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EXTENSION OF TRAUB-LIKE METHODS TO REGULARIZE NONLINEAR ILL-POSED EQUATIONS IN HILBERT SCALES

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Abstract. An extended version of fourth-order Traub-like scheme is applied to approximate nonlinear ill-posed operator equations within the framework of Hilbert scales. A detailed convergence analysis is conducted, yielding an error estimate of optimal order. Furthermore, the adaptive parameter selection strategy proposed by Pereverzev and Schock (2005) is considered for the choice of regularization parameter.

Keywords. Frechet derivative, Traub-like method, Local convergence, Hilbert scale, Ill-posed equations.
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1. Introduction to the Problem Framework

Let X denote a real Hilbert space, and let us consider the task of obtaining an approximate solution to the ill-posed operator equation

$$\mathcal{H}(w) = u, \tag{1.1}$$

where $\mathcal{H} : D(\mathcal{H}) \subseteq X \to X$ is a monotone nonlinear operator (i.e., $\langle \mathcal{H}(w) - \mathcal{H}(u), w - u\rangle \geq 0,\ for\ all\ w, u \in D(\mathcal{H})$).Let the inner product on X be represented by $\langle \cdot, \cdot \rangle$, with the induced norm denoted as $\|\cdot\|$. In this study, the image of the linear operator $\mathcal{H}'(w)$ for $w \in D(\mathcal{H})$ is symbolized by $R(\mathcal{H}'(w))$. The problem described in (1.1) typically lacks well-posedness, meaning it may not possess a unique solution that remains stable under data perturbations. Throughout, we assume that the available noisy data are given by $u^\delta \in X$ with

$$\|u - u^\delta\| \leq \delta \tag{1.2}$$

and $\|\mathcal{H}'(w)\|_{X \to X} \leq M$ for all $w \in D(\mathcal{H})$. Given the ill-posed nature of (1.1), regularization is required to achieve a stable approximate solution. See, for instance, [3, 5, 18, 20, 21, 22, 24, 26, 27] for different regularization techniques applicable to ill-posed operator equations.

A fourth-order Traub-type iteration for solving ill-posed problems within Banach and Hilbert space was investigated in [3]. The current study expands upon

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that framework by implementing the scheme within Hilbert scales, a transition intended to yield superior error bounds.

Let $L : D(L) \subset X \to X$, be a linear, unbounded, self-adjoint, densely defined and strictly positive operator on X. Let X_t be the completion of $D : \bigcap_{k=0}^{\infty} D(L^k)$ with respect to the norm $\|w\|_t$ induced by the inner product

$$\langle u, v\rangle = \langle L^t u, L^t v\rangle,\ u,\ v \in D,$$

then $(X_s)_{s\in\mathbb{R}}$ is called the Hilbert scale induced by L [4, 8, 15, 17]. Moreover, if $\beta \le \gamma$, then the embedding $X_\gamma \to X_\beta$ is continuous, and therefore the norm $\|:\|_\beta$ is also defined in X_γ and there is a constant $c_{\beta,\gamma}$ such that

$$\|X\|_\beta \le c_{\beta,\gamma}\|X\|_\gamma.$$

Several iterative procedures for approximating nonlinear equations are available in the literature [1, 2, 6, 7, 9, 10, 11, 12, 14, 16, 19, 25]. In our work, the proposed sequence $\{w^\delta_{n,\alpha,s}\}$ defined iteratively in the Hilbert scale setting to regularize (1.1) is as follows

$$\begin{aligned} u^\delta_{n,\alpha,s} &= w^\delta_{n,\alpha,s} - [\mathcal{H}'(w^\delta_{n,\alpha,s}) + \alpha L^s]^{-1}[\mathcal{H}(w^\delta_{n,\alpha,s}) + \alpha L^s(w^\delta_{n,\alpha,s} - w_0) - u^\delta] \\ z^\delta_{n,\alpha,s} &= u^\delta_{n,\alpha,s} - [\mathcal{H}'(w^\delta_{n,\alpha,s}) + \alpha L^s]^{-1}[\mathcal{H}(u^\delta_{n,\alpha,s}) + \alpha L^s \\ &\times\ (u^\delta_{n,\alpha,s} - w_0) - u^\delta] \\ w^\delta_{n+1,\alpha,s} &= z^\delta_{n,\alpha,s} - [\mathcal{H}'(w^\delta_{n,\alpha,s}) + \alpha L^s]^{-1}[\mathcal{H}(z^\delta_{n,\alpha,s}) + \alpha L^s(z^\delta_{n,\alpha,s} - w_0) - u^\delta], \end{aligned} \tag{1.3}$$

for $n = 0, 1, 2, 3, ...,$ where $w^\delta_{0,\alpha,s} = w_0$ is an initial guess. For sake of simplicity we consider $u^\delta_{n,\alpha,s} = u_n$, $z^\delta_{n,\alpha,s} = z_n$ and $w^\delta_{n,\alpha,s} = w_n$. Here, the regularization parameter α is chosen in dependence on the noisy data u^δ and the noise bound δ specified in (1.2). We adopt the adaptive parameter–choice rule introduced by Pereverzev and Schock [18].

We select a suitable regularization parameter that accounts for the data perturbations and the error tolerance defined in (1.2). Following the methodology established in [18], an adaptive selection process is utilized to optimize this parameter choice.

2. Preliminaries and Notational Framework

Let $\mathcal{H}'(w) \in L(X)$ be a bounded operator on X that is positive and self-adjoint. Denote $B_s := L^{-\frac{s}{2}}\mathcal{H}'(w)L^{-\frac{s}{2}}$, for every $w \in B_r(w_0)$. Usually, for the analysis of regularization methods in Hilbert scales, it is standard [17] to assume a condition of the following type,

$$\|\mathcal{H}'(\hat{w})w\| \to \|w\|_{-b},\ w \in X, \tag{2.1}$$

characterising the degree of ill-posedness. In contrast, in this work we replace (2.1) with a weaker condition;

$$d_1\|w\|_{-b} \le \|\mathcal{H}'(w_0)w\| \le d_2\|w\|_{-b},\ w \in D(\mathcal{H}), \tag{2.2}$$

for some reals b, d_1 and d_2. It is worth noting that the assumption in (2.2) is considerably simpler than the one stated in (2.1). Further, we define f and g by

$$f(t) = \min\{d_1^t, d_2^t\},\quad g(t) = \max\{d_1^t, d_2^t\},\ t \in \mathbb{R},\ |t| \le 1.$$

One of the key points in establishing the main results of this work is the Proposition stated below.

Proposition 2.1. *(See. [23] Proposition 3.1) For $s > 0$ and $|v| \leq 1$,*

$$f(\frac{\nu}{2})\|w\|_{\frac{\nu(s+b)}{2}} \leq \|B_s^{\frac{\nu}{2}}\| \leq g(\frac{\nu}{2})\|w\|_{\frac{\nu(s+b)}{2}},\ w \in X$$

Lemma 2.2. *(cf. Lemma 2.4 [23]) Let Proposition 2.1 hold. Then for every $h \in X$, the following holds;*

(1) $\|(\mathcal{H}'(w_0) + \beta L^s)^{-1}\mathcal{H}'(w_0)h\| \leq \overline{\psi_2(s)}\|h\|$
(2) $\|(\mathcal{H}'(w_0) + \beta L^s)^{-1}L^s h\| \leq \frac{\psi_2(s)}{\beta}\|h\|$
(3) $\|(\mathcal{H}'(w_0) + \beta L^s)^{-1}h\| \leq \psi_2(s)\beta^{\frac{-s}{2(s+b)}}\|h\|$,

where $\psi_2(s) := \frac{g(\frac{-s}{2(s+b)})}{f(\frac{s}{2(s+b)})}$, $\overline{\psi_2(s)} := \frac{g(\frac{-s}{2(s+b)})}{f(\frac{s}{2(s+b)})}$.
Let $\psi_3(s) = \frac{g(\frac{s}{s+b})}{f(\frac{s}{s+b})}$, $\psi_4(s) := \frac{g(\frac{-s}{(s+b)})}{f(\frac{s}{(s+b)})}$ *and* $\overline{\psi_4(s)} := \frac{g(\frac{-s}{(s+b)})}{f(\frac{-s}{(s+b)})}$.

We first prove that $w^\delta_{n,\alpha,s}$ converges to zero $w^\delta_{\alpha,s}$ of

$$\mathcal{H}(w) + \alpha L^s(w - w_0) = u^\delta \tag{2.3}$$

and then we prove that $w^\delta_{\alpha,s}$ is an approximation for $\hat{w}$. The following assumption is required for the convergence analysis of (1.3).

Assumption 2.1. [21] There exists a constant $\xi > 0$, such that for every $w \in D(\mathcal{H})$ and $v \in X$ there exists an element $\varphi(w, w_0, v) \in X$ satisfying $[\mathcal{H}'(w) - \mathcal{H}'(w_0)]v = \mathcal{H}'(v)\varphi(w, w_0, v)$, $\|\varphi(w, w_0, v)\| \leq \xi\|v\|\|w - w_0\|$.

We present some functions and parameters, that are essential to deduce the main results of this section.
Define functions $A_1, A_2 : [0, \infty) \to [0, \infty)$ by

$$\begin{aligned} A_1(t) &= \psi_3^2(s)\frac{\xi}{2}[1 + \frac{\xi\psi_3(s)}{4}t], \\ A_2(t) &= \psi_3(s)\frac{\xi}{2}[1 + (\frac{\xi\psi_3(s)}{2}t)^2], \end{aligned}$$

where ξ is as in Assumption (2.1).
Let $h_1 : [0, \infty) \to [0, \infty)$ be defined by $h_1(t) = A_2(t)t^2 - 1$.
Then, $h_1(t) = -1 < 0$ and $h_1(t) \to \infty$ as t approaches ∞. So, by intermediate value theorem(IVT), h_1 has a minimal zero r_1.

Define functions $g_1, A_4, A_5, A_6 : [0, r_1) \to [0, \infty)$ by

$$\begin{aligned}
g_1(t) &= \frac{\overline{\psi_2(s)}\xi A_1(t)[1+\psi_3(s)\frac{\xi}{2}t + A_1(t)\frac{t^2}{2}]}{[1-\overline{\psi_2(s)}\xi(A_2(t)t^2+t)]}t^3,\\
A_4(t) &= \frac{\psi_3(s)\xi}{2}[3A_2^2(t)t + \psi_3(s)\xi[A_2(t)t+1]]t^3,\\
A_5(t) &= \psi_3(s)\xi[A_4(t)t + \frac{\psi_3(s)\xi}{2}][A_2(t)+t^2+\frac{1}{2}A_4(t)t^3+\frac{\psi_3(s)\xi}{2}t^2]\\
&+ \frac{1}{2}(\psi_3(s)\xi t)^2[A_2(t)t+1],\\
A_6(t) &= \psi_3(s)\xi A_5(t)[A_1(t)t^2 + \frac{1}{2}A_5(t)t^2]\\
&+ \psi_3(s)\xi A_1(t)[A_2(t)t+1],
\end{aligned}$$

which are continuous, strictly nondecreasing functions.
Let $h_2 : [0, r_1) \to [0, \infty)$ be defined by

$$h_2(t) = g_1(t) - 1.$$

Then, $h_2(t) = -1 < 0$ and $h_2(t) \to \infty$ as $t \to r_1^-$. So, by IVT h_2 has a minimal zero r_0 in $(0, r_1)$. Then, for every $t \in [0, r_0]$, we obtain

$$0 < A_2(t)t^2 < 1 \tag{2.4}$$
$$0 < g_1(t) < 1. \tag{2.5}$$

Let $r = \min\{r_0, r_1\}$ and $R_0 = \eta_0 + \max\{\xi\psi_3(s)\frac{\eta_0^2}{2}, A_2(\eta_0)\eta_0^2\}$. Hereafter, assume that $\|\hat{w} - w_0\| < \rho = R_0$.

Denote $\eta_n := \|w_n - u_n\|$, and $\gamma_\rho := \psi_4(s)\alpha^{\frac{s-b}{s+b}}[M\rho + \delta]$ and now we show that $\eta_0 \leq \gamma_\rho$.

$$\begin{aligned}
\eta_0 &= \|u_0 - w_0\| \leq \|[\mathcal{H}'(w_0) + \alpha L^s]^{-1}(\mathcal{H}(w_0) - u^\delta)\|\\
&\leq \|L^{\frac{-s}{2}}[B_s + \alpha I]^{-1}L^{\frac{-s}{2}}(\mathcal{H}(w_0) - u^\delta)\|\\
&\leq \|[B_s + \alpha I]^{-1}L^{\frac{-s}{2}}(\mathcal{H}(w_0) - u^\delta)\|_{-s}\\
&\leq \frac{1}{f(\frac{s}{s+b})}\|B_s^{\frac{s}{s+b}}[B_s + \alpha I]^{-1}L^{\frac{-s}{2}}(\mathcal{H}(w_0) - u^\delta)\|\\
&\leq \frac{g(\frac{-s}{s+b})}{f(\frac{s}{s+b})}\|[B_s + \alpha I]^{-1}B_s^{\frac{2s}{s+b}}L^{\frac{-s}{2}}(\mathcal{H}(w_0) - u^\delta)\|_s\\
&\leq \frac{g(\frac{-s}{s+b})}{f(\frac{s}{s+b})}\alpha^{\frac{s-b}{s+b}}\|\mathcal{H}(w_0) - u^\delta\|\\
&\leq \psi_4(s)\alpha^{\frac{s-b}{s+b}}[\|\mathcal{H}(w_0) - \mathcal{H}(\hat{w})\| + \|u - u^\delta\|] = \gamma_\rho
\end{aligned}$$

3. The Method and Convergence Analysis

Theorem 3.1. *Let Assumption 2.1 be satisfied. Then $\{w^\delta_{n,\alpha,s}\}$ be as in (1.3) is well defined and $w^\delta_{n,\alpha,s} \in B_r(w_0)$ for all $n \geq 0$. Further $\{w^\delta_{n,\alpha,s}\}$ is a complete sequence in $B_r(w_0)$ and hence converges to $w^\delta_{\alpha,s} \in \overline{B_r(w_0)}$ and $\mathcal{H}(w^\delta_{\alpha,s}) + \alpha L^s(w^\delta_{\alpha,s} - w_0) = u^\delta_\alpha$. In addition, the following bounds apply for all $n \geq 0$:*

(1) $\|z_n - u_n\| \leq \psi_3(s)\frac{\xi}{2}\eta_n^2$
(2) $\|w_{n+1} - z_n\| \leq A_1(\eta_n)\eta_n^3$
(3) $\|w_{n+1} - u_n\| \leq A_2(\eta_n)\eta_n^2$
(4) $\eta_n \leq g_1(\eta_{n-1})\eta_{n-1}$
(5) $g_1(\eta_n) \leq g_1(r)^{4^n}, \forall n \geq 0$
(6) $\eta_n \leq g_1(r)^{\frac{(4^n-1)}{3}}\eta_0$.
(7) $\|w_{n+m} - w_{n+1}\| \leq Ce^{-\gamma_0 4^n}$, *where* $C = \frac{\tau_0}{1-g_1(r)^{4^n}}A_6(r)$ *and* $\gamma_0 = -\log(g_1(r))$.

Proof: *The proof is given by induction. By (1.3) and Assumption 2.1 we have*

$$\begin{aligned} z_0 - u_0 &= u_0 - w_0 - [\mathcal{H}'(w_0) + \alpha L^s]^{-1}[\mathcal{H}(u_0) - \mathcal{H}(w_0) + \alpha L^s(u_0 - w_0)] \\ &= -[\mathcal{H}'(w_0) + \alpha L^s]^{-1}[\int_0^1 (\mathcal{H}'(w_0 + t(u_0 - w_0)) - \mathcal{H}'(w_0))(u_0 - w_0)dt] \\ &= -[\mathcal{H}'(w_0) + \alpha L^s]^{-1}\mathcal{H}'(w_0)\int_0^1 \varphi(w_0 + t(u_0 - w_0), w_0, u_0 - w_0)dt \end{aligned}$$

and further

$$\begin{aligned} \|z_0 - u_0\| &\leq \|L^{\frac{-s}{2}}[B_s + \alpha I]^{-1}B_s L^{\frac{s}{2}}\int_0^1 \varphi(w_0 + t(u_0 - w_0), w_0, u_0 - w_0)dt\| \\ &\leq \frac{1}{f(\frac{s}{s+b})}\|B_s^{\frac{s}{s+b}}[B_s + \alpha I]^{-1}B_s L^{\frac{s}{2}}\int_0^1 \varphi(w_0 + t(u_0 - w_0), w_0, u_0 - w_0)dt\| \\ &\leq \frac{g(\frac{s}{s+b})}{f(\frac{s}{s+b})}\|L^{\frac{s}{2}}\int_0^1 \varphi(w_0 + t(u_0 - w_0), w_0, u_0 - w_0)dt\|_{-s} \\ &\leq \frac{g(\frac{s}{s+b})}{f(\frac{s}{s+b})}\frac{\xi}{2}\|u_0 - w_0\|^2 \\ &\leq \psi_3(s)\frac{\xi}{2}\eta_0^2. \end{aligned}$$

Next to get estimate $\|w_{n+1} - z_n\| \leq A_1(\eta_n)\eta_n^3$, consider $w_1 - z_0$ for n=0 from (1.3).

$$\begin{aligned} w_1 - z_0 &= z_0 - u_0 - [\mathcal{H}'(w_0) + \alpha L^s]^{-1}[\mathcal{H}(z_0) - \mathcal{H}(u_0) + \alpha L^s(z_0 - u_0)] \\ &= -[\mathcal{H}'(w_0) + \alpha L^s]^{-1}[\int_0^1 (\mathcal{H}'(u_0 + t(z_0 - u_0)) - \mathcal{H}'(w_0))(z_0 - u_0)dt] \\ &= -[\mathcal{H}'(w_0) + \alpha L^s]^{-1}\mathcal{H}'(w_0)\int_0^1 \varphi(u_0 + t(z_0 - u_0), w_0, z_0 - u_0)dt \end{aligned}$$

and

$$\begin{aligned}
\|w_1 - z_0\| &\leq \|L^{\frac{-s}{2}}[B_s+\alpha I]^{-1}B_s L^{\frac{s}{2}}\int_0^1 \varphi(u_0+t(z_0-u_0),w_0,z_0-u_0)dt\| \\
&\leq \frac{1}{f(\frac{s}{s+b})}\|B_s^{\frac{s}{s+b}}[B_s+\alpha I]^{-1}B_s L^{\frac{s}{2}}\int_0^1 \varphi(u_0+t(z_0-u_0),w_0,z_0-u_0)dt\| \\
&\leq \frac{g(\frac{s}{s+b})}{f(\frac{s}{s+b})}\|L^{\frac{s}{2}}\int_0^1 \varphi(u_0+t(z_0-u_0),w_0,z_0-u_0)dt\|_{-s} \\
&\leq \frac{g(\frac{s}{s+b})}{f(\frac{s}{s+b})}\xi\|z_0-u_0\|[\|u_0-w_0\|+\frac{1}{2}\|z_0-u_0\|] \\
&\leq \psi_3^2(s)\frac{\xi}{2}[1+\frac{\xi\psi_3(s)}{4}\eta_0]\eta_0^3 = A_1(\eta_0)\eta_0^3.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|w_1 - u_0\| &\leq \|L^{\frac{-s}{2}}[L^{\frac{-s}{2}}\mathcal{H}'(w_0)L^{\frac{-s}{2}}+\alpha I]^{-1}L^{\frac{-s}{2}}\mathcal{H}'(w_0)L^{\frac{-s}{2}}L^{\frac{s}{2}} \\
&\times \int_0^1 \varphi(u_0+t(z_0-w_0),w_0,z_0-w_0)dt\| \\
&\leq \frac{g(\frac{s}{s+b})}{f(\frac{s}{s+b})}\frac{\xi}{2}\|z_0-w_0\|^2 \\
&\leq \psi_3(s)\frac{\xi}{2}[1+(\frac{\xi\psi_3(s)}{2}\eta_0)^2]\eta_0^2 = A_2(\eta_0)\eta_0^2.
\end{aligned}$$

And again by using Assumption (2.1)

$$\begin{aligned}
w_1 - u_1 &= [\mathcal{H}'(w_0)+\alpha L^s]^{-1}[\mathcal{H}'(w_0)(z_0-w_1)-[\mathcal{H}(z_0)-\mathcal{H}(w_1)]] \\
&+ [[\mathcal{H}'(w_1)+\alpha L^s]^{-1}-[\mathcal{H}'(w_0)+\alpha L^s]^{-1}][\mathcal{H}(w_1)-u^\delta+\alpha L^s(w_1-w_0) \\
&= [\mathcal{H}'(w_0)+\alpha L^s]^{-1}[\int_0^1 \mathcal{H}'(z_0+t(w_1-z_0))-\mathcal{H}'(w_0)]dt(w_1-z_0) \\
&+ [\mathcal{H}'(w_0)+\alpha L^s]^{-1}[\mathcal{H}'(w_1)-\mathcal{H}'(w_0)](u_1-w_1) \\
&= -[\mathcal{H}'(w_0)+\alpha L^s]^{-1}\mathcal{H}'(w_0)\int_0^1 \varphi(z_0+t(w_1-z_0),w_0,w_1-z_0)dt \\
&+ [\mathcal{H}'(w_0)+\alpha L^s]^{-1}\mathcal{H}'(w_0)\varphi(w_0,w_1,u_1-w_1).
\end{aligned}$$

Using the above deduced estimates,

$$\begin{aligned}
\|w_1-u_1\| &\leq \xi\overline{\psi_2(s)}\|w_1-z_0\|[\|z_0-w_0\|\frac{\|w_1-z_0\|}{2}] \\
&+ \xi\overline{\psi_2(s)}\|w_0-u_0\|\|w_1-u_1\| \\
[1-\overline{\psi_2(s)}\xi(A_2(\eta_0)\eta_0^2+\eta_0)]\|w_1-u_1\| &\leq \overline{\psi_2(s)}\xi A_1(\eta_0)[1+\psi_3(s)\frac{\xi}{2}\eta_0+A_1(\eta_0)\frac{\eta_0^2}{2}]\eta_0^4.
\end{aligned}$$

Thus, we have

$$\eta_1 = \|w_1-u_1\| \leq g_1(\eta_0)\eta_0 \leq \eta_0. \tag{3.1}$$

The induction for (1), (2), (3) *and* (4), *is complete, if we simply replace,* $w_0,\ u_0,\ z_0,\ w_1,\ u_1$ *by* $w_n,\ u_n,\ z_n,\ w_{n+1},\ u_{n+1}$ *in the preceding arguments. To prove* (5) *and* (6), *observe that*

$$g_1(\mu t) \leq \mu^3 g_1(t) \text{ for } 0 < \mu < 1$$

$$g_1(\eta_n) \leq g_1(\eta_{n-1})^4 \leq \cdots \leq g_1(\eta_0)^{4^n} \leq g_1(r)^{4^n}. \tag{3.2}$$

Notice that,

$$\begin{aligned}
\eta_n &\leq g_1(\eta_{n-1})\eta_{n-1} \\
&\leq g_1(\eta_0)^{4^{n-1}} g_1(\eta_{n-2})\eta_{n-2} \\
&\leq g_1(\eta_0)^{4^{n-1}+4^{n-2}+\cdots+1}\eta_0 \\
&\leq g_1(r)^{\frac{4^n-1}{3}}\eta_0.
\end{aligned}$$

Clearly, $w_0,\ u_0 \in B(w_0, R_0)$. *With triangle inequality,* $\|z_0 - w_0\| \leq \|z_0 - u_0\| + \|u_0 - w_0\| \leq \frac{\psi_3(s)\xi}{2}\eta_0^2 + \eta_0 \leq R_0$, *so* $z_0 \in B(w_0, R_0)$. *Also*

$$\|w_1 - w_0\| \leq \|w_1 - u_0\| + \|u_0 - w_0\| \leq A_2(\eta_0)\eta_0^2 + \eta_0 \leq R_0$$

If we replace $u_0,\ z_0,\ w_1$ *by* $u_n,\ z_n,\ w_{n+1}$ *in the above argument, we obtain* $u_n,\ z_n,\ w_{n+1} \in B(w_0, R_0)$, *for all* $n \geq 0$. *To prove the completeness we further deduce the following estimates using Assumption 2.1.*

$$\begin{aligned}
u_1 - z_0 &= -[\mathcal{H}'(w_1) + \alpha L^s]^{-1}\mathcal{H}'(w_0)\int_0^1 \varphi(u_0 + t(w_1 - u_0), w_1, w_1 - u_0)dt \\
&+ [\mathcal{H}'(w_1) + \alpha L^s]^{-1}\mathcal{H}'(w_1)\varphi(w_0, w_1, z_0 - u_0) \\
\|u_1 - z_0\| &\leq \frac{\psi_3(s)\xi}{2}[3A_2^2(\eta_0)\eta_0 + \psi_3(s)\xi[A_2(\eta_0)\eta_0 + 1]]\eta_0^3 = A_4(\eta_0)\eta_0^3,
\end{aligned}$$

$$\begin{aligned}
z_1 - z_0 &= -[\mathcal{H}'(w_1) + \alpha L^s]^{-1}\mathcal{H}'(w_0)\int_0^1 \varphi(u_0 + t(w_1 - u_0), w_1, u_1 - u_0)dt \\
&+ [\mathcal{H}'(w_1) + \alpha L^s]^{-1}\mathcal{H}'(w_1)\varphi(w_0, w_1, z_0 - u_0) \\
\|z_1 - z_0\| &\leq \psi_3(s)\xi[A_4(\eta_0)\eta_0^3 + \frac{\psi_3(s)\xi}{2}\eta_0^2][A_2(\eta_0) + \eta_0^2 + \frac{1}{2}A_4(\eta_0)\eta_0^3 + \frac{\psi_3(s)\xi}{2}\eta_0^2] \\
&+ \frac{1}{2}(\psi_3(s)\xi\eta_0)^2[A_2(\eta_0)\eta_0^2 + \eta_0] \\
&\leq A_5(\eta_0)\eta_0^3,
\end{aligned}$$

and

$$\begin{aligned}
w_2 - w_1 &= -[\mathcal{H}'(w_1) + \alpha L^s]^{-1}\mathcal{H}'(w_1)\int_0^1 \varphi(z_0 + t(z_1 - z_0), w_1, z_1 - z_0)dt \\
&+ [\mathcal{H}'(w_1) + \alpha L^s]^{-1}\mathcal{H}'(w_1)\varphi(w_0, w_1, w_1 - z_0) \\
\|w_2 - w_1\| &\leq \xi\|z_1 - z_0\|[\|w_1 - z_0\| + \frac{\|z_1 - z_0\|}{2}] + \xi\|w_1 - z_0\|\|w_1 - w_0\| \\
&\leq \psi_3(s)\xi A_5(\eta_0)\eta_0^3[A_1(\eta_0)\eta_0^3 + \frac{1}{2}A_5(\eta_0)\eta_0^3] \\
&+ \psi_3(s)\xi A_1(\eta_0)\eta_0^3[A_2(\eta_0)\eta_0^2 + \eta_0] \\
&= A_6(\eta_0)\eta_0^4.
\end{aligned}$$

Now, simply replace $u_0, u_1, z_0, z_1, w_1, w_2$ *in the above argument by* $u_n, u_{n+1}, z_n, z_{n+1}, w_{n+1}, w_{n+2}$ *to complete the induction, and hence we have*

$$\|w_{n+2} - w_{n+1}\| \leq A_6(\eta_n)\eta_n^4 \leq A_6(r)\eta_n^4 \tag{3.3}$$

Therefore,

$$\begin{aligned}
\|w_{n+m} - w_{n+1}\| &\leq \sum_{i=0}^{m-2}\|w_{n+i+2} - w_{n+i+1}\| \\
&\leq \sum_{i=0}^{m-2} A_6(r)\eta_{n+i}^4 \\
&\leq A_6(r)[\eta_n^4 + \eta_{n+1}^4 + \cdots + \eta_{n+m-2}^4] \qquad (3.4) \\
&\leq A_6(r)[g_1(r)^{\frac{(4^n-1)}{3}}\eta_0 + g_1(r)^{\frac{(4^{n+1}-1)}{3}}\eta_0 + \cdots + g_1(r)^{\frac{(4^{n+m-2}-1)}{3}}\eta_0] \\
&\leq \frac{1}{1 - g_1(r)^{4^n}}A_6(r)g_1(r)^{\frac{(4^n-1)}{3}}\eta_0 \\
&\leq \frac{\eta_0}{1 - g_1(r)^{4^n}}A_6(r)g_1(r)^{4^n} \\
&\leq Ce^{-\gamma_0 4^n}. \qquad (3.5)
\end{aligned}$$

Hence, the sequence $\{w_n\}$ *forms a Cauchy's sequence in* $B(w_0, R_0)$*. Therefore it is convergent, and from the initial step of the iteration from the method (1.3) of Theorem (3.1), we have* $\{w_n = w^\delta_{n,\alpha,s}\}$ *converges to* $\{w^\delta_{\alpha,s}\}$*.*

4. Error Bounds Within the Framework of Source Conditions

The following condition is essential for establishing the error estimate of $\|\hat{w} - w^\delta_{\alpha,s}\|$.

Assumption 4.1. (cf.[23]) Let n $\psi : (0, \|B_s\|] \to (0, \infty)$ be a continuous, index function, provided that the conditions below are satisfied:

- $\lim_{\lambda \to 0} \psi(\lambda) = 0$,
- $\sup_{\lambda > 0} \frac{\alpha\psi(\lambda)}{\lambda + \alpha} \leq \psi(\alpha)$ for all $\lambda \in (0, \|B_s\|]$ and

- there exists $w \in X$ with $\|w\| \leq E_2$, such that

$$B_s^{\frac{s}{(s+b)}} L^{\frac{s}{2}}(w_0 - \hat{w}) = \psi(B_s)w$$

Remark 4.1. If the initial error satisfies $w_0 - \hat{w} \in X_t$(equivalently, $\|w_0 - \hat{w}\|_t \leq E$ for some $E > 0$ and $0 \leq t \leq s + b$, then the preceding assumption holds.

Theorem 4.2. *Suppose $w_{\alpha,s}^\delta$ is the solution of (2.3) and Assumption 2.1 and 4.1 hold. Then*

$$\|\hat{w} - w_{\alpha,s}^\delta\| \leq \frac{1}{1 - \psi_3(s)\xi r}\left(\frac{1}{f(\frac{s}{(s+b)})}\psi(\alpha) + \psi_2(s)\alpha^{\frac{s-b}{s+b}}\delta\right)$$

Proof. *Note that* $(\mathcal{H}(w_{\alpha,s}^\delta) - u^\delta) + \alpha L^s(w_{\alpha,s}^\delta - w_0) = 0$, *so*

$$\begin{aligned}(\mathcal{H}'(w_0) + \alpha L^s)(w_{\alpha,s}^\delta - \hat{w}) &= (\mathcal{H}'(w_0) + \alpha L^s)(w_{\alpha,s}^\delta - \hat{w}) - (\mathcal{H}(w_{\alpha,s}^\delta) - u^\delta) - \alpha L^s(w_{\alpha,s}^\delta - w_0) \\ &= \alpha L^s(w_0 - \hat{w}) + \mathcal{H}'(w_0)(w_{\alpha,s}^\delta - \hat{w}) - [\mathcal{H}(w_{\alpha,s}^\delta) - \mathcal{H}(\hat{w}) + \mathcal{H}(\hat{w}) - u^\delta] \\ &= \alpha L^s(w_0 - \hat{w}) - (\mathcal{H}(\hat{w}) - u^\delta) + \mathcal{H}'(w_0)(w_{\alpha,s}^\delta - \hat{w}) - [\mathcal{H}(w_{\alpha,s}^\delta) - \mathcal{H}(\hat{w})].\end{aligned}$$

Thus

$$\begin{aligned}\|w_{\alpha,s}^\delta - \hat{w}\| &\leq \|\alpha(\mathcal{H}'(w_0) + \alpha L^s)^{-1}L^s(w_0 - \hat{w})\| + \|(\mathcal{H}'(w_0 + \alpha L^s)^{-1}(\mathcal{H}(\hat{w}) - u^\delta)\| \\ &+ \|(\mathcal{H}'(w_0) + \alpha L^s)^{-1}[\mathcal{H}'(w_0)(w_{\alpha,s}^\delta - \hat{w}) - (\mathcal{H}(w_{\alpha,s}^\delta) - \mathcal{H}(\hat{w}))]\| \\ &\leq \|\alpha(\mathcal{H}'(w_0) + \alpha L^s)^{-1}L^s(w_0 - \hat{w})\| + \psi_2(s)\alpha^{\frac{s-b}{s+b}}\delta + \Gamma \qquad (4.1)\end{aligned}$$

where $\Gamma := \|(\mathcal{H}'(w_0 + \alpha L^s)^{-1}\int_0^1[\mathcal{H}'(w_0) - \mathcal{H}'(\hat{w} + t(w_{\alpha,s}^\delta - \hat{w}))](w_{\alpha,s}^\delta - \hat{w})dt\|$. *Note that by Assumption 4.1, we obtain*

$$\begin{aligned}\|\alpha(\mathcal{H}'(w_0) + \alpha L^s)^{-1}L^s(w_0 - \hat{w})\| &= \|\alpha L^{\frac{-s}{2}}(B_s + \alpha)^{-1}L^{\frac{s}{2}}(w_0 - \hat{w})\| \\ &\leq \frac{1}{f(\frac{s}{(s+b)})}\|\alpha(B_s + \alpha)^{-1}B_s^{\frac{s}{(s+b)}}L^{\frac{s}{2}}(w_0 - \hat{w})\| \\ &\leq \frac{1}{f(\frac{s}{(s+b)})}\sup_{\lambda \in \sigma(\mathcal{H}'(w_0))}\frac{\alpha\psi_1(\lambda)}{\lambda + \alpha} \\ &\leq \frac{1}{f(\frac{s}{(s+b)})}\psi(\lambda)\end{aligned}$$

and by Assumption 4.1 and Lemma 2.4 we obtain

$$\begin{aligned}\Gamma &\leq \|(\mathcal{H}'(w_0) + \alpha L^s)^{-1}\int_0^1[\mathcal{H}'(w_0) - \mathcal{H}'(\hat{w} + t(w_{\alpha,s}^\delta - \hat{w})) \\ &\times (w_{\alpha,s}^\delta - \hat{w})dt\| \\ &\leq \psi_3(s)\xi r\|w_{\alpha,s}^\delta - \hat{w}\| \qquad (4.2)\end{aligned}$$

and hence by (4.2), one gets

$$\|w_{\alpha,s}^\delta - \hat{w}\| \leq \frac{1}{1 - \psi_3(s)\xi r}\left(\frac{1}{f(\frac{s}{(s+b)})}\psi(\alpha) + \psi_2(s)\alpha^{\frac{s-b}{s+b}}\delta\right).$$

Combining the results of Theorems 3.1 and 4.2 leads directly to the following theorem.

Corollary 4.3. *Let $\{w^\delta_{n,\alpha,s}\}$ denote the sequence defined in (1.3) with the choice $\alpha = \alpha(\delta)$ and $\delta \in (0, \delta_0]$. We assume that the conditions of Theorem 3.1 and Theorem 4.2 hold. Then*

$$\|\hat{w} - w^\delta_{n,\alpha,s}\| \leq Ce^{-\gamma_0 4^n} + \frac{1}{1-\psi_3(s)\xi r}\left(\frac{1}{f(\frac{s}{(s+b)})}\psi(\alpha) + \psi_2(s)\alpha^{\frac{s-b}{s+b}}\delta\right) \tag{4.3}$$

Further, let

$$n_k := \min\{n : Ce^{-\gamma_0 4^n} \leq \alpha^{\frac{s-b}{s+b}}\delta\}.$$

Then

$$\|\hat{w} - w^\delta_{n_k,\alpha,s}\| \leq \frac{\max\{\frac{1}{f(\frac{s}{2(s+b)})}, \psi_2(s) + \frac{\gamma_\rho}{1-q}\}}{1-\overline{\psi_2(s)}\xi r}(\psi(\alpha) + \alpha^{\frac{-b}{s+b}}\delta). \tag{4.4}$$

The error estimate $\psi(\alpha) + \alpha^{\frac{s-b}{s+b}}\delta$ in (4.4) in Theorem 4.3 attains minimum for the choice $\alpha := \alpha(\delta, s, b)$ which satisfies $\psi(\alpha) = \alpha^{\frac{s-b}{s+b}}\delta$. Clearly $\alpha(\delta, s, b) = \psi^{-1}(\psi^{-1}_{s,b}(\delta))$, where

$$\psi_{s,b}(\lambda) = \lambda[\psi^{-1}(\lambda)]^{\frac{b}{s+b}}, \quad 0 < \lambda \leq \|B_s\| \tag{4.5}$$

and in this case

$$\|\hat{w} - w^\delta_{\alpha,s}\| \leq C_s\psi^{-1}_{s,b}(\delta) \tag{4.6}$$

where $C_s = \frac{\max\{\frac{1}{f(\frac{s}{2(s+b)})}, \psi_2(s)+\frac{\gamma_\rho}{1-q}\}}{1-\overline{\psi_2(s)}\xi r}$, the above error estimate has at least optimal order with respect to δ, s and b.

4.1. **Adaptive Scheme and stopping rule.** This study employs a modified version of the balancing principle introduced by Pereverzev and Schock [18] to determine the regularization parameter α. A key feature of this selection criterion is its independence from the specific regularization technique used. Let

$$l := \max\{i : \psi(\alpha_i) \leq \alpha_i^{\frac{s-b}{s+b}}\delta\} < N \tag{4.7}$$

and

$$k := \max\{i : \|w^\delta_{n_i,\alpha_i,s} - w^\delta_{n_j,\alpha_j,s}\| \leq 4\alpha_j^{\frac{s-b}{s+b}}\delta, j = 0, 1, 2, \cdots, i\} \tag{4.8}$$

Then, with $\alpha = \alpha_k,\ n = n_k$, it implies that $l \leq k$ ((cf. Theorem 3.10, [23])) and

$$\|\hat{w} - w^\delta_{n_k,\alpha_k,s}\| \leq C_s(2 + \frac{4\eta}{\eta - 1})\eta\psi^{-1}_{s,b}(\delta),$$

where C_s is as in (4.6).

5. Numerical Example

Example 5.1. Nonlinear Integral Equation: Consider the nonlinear integral equation of the form:

$$\mathcal{H}(w)(t) = \int_0^1 \kappa(t,\tau)w(\tau)^3\,d\tau = u(t), \quad t \in [0,1]$$

Consider a smooth kernel, $\kappa(t,\tau) = t(1-\tau)$ for $\tau \geq t$ and $\tau(1-t)$ for $\tau < t$. This kernel is associated with the operator $L = -d^2/dt^2$, which defines the Hilbert

TABLE 1. Error comparison $||w_n - w_{true}||$ of regularization methods in Hilbert scales

Iteration	Newton	Newton-Landweber	Levenberg-Marquardt	Traub (2-step)	Proposed
1	86.883	1.8166	1.8166	1.1058e+07	2.1683e+22
2	57.318	1.8166	1.8166	6.2801e+06	1.099e+22
3	37.608	1.8166	1.8166	3.5665e+06	5.5704e+21
4	24.469	1.8166	1.8166	2.0254e+06	2.8233e+21
5	15.711	1.8166	1.8166	1.1502e+06	1.431e+21

scale L_s. Let $\hat{w}(t) = t(1-t)$. Calculate $u = \mathcal{H}(\hat{w})$ and noise is added to create u^δ such that $||u - u^\delta|| \leq \delta$ Next to demonstrate the efficiency of the fourth-order nature of the Traub-like scheme, it is compared with the following existing methods as follows: The iteration for a fixed α is run for very small δ. The error $e_n = ||w_n - w^\delta_{\alpha,s}||$ is plotted against the iteration count n.

(1) Iteratively Regularized Gauss-Newton Method (IRGNM)

$$w_{k+1} = w_k + (\mathcal{H}'(w_k)^*\mathcal{H}'(w_k) + \alpha_k L^{2s})^{-1}\left(\mathcal{H}'(w_k)^*(u - \mathcal{H}(w_k)) - \alpha_k L^{2s}(w_k - w_0)\right)$$

(2) Newton-Landweber Method

$$w_{k,j+1} = w_{k,j} + \omega L^{-2s}\mathcal{H}'(w_k)^*(u - \mathcal{H}(w_k) - \mathcal{H}'(w_k)(w_{k,j} - w_k))$$

(3) Levenberg-Marquardt Method

$$w_{k+1} = w_k + (\mathcal{H}'(w_k)^*\mathcal{H}'(w_k) + \alpha_k L^{2s})^{-1}\mathcal{H}'(w_k)^*(u - \mathcal{H}(w_k))$$

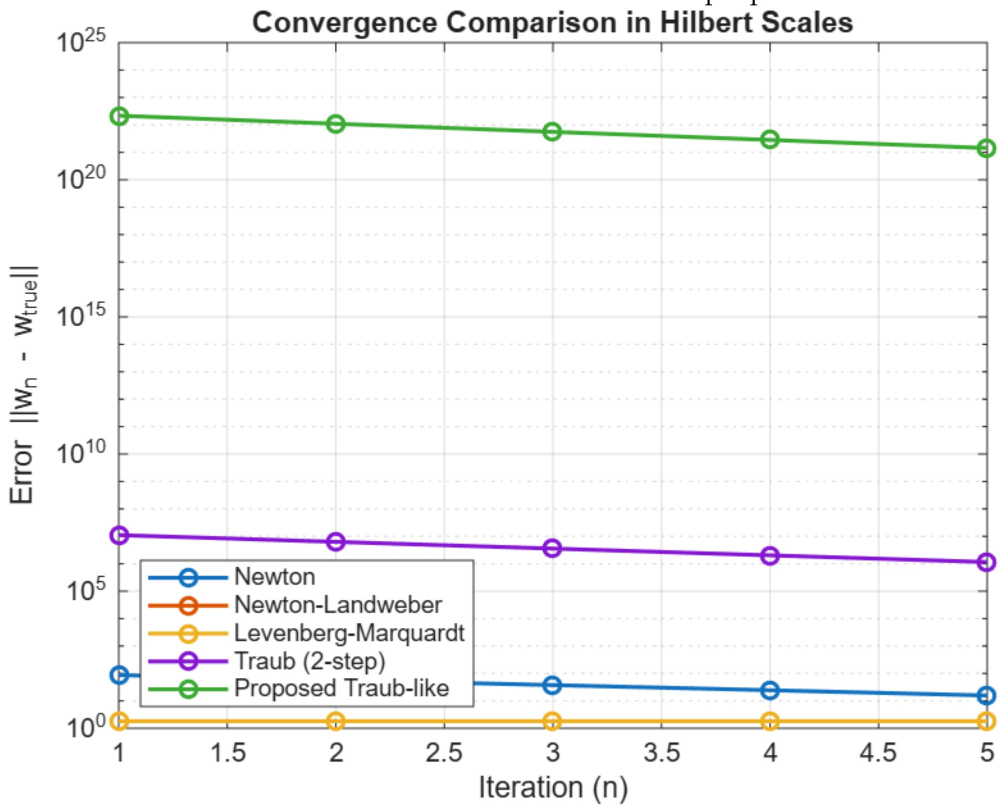
(4) Traub (2-step) Method: Predictor (Newton Step):

$$\lambda_k = w_k - (\mathcal{H}'(w_k)^*\mathcal{H}'(w_k) + \alpha_k L^{2s})^{-1}\mathcal{H}'(w_k)^*(\mathcal{H}(w_k) - u)$$

Corrector:

$$w_{k+1} = \lambda_k - (\mathcal{H}'(w_k)^*\mathcal{H}'(w_k) + \alpha_k L^{2s})^{-1}\mathcal{H}'(w_k)^*(\mathcal{H}(\lambda_k) - u)$$

Form the table1 it can be observed that the error has decreased drastically (quadratically or higher) compared to standard Newton methods. It can be observed that the α_k selected by the adaptive algorithm yields an error $||\hat{w} - w^\delta_{n_k,\alpha_k,s}||$ that is very close to the theoretical optimal error $C_s\psi^{-1}_{s,b}(\delta)$ which contributes to the numerical verification of the proposed method.



6. Conclusion

In this work, a Traub-type iterative regularization method was developed to obtain stable approximate solutions of ill-posed operator equations within the framework of Hilbert scales. The convergence of the proposed scheme has been rigorously established, leading to optimal error estimates. By incorporating general source conditions, we derived corresponding error bounds, where the regularization parameter is determined through the adaptive strategy introduced by Pereverzev and Schock [18].

The use of Hilbert scales provides a notable advantage over conventional regularization in Hilbert spaces. The scaling framework allows the method to exploit the intrinsic smoothness of the exact solution more effectively, producing sharper error estimates and improving the robustness of regularization in severely ill-posed settings. Hilbert scales also enable a more flexible treatment of operators with varying degrees of ill-posedness, thereby yielding stability and convergence results that are typically stronger than those obtained in the standard Hilbert-space setting.

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