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IMPROVED SHOOTING METHOD FOR SOLVING A BOUNDARY VALUE PROBLEM WITH A POWER-LAW NONLINEARITY

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ABSTRACT. This work investigates a class of nonlinear second-order boundary value problems with a power-type nonlinearity. To solve such problems, an improved shooting method is proposed that combines the classical shooting approach with a numerical–analytical technique based on power series expansions and analytic continuation. The method allows one to control the domain of analyticity of the solution and to obtain a priori error estimates at each continuation step. Numerical experiments are presented to demonstrate the effectiveness and accuracy of the proposed approach. A comparative analysis with the classical numerical shooting method based on Runge–Kutta integration is also provided.

Keywords. Approximate analytical solution, nonlinear boundary value problem, numerical–analytical continuation, a priori error estimates, majorant method, Shooting method
2020 Mathematics Subject Classification. Primary 34B15; Secondary 65L10

1. Introduction

In the present work, we consider the nonlinear boundary value problem

$$\begin{cases} aw'' - pw' + qw^k = 0, \quad z \in (0, L), \\ w(0) = w_-, \quad w(L) = w_+, \end{cases} \tag{1.1}$$

where p and q are given real parameters and $k > 1$ characterizes the strength of the nonlinear source term. Boundary value problems of this type naturally arise in various applied models, including reaction–diffusion processes with advection, nonlinear transport phenomena, and the study of stationary profiles of traveling waves in nonlinear media. In such models, the second-order term w'' typically represents diffusion or viscosity effects, the first-order term $-pw'$ corresponds to drift or advective transport, while the nonlinear term qw^k describes reaction kinetics or nonlinear source mechanisms [12, 16].

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Boundary value problems for nonlinear second-order ordinary differential equations have been extensively investigated in the literature, primarily with respect to existence, uniqueness, and qualitative properties of solutions. However, for equations with power-type nonlinearities, explicit analytical solutions are rarely available. As a result, numerical methods have become the main tool for studying such problems. Classical references on numerical boundary value problems describe a wide range of approaches, including finite difference schemes, shooting methods, and collocation techniques [1, 11].

Finite difference methods are among the most commonly used techniques for solving nonlinear boundary value problems of second order. By discretizing the interval $(0, L)$, the differential equation (1.1) is reduced to a nonlinear system of algebraic equations, which is typically solved using iterative procedures such as Newton's method. While finite difference schemes are relatively straightforward to implement and can achieve high accuracy for smooth solutions, they may suffer from serious drawbacks in the presence of strong nonlinearities. In particular, convergence may depend sensitively on the choice of the initial guess, and numerical instability may occur unless sufficiently fine meshes are employed [15].

Another classical approach is the shooting method, which transforms the boundary value problem into an initial value problem by iteratively adjusting unknown initial data to satisfy the boundary condition at $z = L$. Despite its conceptual simplicity, the shooting method is well known to be unreliable for strongly nonlinear or stiff problems. In equations involving power-type nonlinearities, small perturbations in the initial data can lead to large deviations in the solution, resulting in numerical instability or divergence of the method [1].

Collocation and finite element methods provide a more flexible framework for the numerical approximation of nonlinear boundary value problems. These methods approximate the solution in suitable finite-dimensional function spaces and reduce the original problem to a system of nonlinear algebraic equations. Although collocation and finite element techniques often exhibit improved stability properties compared to shooting methods, their practical implementation requires careful selection of basis functions and numerical integration procedures. Moreover, for large values of the exponent k or in parameter-dependent problems, the resulting nonlinear systems can be computationally expensive and difficult to solve reliably [2].

Despite the widespread use of numerical methods, their application to nonlinear boundary value problems such as (1.1) is subject to inherent limitations. Numerical solutions typically provide only approximate information about the solution structure and may fail to detect multiple solutions or bifurcation phenomena. Furthermore, numerical convergence and stability often depend critically on discretization parameters and solver tolerances. These limitations motivate the development of analytical and semi-analytical approaches that can complement numerical studies by providing qualitative insight into the structure of solutions and their dependence on parameters.

2. Preliminaries

2.1. **Description of the numerical–analytical shooting method.** We present the algorithm for the improved shooting method.

(1) As in the classical shooting method, reformulate the boundary-value problem as a Cauchy (initial-value) problem. Thus the boundary conditions are transformed into initial conditions as follows

$$\left\{ \begin{array}{l} w(0) = w_- \\ w(L) = w_+ \end{array} \right. \to \left\{ \begin{array}{l} w(0) = w_- \\ w'(0) = \varpi \end{array} \right. \tag{2.1}$$

where ϖ is an unknown parameter chosen so that the conditions at the right endpoint are satisfied.

(2) We now turn to the study of the Cauchy problem

$$w'' - pw' + qw^k = 0 \tag{2.2}$$

$$\left\{ \begin{array}{l} w(0) = w_- \\ w'(0) = \varpi \end{array} \right. \tag{2.3}$$

(3) As a method for solving the Cauchy problem, we use not classical numerical methods (such as, for example, RK4), but a numerical-analytical method employed by the author in previous works [4, 5, 3, 6], and partially applied by other researchers in works [9, 10, 13, 14].

The advantages of the proposed method compared to classical numerical methods are as follows:

a. Guarantee of absence of singular points in the domain of solution investigation, since there is a formula for calculating the domain of analyticity. This allows performing numerical-analytical continuation of the solution.

b. Advantage in accuracy. Classical numerical methods can indicate the order of accuracy, but do not provide an a priori error estimate. This method allows finding the solution at each step of the iterative process with any predetermined accuracy, since there is a formula for calculating the a priori error estimate.

(4) By performing the numerical-analytical continuation up to the right boundary, the residual at the right end is computed using Newton's method.

(5) The parameter is recalculated, and the process is repeated until convergence.

2.2. **The main idea of the numerical-analytical method for solving the Cauchy problem.** Instead of the Cauchy problem (2.2)–(2.3), we use a more general formulation

$$w'' - pw' + qw^k = 0 \tag{2.4}$$

$$\left\{ \begin{array}{l} w(z_0) = w_0 \\ w'(z_0) = w_1 \end{array} \right. \tag{2.5}$$

Next, in the vicinity of the initial conditions, we seek a solution representable in the form of a Taylor series

$$w(z) = \sum_{n>0} C_n \left(z - z_0\right)^n . \tag{2.6}$$

The theory of power series is a local theory; therefore, to move from formal series to convergent ones, it is necessary to determine the domain of convergence of the solution represented in the form of (2.6). Next, after finding the domain of convergence, it is necessary to go beyond the obtained domain. For this, analytic continuation must be performed. However, in this case, it is practically impossible to use the analytic solution (2.6) directly, since it is generally unclear to which function the series (2.6) converges pointwise. Therefore, it is necessary to use an approximate analytic solution of the form

$$w_N(z) = \sum_{n=0}^{N} C_n \left(z - z_0\right)^n , \tag{2.7}$$

that is, a partial sum of the series (2.6). To use (2.7), it is necessary to estimate the error of such a solution. Then, using the form (2.7), a numerical-analytical solution can be performed. After obtaining the first domain near the initial point, a point close to the boundary of the obtained domain can be chosen. At this point, new initial conditions are determined, obtained from the approximate analytic solution (2.7). Thus, at all subsequent steps of the iterative process, the solution in the form of (2.7) can be used to determine new initial conditions. It should be understood that the obtained initial conditions contain some error, which raises the question of the influence of this error on the domain of convergence of the approximate analytic solution (2.7).

The method described above, combined with the Newton method, allows solving boundary value problems with higher accuracy (any predetermined accuracy) compared to classical numerical methods, which only guarantee the order of accuracy.

Next, we move on to the main results obtained in the course of the study.

3. Main results

To begin, we formulate a theorem that permits using a solution in the form of Taylor series in the neighborhood of the initial point.

Theorem 3.1. *Let $|u_0|, |u_1|, |\xi_0| \neq \infty$, then for the Cauchy problem (2.4) – (2.5), there exists a unique solution, which can be represented as:*

$$w(z) = \sum_{n\geq 0} W_n (z - z_0)^n, \tag{3.1}$$

in the disk

$$D_{z_0} = \left\{ z : |z - z_0| < \frac{1}{\gamma(M+1)} \right\},$$

where $\gamma = \max\left\{|p|, |q|\right\}$, $M = \max\left\{\left|W_0\right|, \left|W_1\right|\right\}$.

Proof. In the domain of analyticity, the solution can be represented as a Taylor series (3.1). Therefore, we substitute (3.1) into equation (2.4)

$$\left(\sum_{n\geq 0} W_n(z-z_0)^n\right)'' - p\left(\sum_{n\geq 0} W_n(z-z_0)^n\right)' + q\left(\sum_{n\geq 0} W_n(z-z_0)^n\right)^k = 0,$$

$$\sum_{n\geq 2} n(n-1)W_n(z-z_0)^{n-2} - p\sum_{n\geq 1} nW_n(z-z_0)^{n-1} \\ + q\sum_{n\geq 0} W_{n,k}(z-z_0)^n = 0.$$

From the last equality, we obtain a recurrence relation for the coefficients

$$(n+2)(n+1)W_{n+2} = p(n+1)W_{n+1} - qW_{n,k} \tag{3.2}$$

for all $n \geq 0$. To determine $W_{n,k}$, we use the results from [7]:

$$W_{0,0} = W_0^k, W_{n,k} = \frac{1}{nW_0}\sum_{m=1}^{n}(mk-n+m)\, W_m W_{n-m,k}. \tag{3.3}$$

Equality (3.3) may also have a non-recursive representation. From a variation of Fa‘a di Bruno's formula [8], the following representations can be obtained:

$$W_{n,k} = \sum_{\alpha\in A_n} W_{\alpha_1}\cdots W_{\alpha_p} \tag{3.4}$$

where

$$A_n = \{\alpha = (\alpha_0,\ldots,\alpha_p) : 0\leq p\leq n, \alpha_0+\ldots+\alpha_p = n\}.$$

$$W_{n,k} = \sum_{\sigma\in\Pi_{n,k}} \left(\begin{array}{c} k \\ \sigma_1,\ldots,\sigma_n \end{array}\right) W_1^{\sigma_1}\cdot\ldots\cdot W_n^{\sigma_n}, \tag{3.5}$$

where

$$\Pi_{n,k} = \{\sigma = (\sigma_1,\ldots,\sigma_n) : \sigma_1+\ldots+\sigma_n = k,\ \sigma_1+2\sigma_2+\ldots+n\sigma_n = n\}$$

To determine the domain of convergence of the series (3.1), it is necessary to obtain an estimate for the coefficients W_n. For this, we formulate the following lemma.

Lemma 3.2. *Let the recurrence relation* (3.2) *hold and the initial conditions* W_0 *and* W_1 *be given; then the following estimate is valid*

$$|W_{n+2}| \leq \frac{\gamma^{n+2}(M+1)^{n+2}}{n+2}, \tag{3.6}$$

where $\gamma = \max\{|p|,|q|\}$, $M = \max\{|W_0|,|W_1|\}$.

Proof. To obtain the estimate, we use the triangle inequality

$$|W_{n+2}| = \left|\frac{p(n+1)W_{n+1} - qW_{n,k}}{(n+2)(n+1)}\right| \leq \frac{|p|(n+1)|W_{n+1}| + |q||W_{n,k}|}{(n+2)(n+1)}$$
$$\leq \gamma\frac{(n+1)\,|W_{n+1}| + |U_{n,k}|}{(n+1)\,(n+2)}$$
$$\leq \gamma\frac{\gamma^{n+1}\,(M+1)^{n+1} + \sum_{\pi\in\Pi_{n,k}}\begin{pmatrix} k \\ \sigma_1,\dots,\sigma_n\end{pmatrix} W_1^{\sigma_1}\cdot\ldots\cdot W_n^{\sigma_n}}{(n+1)\,(n+2)}$$
$$\leq \gamma\frac{\gamma^{n+1}\,(M+1)^{n+1} + \left(\frac{\gamma^n(M+1)^n}{n!}\right)^k \sum_{\sigma\in\Pi_{n,k}}\begin{pmatrix} k \\ \sigma_1,\dots,\sigma_n\end{pmatrix}}{(n+1)\,(n+2)}$$
$$\leq \frac{\gamma^{n+2}\,(M+1)^{n+1} + \gamma\left(\frac{\gamma^n(M+1)^n}{(n-1)!}\right)^k}{(n+1)\,(n+2)}.$$

Since $\lim\limits_{n\to\infty}\frac{\gamma^n(M+1)^n}{(n-1)!} = 0$, for some $n \geq N_0 \in \mathbb{N}$, the condition $\gamma\left(\frac{\gamma^n(M+1)^n}{(n-1)!}\right)^k \ll 1$ will hold. From this, we obtain the final estimate for $|W_{n+2}|$:

$$|W_{n+2}| \leq \frac{\gamma^{n+2}\,(M+1)^{n+1} + 1}{(n+1)\,(n+2)} \leq \frac{\gamma^{n+2}\,(M+1)^{n+2}}{n+2}.$$

□

The proven lemma allows us to find the region of convergence for the series (3.1), which is the solution to the Cauchy problem (2.4) – (2.5). To do this, we consider the series

$$\sum_{n=N_0}^{\infty}\frac{\gamma^n\,(M+1)^n}{n}\,|z-z_0|^n\,, \tag{3.7}$$

which is majorizing the series (3.1). The region of convergence for the series (3.7) is determined by the inequality

$$|z-z_0| < \frac{1}{\gamma\,(M+1)},$$

where $M = \max\{|W_0|\,,|W_1|\}$.

Formally, the solution to (3.1) is an analytical solution, but it cannot be used due to the general nature of the problem and the absence of a general formula for the coefficients. This leads to the problem of finding an analytical approximate solution, which takes the form of a partial sum of the series (3.1)

$$w_N\,(\xi) = \sum_{n=0}^{N} W_n\,(z-z_0)^n\,, \tag{3.8}$$

and determining estimates for the remainder error of the series (3.1). To do this, we will formulate the following theorem.

□

Theorem 3.3. *Let all the conditions of Theorem 3.1 be satisfied. Then, for the analytical approximate solution (3.8) of the Cauchy problem (2.4) – (2.5), in the domain D_{z_0} the error estimate holds:*

$$\delta w_N(z) \le \frac{\left(\gamma\left(M+1\right)\left|z-z_0\right|\right)^{N+1}}{\left(N+1\right)\left(1-\gamma\left(M+1\right)\left|z-z_0\right|\right)},$$

for all $N \ge N_0 \in \mathbb{N}$.

Proof. The error of the analytical approximate solution $\delta W_N(\xi)$ is the absolute value of the difference between $W(\xi)$ and $W_N(\xi)$:

$$\begin{aligned}\delta W_N\left(z\right) &= \left|w\left(z\right)-w_N\left(z\right)\right| \\ &= \left|\sum_{n\ge 0} W_n\left(z-z_0\right)^n - \sum_{n=0}^{N} W_n\left(z-z_0\right)^n\right| = \left|\sum_{n=N+1}^{\infty} W_n\left(z-z_0\right)^n\right|.\end{aligned}$$

To estimate the expression $\left|\sum_{n=N+1}^{\infty} U_n(\xi-\xi_0)^n\right|$, we will use the triangle inequality $\left|\sum a_i\right| \le \sum \left|a_i\right|$:

$$\begin{aligned}&\left|\sum_{n=N+1}^{\infty} W_n\left(z-z_0\right)^n\right| \le \sum_{n=N+1}^{\infty} \left|W_n\right|\left|z-z_0\right|^n \le \sum_{n=N+1}^{\infty} \frac{\gamma^n\left(M+1\right)^n}{n}\left|z-z_0\right|^n \\ &\le \frac{1}{N+1}\sum_{n=N+1}^{\infty}\left(\gamma\left(M+1\right)\left|z-z_0\right|\right)^n \le \frac{\left(\gamma\left(M+1\right)\left|z-z_0\right|\right)^{N+1}}{\left(N+1\right)\left(1-\gamma\left(M+1\right)\left|z-z_0\right|\right)}.\end{aligned}$$

□

3.1. **Numerical-analytical continuation.** Theorem 3.1, which was established in the previous section, enables the construction of an approximate solution in analytic form, but only within a limited neighborhood around the initial conditions. This limitation naturally leads to the need for analytic continuation beyond this local region. In our approach, the solution is determined using formula (3.8) rather than (3.1), which shifts the focus of analytic continuation to understanding the sensitivity of the approximate analytic solution to small perturbations in the initial data.

To solve this problem, instead of the Cauchy problem (2.4) – (2.5), the following Cauchy problem is considered:

$$w'' - pw' + qw^k = 0, \tag{3.9}$$

$$\begin{cases} w(z_1) = \tilde{w}_0 \\ w'(z_1) = \tilde{w}_1 \end{cases} \tag{3.10}$$

with the corresponding analytical approximate solution

$$\tilde{w}_N\left(\xi\right) = \sum_{n=0}^{N} \tilde{W}_n\left(z-z_1\right)^n. \tag{3.11}$$

Theorem 3.4. *Let all the conditions of Theorems 3.1 and 3.3 be satisfied, and the condition $\delta M = M \cdot \delta w$ holds, where $\delta w = \max\{\delta w_0, \delta w_1\}$, $\delta w_0 = w_0 - \tilde{w}_0$, $\delta w_1 = w_1 - \tilde{w}_1$, then for the analytical approximate solution (3.11), the following error estimate holds:*

$$\delta \tilde{w}_N(\xi) \leq \sum_{n=0}^{N_0} \delta W_n |z - z_1|^n + \delta M \frac{(\gamma(M + \delta M + 1)|z - z_1|)^{N_0+1}}{(N_0 + 1)(1 - \gamma(M + \delta M + 1)|z - z_1|)} + \frac{(\gamma(M + 1)|z - z_1|)^{N+1}}{(N + 1)(1 - \gamma(M + 1)|z - z_1|)}, \quad (3.12)$$

in the disk

$$D_{z_1} = \left\{ z : |z - z_1| < \frac{1}{\gamma(M + \delta M + 1)} \right\},$$

where $D_{z_0} \cap D_{z_1} \neq \emptyset$.

Proof. We will use the triangle inequality $|a - b| \leq |a - c| + |c - b|$ to estimate the error of the analytical approximate solution (3.11):

$$\begin{aligned}
&\delta \tilde{w}_N(z) \leq |w(z) - \tilde{w}_N(z)| \leq |w(z) - \tilde{w}(z)| + |\tilde{w}(z) - \tilde{w}_N(z)| \\
&= \left| \sum_{n \geq 0} W_n (z - z_1)^n - \sum_{n \geq 0} \tilde{W}_n (z - z_1)^n \right| + \left| \sum_{n \geq 0} \tilde{W}_n (z - z_1)^n - \sum_{n=0}^{N} \tilde{W}_n (z - z_1)^n \right| \\
&= \left| \sum_{n \geq 0} \delta W_n (z - z_1)^n \right| + \left| \sum_{n=N+1}^{\infty} \tilde{W}_n (z - z_0)^n \right| \\
&\leq \sum_{n \geq 0} |\delta W_n| |z - z_1|^n + \sum_{n=N+1}^{\infty} \left| \tilde{W}_n \right| |z - z_1|^n .
\end{aligned}$$

To estimate the expression $\sum_{n=N+1}^{\infty} \left|\tilde{W}_n\right| |z - z_1|^n$, we will use Theorem 3.3. For estimating the expression $\sum_{n \geq 0} |\delta W_n| |z - z_1|^n$, estimates for $|\delta W_n|$ are required. Similarly to the results obtained in Lemma 3.2, the following estimates can be derived:

$$|\delta W_n| \leq \frac{\gamma^n (M + \delta M + 1)^n}{n} \delta M. \quad (3.13)$$

Taking into account the estimate (3.13) and the results obtained earlier in the work, we get the estimate for $\delta\tilde{w}_N(\xi)$:

$$\begin{aligned}\delta\tilde{w}_N(z) \leq \sum_{n=0}^{N_0} \delta W_n |z-z_1|^n + \sum_{n\geq N_0+1} \frac{\gamma^n (M+\delta M+1)^n}{n} \delta M |z-z_1|^n \\ + \frac{(\gamma(M+1)|z-z_1|)^{N+1}}{(N+1)(1-\gamma(M+1)|z-z_1|)} \\ \leq \sum_{n=0}^{N_0} \delta W_n |z-z_1|^n + \delta M \frac{(\gamma(M+\delta M+1)|z-z_1|)^{N_0+1}}{(N_0+1)(1-\gamma(M+\delta M+1)|z-z_1|)} \\ + \frac{(\gamma(M+1)|z-z_1|)^{N+1}}{(N+1)(1-\gamma(M+1)|z-z_1|)}.\end{aligned}$$

□

4. Numerical experiment

We consider the boundary value problem for a second-order nonlinear differential equation:

$$w'' - pw' + qw^k = 0, \quad z \in (0, L)$$

with the boundary conditions

$$w(0) = w_-, \quad w(L) = w_+$$

where (p, q, k, L, w_-, w_+) are the problem parameters. The experiment is conducted using two methods: a numerical-analytical method and a purely numerical method.

The numerical-analytical method consists of two parts: first, the Cauchy problem is solved analytically using power series. Then, an analytic continuation is performed along a chain of curves up to the right boundary. Next, a numerical method is applied to find the residual using Newton's method. The numerical method is the classical shooting method.

The advantage of the numerical-analytical method is its high accuracy; moreover, at each step of the analytic continuation, the accuracy can be controlled using the theorems established in this work.

Next, a comparative analysis of the two methods is presented, along with the error of both methods.

A comparative analysis of Figures (1), (2), (3) and (4) shows that changing the number of terms in the series expansion affects the degree of accuracy. Thus, the theorems established in this work allow one to control the error, which classical numerical methods cannot do (as they only specify the order of accuracy). Additionally, they guarantee the absence of singular points in the considered domain, which represents an improvement over the Parker–Sochacki method.

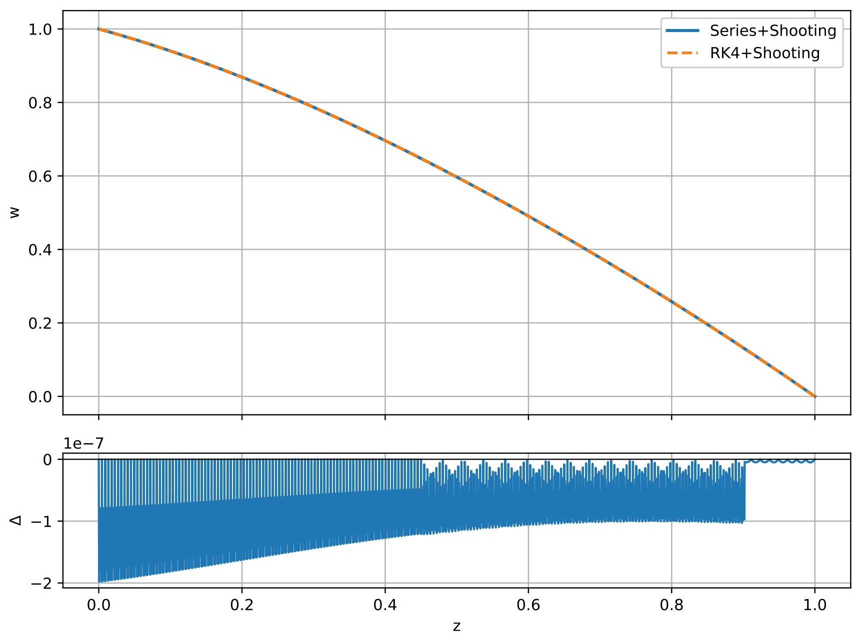

FIGURE 1. Comparison of the boundary value solution $w(z)$ obtained using the Taylor step-by-step marching method with the shooting method and the fourth-order Runge–Kutta (RK4) shooting method. The error $\Delta w = z_{\text{Taylor}} - z_{\text{RK4}}$ is shown below the main graph. Problem parameters: $p = 0.5$, $q = 1$, $k = 3$, $L = 1$, $w(0) = 1$, $w(L) = 0$.

5. Conclusion

We have presented an improved numerical–analytical shooting method for nonlinear second-order boundary value problems with power-type nonlinearities. By combining local Taylor series expansions with stepwise analytic continuation and iterative correction of the shooting parameter, the method provides rigorous control over the solution domain and explicit a priori error estimates.

Numerical experiments demonstrate that the approach achieves high accuracy and robust convergence compared to classical Runge–Kutta-based shooting, even for strongly nonlinear or stiff problems. The methodology is versatile and can be extended to broader classes of nonlinear ordinary differential equations, including systems with multiple shooting parameters, parameter-dependent nonlinearities, or higher-order derivatives. Future work may focus on adaptive continuation strategies and hybrid analytic–numerical schemes for enhanced efficiency.

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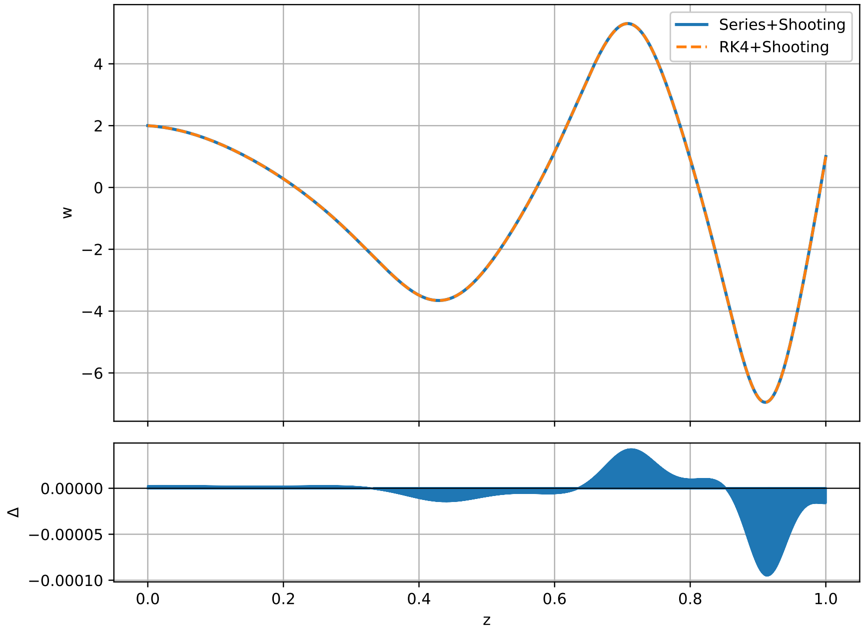

FIGURE 2. Comparison of the boundary value solution $w(z)$ obtained using the Taylor step-by-step marching method with the shooting method and the fourth-order Runge–Kutta (RK4) shooting method. The error $\Delta w = z_{\text{Taylor}} - z_{\text{RK4}}$ is shown below the main graph. Problem parameters: $p = 4$, $q = 9$, $k = 3$, $L = 1$, $w(0) = 2$, $w(L) = 1$.

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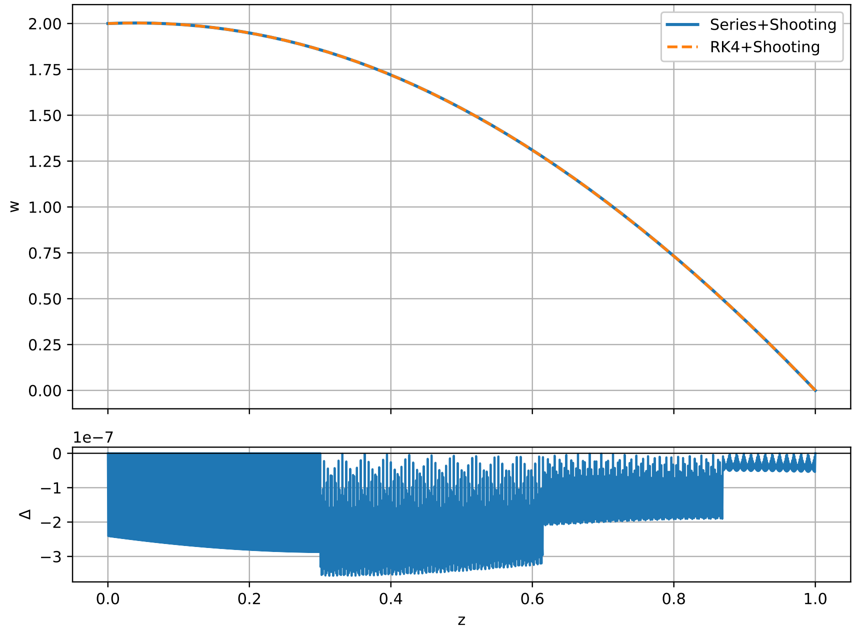

FIGURE 3. Comparison of the boundary value solution $w(z)$ obtained using the Taylor step-by-step marching method with the shooting method and the fourth-order Runge–Kutta (RK4) shooting method. The error $\Delta w = z_{\text{Taylor}} - z_{\text{RK4}}$ is shown below the main graph. Problem parameters: $p = 1$, $q = 1$, $k = 2$, $L = 1$, $w(0) = 2$, $w(L) = 1$.

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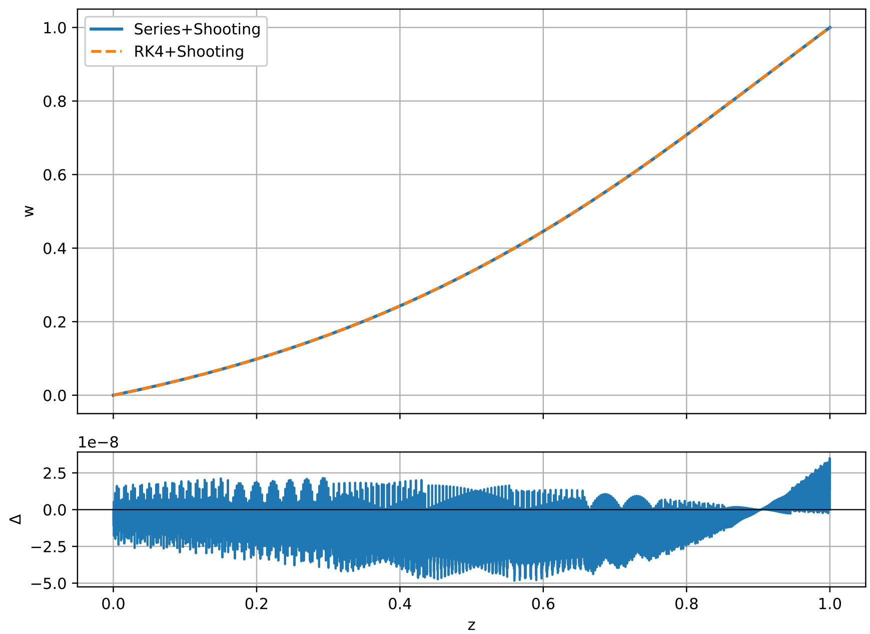

FIGURE 4. Comparison of the boundary value solution $w(z)$ obtained using the Taylor step-by-step marching method with the shooting method and the fourth-order Runge–Kutta (RK4) shooting method. The error $\Delta w = z_{\text{Taylor}} - z_{\text{RK4}}$ is shown below the main graph. Problem parameters: $p = 2$, $q = 4$, $k = 2$, $L = 1$, $w(0) = 0$, $w(L) = 1$.