels:

$$b^{(1)}(t,l) = \alpha(t,l)1_{[t+\delta,T]}, \quad b^{(2)}(t,l) = \int_t^l b^{(1)}(t,s)b(s,l)ds,$$

and recursively for $n \ge 2$:

$$b^{(n)}(t,l) = \int_t^l b^{(n-1)}(t,s)b(s,l)ds.$$

Note that if $|b^{(1)}(t,l)| \le C$ (constant) for all t,l, then by induction on $n \in \mathbb{N}$

$$|b^{(n)}(t,l)| \le \tfrac{C^n T^n}{n!},$$

for all t,l,n. Hence,

$$\Psi(t,l) := \Sigma_{n=1}^{\infty}|b^{(n)}(t,l)| < \infty,$$

for all t, l. By changing of measure, we can rewrite Equation (5.6) as:

$$\begin{array}{rl} \hat{p}(t) = & \theta + \int_t^T E\left[\alpha(t,s)\hat{p}(s)1_{[t+\delta,T]}|\mathcal{F}_{s-\delta}\right] ds - \int_t^T \hat{q}(t,s)dW_{\mathbb{Q}}(s) \\ & - \int_t^T \int_{\mathbb{R}_0} \hat{r}(t,s,\zeta)\tilde{N}_{\mathbb{Q}}(dt,d\zeta); \quad 0 \le t \le T, \end{array} \tag{5.9}$$

where the processes $B_{\mathbb{Q}}$ and $\tilde{N}_{\mathbb{Q}}$ are defined by (5.7)-(5.8). Taking the conditional $\mathbb{Q}$-expectation on $\mathcal{F}_t$, we get:

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[\hat{p}(t)|\mathcal{F}_t\right] &= \mathbb{E}_{\mathbb{Q}}[\theta + \int_t^T \alpha(t,s)1_{[t+\delta,T]}(s)\hat{p}(s)ds|\mathcal{F}_t] \\ &= \tilde{F}(t,t) + \int_t^T \alpha(t,s)1_{[t+\delta,T]}\mathbb{E}_{\mathbb{Q}}[\hat{p}(s)|\mathcal{F}_t]ds\,, \quad 0 \le t \le T, \end{aligned} \tag{5.10}$$

where

$$\tilde{F}(t,s) = \mathbb{E}_{\mathbb{Q}}[\theta(t)|\mathcal{F}_s\,].$$

Fix $l \in [0,t]$. Taking the conditional $\mathbb{Q}$-expectation on $\mathcal{F}_l$ of (5.10), we get:

$$\mathbb{E}_{\mathbb{Q}}\left[\hat{p}(t)|\mathcal{F}_l\right] = \tilde{F}(t,l) + \int_t^T \alpha(t,s)1_{[t+\delta,T]}\mathbb{E}_{\mathbb{Q}}[\hat{p}(s)|\mathcal{F}_l]ds\,, \quad l \le t \le T\,.$$

Denote

$$\tilde{p}(s) = \mathbb{E}_{\mathbb{Q}}\left[\hat{p}(s)|\mathcal{F}_l\right]\,, \quad l \le s \le T\,.$$

Then the above equation can be written as:

$$\tilde{p}(t) = \tilde{F}(t,l) + \int_t^T \alpha(t,s)1_{[t+\delta,T]}\tilde{p}(s)ds\,, \quad l \le t \le T\,.$$

Substituting $\tilde{p}(s) = \tilde{F}(s,l) + \int_s^T \alpha(s,\lambda)1_{[t+\delta,T]}\tilde{p}(\lambda)d\lambda$ in the above equation, we obtain:

$$\begin{aligned} \tilde{p}(t) &= \tilde{F}(t,l) + \int_t^T \alpha(t,s)1_{[t+\delta,T]}\{\tilde{F}(s,l) + \int_s^T \alpha(s,\lambda)1_{[t+\delta,T]}\tilde{p}(\lambda)d\lambda\}ds \\ &= \tilde{F}(t,l) + \int_t^T \alpha(t,s)1_{[t+\delta,T]}\tilde{F}(s,l)ds + \int_t^T b^{(2)}(t,\lambda)\tilde{p}(\lambda)d\lambda\,, \quad l \le t \le T\,. \end{aligned}$$

Repeating this, we get by induction:

$$\begin{aligned} \tilde{p}(t) &= \tilde{F}(t,l) + \textstyle\sum_{n=1}^{\infty} \displaystyle\int_t^T b^{(n)}(t,\lambda)\tilde{F}(\lambda,l)d\lambda \\ &= \tilde{F}(t,l) + \int_t^T \Psi(t,\lambda)\tilde{F}(\lambda,l)d\lambda\,. \end{aligned}$$

Now for $s > l = t$ we have $\mathbb{E}_{\mathbb{Q}}(\hat{p}(s)|\mathcal{F}_t) = \tilde{p}(s)$. Hence for $s = t$ we obtain $\hat{p}(t) = \tilde{p}(t)$, which implies that:

$$\begin{aligned} \hat{p}(t) &= \tilde{F}(t,t) + \int_t^T \Psi(t,\lambda)\tilde{F}(t,t)d\lambda \\ &= \mathbb{E}_{\mathbb{Q}}[\theta(t) + \theta(t)\int_t^T \Psi(t,\lambda)d\lambda|\mathcal{F}_t]. \end{aligned} \tag{5.11}$$

Now we show that the assumption A2 is satisfied.

Define

$$V(t) := \theta + \int_t^T b^{(1)}(t,r)\hat{p}(r)dr - \hat{p}(t);\ 0 \le t \le T,$$

then

$$V(t) := \int_t^T \hat{q}(t,s)dW_{\mathbb{Q}}(s) + \int_t^T \int_{\mathbb{R}_0} \hat{r}(t,s,\zeta)\tilde{N}_{\mathbb{Q}}(dt,d\zeta), \quad 0 \le t \le T,$$

Note that by the Clark-Ocone formula under change of measure, extended to $\mathbb{L}^2(\mathbb{Q},\mathcal{F}_T)$ as in [18], we get:

$$\hat{q}(t,s) = \mathbb{E}_{\mathbb{Q}}\left[D_s V(t) - V(t)\int_s^T D_s\beta\,(r)\,dW_{\mathbb{Q}}(r)|\mathcal{F}_s\right];t \le s \le T,$$

and

$$\hat{r}(t,s,\zeta) = \mathbb{E}_{\mathbb{Q}}\left[V(t)\left(L_s - 1\right) + L_s D_{s;\zeta}V(t)|\mathcal{F}_s\right];t \le s \le T,$$

where

$$L_s = \exp(\int_0^s \int_{\mathbb{R}_0}\left[D_{s;\zeta}\pi(r,\zeta)dW(r) + \ln\left(1 - \frac{D_{s;\zeta}\pi(r,\zeta)}{1-\pi(r,\zeta)}\right)(1-\pi(r,\zeta))\right]\nu(d\zeta)dr$$
$$+ \int_0^s \int_{\mathbb{R}_0} \ln\left(1 - \frac{D_{s;\zeta}\pi(r,\zeta)}{1-\pi(r,\zeta)}\right)\tilde{N}_{\mathbb{Q}}(dr,d\zeta).$$

we have β, π are deterministic and $b^{(1)}(t,s)$ is C^1 with respect to t, by [12, Theorem 2.1], we have that $\hat{q}(t,s)$ and $\hat{r}(t,s,\zeta)$ are C^1 with respect to t and

$$\mathbb{E}_{\mathbb{Q}}[\int_0^T\int_0^T(\tfrac{\partial\hat{q}(t,s)}{\partial t})^2 dsdt + \int_0^T\int_0^T\int_{\mathbb{R}_0}(\tfrac{\partial\hat{r}(t,s,\zeta)}{\partial t})^2\nu(d\zeta)dsdt] < \infty.$$

Substituting this expression for $\hat{p}(t)$ into the expression of $\hat{v}(t)$ in (5.5) and using Bayes' rule for conditional expectation under change of measure, we obtain, by concavity of the Hamiltonian, the following result:

Result: The optimal consumption rate $\hat{v}(t)$ for the problem (5.1), is given by:

$$\hat{v}(t) = \frac{1}{\mathbb{E}[\mathbb{E}_{\mathbb{Q}}[\theta+\theta\int_t^T\Psi(t,\lambda)d\lambda|\mathcal{F}_t]|\mathcal{G}_t]]} = \frac{1}{\mathbb{E}\left[\frac{\mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}(\theta+\theta\int_t^T\Psi(t,\lambda)d\lambda)|\mathcal{F}_t]}{\mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t]}|\mathcal{G}_t\right]},$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}} = M(T)$ is the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{F}_T$.

6. Conclusion

This paper develops a comprehensive framework for the optimal control of forward stochastic Volterra equations incorporating delays, Lévy-type jumps, and partial information. Employing the Hida–Malliavin calculus, we have derived a stochastic maximum principle providing both necessary and sufficient conditions for optimality. A key feature of our approach is that the adjoint processes are governed by anticipated backward stochastic Volterra integral equations (ABSVIEs). The practical relevance of these results is illustrated through an application to optimal control of linear Volterra systems with delays.

Several promising avenues for future research emerge from this work. These include extensions to mean-field stochastic Volterra systems with delays and jumps;

the development of stochastic maximum principles for fully coupled forward-backward Volterra systems incorporating memory and regime-switching; the design of robust numerical schemes and approximation theory for high-dimensional ABSVIEs; and applications to complex real-world systems such as energy markets, pension fund management with legacy assets, and epidemic control with hereditary effects.

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