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QUILLEN K-THEORY AND REPRESENTATION RINGS OF THE QUATERNION GROUP Q_8 AND THE DIHEDRAL GROUP D_8

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ABSTRACT. In this paper, we show that the quaternion group Q_8 and the dihedral group D_8 have isomorphic complex representation rings. As their complex group algebras are Morita equivalent, their Quillen K-theory groups are isomorphic.

Keywords: quaternion group,dihedral group,Morita equivalence, representation ring, Quillen K-theory.
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1. Introduction

Representation theory of finite groups, created by Frobenius and Schur and later expanded by Serre (see [10]) and others, has found numerous applications across various fields of mathematics, such as cohomology theory, algebraic topology, algebraic geometry, theoretical physics, topological K-theory, connective K-theory, and Quillen K-theory. In [6], the irreducible representations of G were described and it was shown that the integral cohomology is generated by the Chern classes of these irreducible representations, where G denotes the finite split metacyclic group with presentation

$$G = \langle A, B \mid A^m = B^n = e,\ BAB^{-1} = A^r,\ r^n \equiv 1 \pmod{m},\ (n(r-1), m) = 1\rangle.$$

Reference [3] provides the explicit construction of the representation ring of $SL_d(k)$ and its associated completion ring; subsequently, in [4], the structure of its topological K-theory ring was deduced by means of Atiyah's motivic theorem. In [7], the authors determined the structure of the representation ring of $SU(2)$ and its completion and then the ring of topological and connective K-theory of its classifying space.

In the present work, we consider the quaternion group Q_8, the dihedral group D_8 and their associated algebras $\mathbb{C}[Q_8]$ and $\mathbb{C}[D_8]$. In the first section, we describe

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the characterization of finite groups G and H such that the complex representation groups $R_{\mathbb{C}}(G)$ and $R_{\mathbb{C}}(H)$ are isomorphic. More precisely we establish:

Proposition (2.9)**.** The algebras $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are two Morita-equivalent algebras if and only if the representation groups $R_{\mathbb{C}}(G)$ and $R_{\mathbb{C}}(H)$ are isomorphic.

In Section 3, we establish Proposition 3.10.

Proposition (3.10)**.** The character table for the group Q_8:

	1	-1	$\pm i$	$\pm j$	$\pm k$
1	1	1	1	1	1
c_1	1	1	-1	1	-1
c_2	1	1	1	-1	-1
c_3	1	1	-1	-1	1
c_4	2	-2	0	0	0

with relations:

$$c_1^2 = c_2^2 = c_3^2 = 1$$

$$c_1c_2 = c_3,\ c_2c_3 = c_1,\ c_3c_1 = c_2,\ c_1 = c_4c_2 = c_4c_3 = c_4$$

In the section 4 we establish proposition 4.9

Proposition (4.9)**.** Table of characters

1	1	x^2	x, x^3	y, x^2y	xy, x^3y
1	1	1	1	1	1
a	1	1	1	-1	-1
b	1	1	-1	1	-1
c	1	1	-1	-1	1
d	2	-2	0	0	0

In Section 5, we define an isomorphism of rings between $R_{\mathbb{C}}(Q_8)$ and $R_{\mathbb{C}}(D_8)$.

Theorem (5.1)**.** There exists an augmented ring isomorphism

$$f : R_{\mathbb{C}}(Q_8) \longrightarrow R_{\mathbb{C}}(D_8)$$

defined on the irreducible characters by

$$f(1) = 1, \qquad f(c_1) = a, \qquad f(c_2) = b, \qquad f(c_3) = c, \qquad f(c_4) = d.$$

In fact we have:

Proposition. The group algebras $\mathbb{C}[Q_8]$ and $\mathbb{C}[D_8]$ are isomorphic.

Corollary. The algebraic K-theory groups of $\mathbb{C}[Q_8]$ and $\mathbb{C}[D_8]$ are isomorphic.

To the augmented complex representation ring $R_{\mathbb{C}}(G)$ is associated its completion $\widehat{R}_{\mathbb{C}}(G)$, which identifies with $K_0(BG)$ via the Atiyah isomorphism (see [1]), where BG denotes the classifying space of G and K_0 the topological K-theory.

The computation of the completed representation ring $\widehat{R}_{\mathbb{C}}(G)$ provides an alternative approach to the spectral sequence

$$H^p(BG, K^q(*)) \implies K^{p+q}(BG).$$

This work fits into the continuation of the results developed in the articles [6], [7], [4] and [3] and requires detailed computations of the complex representations of the groups Q_8 and D_8 (see [10] for instance.). In [8], we will determine explicitly the completion of the representation ring $R(Q_8)$.

2. Morita equivalence of group algebras

2.1. **Morita equivalence.** Let A and B be two rings; we denote by Mod_A the category of left A-modules.

Definition 2.1. (see [2]) We say that two rings A and B are Morita equivalent if there exist two functors satisfying $T : Mod_A \to Mod_B$ and $T' : Mod_B \to Mod_A$ such that: $T \circ T'$ is naturally isomorphic to 1_{Mod_B} and $T' \circ T$ is naturally isomorphic to 1_{Mod_A}.

Definition 2.2. M is a projective A-module if for every A-epimorphism $\pi : P \to S$ and every A-morphism $f : M \to S$, there exists a morphism $\tilde{f} : M \to P$ lifting f, that is, such that $\pi \circ \tilde{f} = f$.

$$\begin{tikzcd} & M \arrow[dl, dashed, "\tilde{f}"'] \arrow[d, "f"] & \\ P \arrow[r] & S \arrow[r] & O \end{tikzcd}$$

Proposition 2.3. *If A and B are Morita equivalent, the category of projective A-modules is equivalent to the category of projective B-modules.*

Let α be the natural isomorphism between $T' \circ T$ and the identity of Mod_A, and β the natural isomorphism between $T \circ T'$ and the identity of Mod_B

Proof. Let M be a projective A-module. Let us show that T(M) is a projective B-module. To do this, we consider:

$$\begin{tikzcd} & T(M) \arrow[d, "g"] & \\ P' \arrow[r, "\pi'"] & S' \arrow[r] & O \end{tikzcd}$$

(1)

(in Mod_B)

We apply the functor T' and obtain:

$$\begin{tikzcd} & T'(T(M)) \arrow[d, "T'(g)"] & \\ T'(P') \arrow[r, "T'(\pi')"] & T'(S') \arrow[r] & O \end{tikzcd}$$

(2)

We obtain:

$$\begin{tikzcd}
T'(T(M)) \arrow[d, "T'(g)"'] & M \arrow[l, "\alpha(M)"'] \arrow[d, "f=T'(g)\circ\alpha(M)"] & \\
T'(P') \arrow[r, "T'(\pi')"'] & T'(S') \arrow[r] & 0
\end{tikzcd}$$

(3) (in Mod_A)

Since M is A-projective, then: There exists $\tilde{f}$ making this diagram commutative.:

$$\begin{tikzcd}
M \arrow[r, "\alpha(M)"] \arrow[d, "\tilde{f}"'] & T'(T(M)) \arrow[d, "T'(g)"] & \\
T'(P') \arrow[r, "T'(\pi')"'] & T'(S') \arrow[r] & 0
\end{tikzcd}$$

(4)

We return to Mod_B:

$$\begin{tikzcd}
T(M) \arrow[r, "T(\alpha(M))"] \arrow[d, "T(\tilde{f})"] & T(T'(T(M))) \arrow[d, "T(T'(g))"] & \\
T(T'(P')) \arrow[r, "T(T'(\pi'))"] \arrow[d, "\beta(P')"] & T(T'(S')) \arrow[r] \arrow[d, "\beta(S')"] & 0 \\
P' \arrow[r, "\pi'"] & S' \arrow[r] & 0
\end{tikzcd}$$

(5)

We verify that $\tilde{g}$ is a lifting of g, that means that we should verify that the diagram (5) is commutative

Thanks to (4):

$$T'(g) \circ \alpha(M) = T'(\pi') \circ \tilde{f}$$

after applying T, we get :

$$T(T'(g)) \circ T(\alpha(M)) = T(T'(\pi')) \circ T(\tilde{f})$$

By the naturality:

$$\beta(S') \circ T(T'(\pi')) = \pi' \circ \beta(P')$$

Then:

$$\beta(S') \circ T(T'(\pi')) \circ T(\tilde{f}) = \pi' \circ \beta(P') \circ T(\tilde{f}) = \pi' \circ \tilde{g}$$

Hence the diagram 5 is commutative.

$\square$

2.2. **Quillen algebraic K-theory.**

- **$\mathbf{K}_0$ group:** To each ring A, we denote by P_A the category of projective left A-modules of finite type. Two A-modules P and Q are said equivalent if there exist $f : P \to Q$ and $g : Q \to P$ such that $f \circ g = 1_Q$ and $g \circ f = 1_P$. We denote by F the free $\mathbb{Z}$-module generated by the equivalence classes and H the free $\mathbb{Z}$-module generated by $[P \oplus Q] - [P] - [Q]$ and
$$K_0(A) = \frac{F}{H}$$
- **Higher Quillen groups:**
In [9], Quillen defines
$$K_n(A) = \pi_{n+1}\big(B(P_A), *\big)$$
where $B(P_A)$ is the classifying space of P_A and $\pi_{n+1}(B(P_A))$ is the $(n+1)$ homotopy group of $(B(P_A))$.

Lemma 2.4. *If P_A and P_B are equivalent, then*
$$K_i(A) = K_i(B), \quad i \geq 0.$$

In fact, the corresponding classifying spaces to equivalent categories are homotopically equivalent. Hence, they have the same homotopy groups.

Proposition 2.5. *If A and B are two equivalent Morita rings, then $K_n(A) \cong K_n(B)$.*

Proof. Suppose that A and B are Morita equivalent rings, then we have: $T : Mod_A \to Mod_B$ and $T' : Mod_B \to Mod_A$ satisfying $T \circ T' = 1_{Mod_B}$ and $T' \circ T = 1_{Mod_A}$. From the proposition 2.3 yields $T(P_A) \subset P_B$ and $T'(P_B) \subset P_A$. Then P_A and P_B are two equivalent categories, which implies that $K_n(A) \cong K_n(B)$. □

2.3. **Group algebras.** Let G be a finite group of order n.

Definition 2.6. We denote by $\mathbb{C}[G]$ the elements of form $\sum\limits_{fini} l_i g_i$ with $l_i \in \mathbb{C}$ and $g_i \in G$ (in a unique way).
$(\mathbb{C}[G], +, \times, .)$ is an $\mathbb{C}$-algebra with:

- The sum is defined by: $\sum\limits_{I} l_i g_i + \sum\limits_{J} l'_i g'_i = \sum\limits_{U \cup J} l''_i g''_i$
- The product is defined by: $\sum\limits_{I} l_i g_i \times \sum\limits_{J} l'_j g'_j = \sum\limits_{I \times J} l_i l'_j . g_i g_j$
- The neutral element for $+$ of $\mathbb{C}[G]$ is given by: $0_{\mathbb{C}[G]} = 0_{\mathbb{C}} e_G$
- $1_{L[G]} = 1_L e_G$ is the neutral element for $\times$
- The scalar multiplication of the algebra is defined by: $a.\sum\limits_{I} u_i g_i = \sum\limits_{I} a u_i g_i$

We denote by $U(\mathbb{C}(G))$ the group of invertible elements of the algebra $\mathbb{C}[G]$

Proposition 2.7. *The group G is a subgroup of $U(\mathbb{C}[G])$*

In fact, let g be an element of G,

- if $g = e$, then $g = 1_{\mathbb{C}[G]}$

- Otherwise, according to Lagrange, there exists a $d \neq 1$ dividing n such that $g^d = e$
  $g.g^{d-1} = e$
  so g is invertible in $\mathbb{C}[G]$.

Let G and H be finite groups.

Proposition 2.8. *The algebras $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are $\mathbb{C}$-isomorphic if and only if the groups G and H have the same number of conjugacy classes and the same dimensions of irreducible characters.*

Proof. (1) Suppose that $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are $\mathbb{C}$-algebra isomorphic

$$f : \mathbb{C}[G] \longrightarrow \mathbb{C}[H]$$

is a $\mathbb{C}$-algebra isomorphism, with inverse

$$g : \mathbb{C}[H] \longrightarrow \mathbb{C}[G].$$

$\rightarrow$ Let W_i be a simple $\mathbb{C}[H]$-module. Then the $\mathbb{C}[G]$-module induced via f, denoted $f^*(W_i)$, is also simple.
Hence, the number of simple modules is equal to the number of simple $\mathbb{C}[G]$-modules.

$\rightarrow$ On the other hand, the number of simple $\mathbb{C}[G]$-modules is equal to the number of conjugacy classes of G (see [10]).

(2) $\Rightarrow$ We assume that G and H have the same number r of conjugacy classes and that the irreducible characters have the same dimensions n_i, for $i = 1, 2, \dots, r$. From the Wedderburn decomposition, we have

$$\mathbb{C}[G] \simeq M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C}),$$

and

$$\mathbb{C}[H] \simeq M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C}).$$

Therefore, $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are two isomorphic $\mathbb{C}$-algebras.
$\Leftarrow$ We suppose that $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are $\mathbb{C}$-isomorphic. Then

$$\mathbb{C}[G] \simeq M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C}),$$

and

$$\mathbb{C}[H] \simeq M_{m_1}(\mathbb{C}) \oplus Mm_2(\mathbb{C}) \oplus \cdots \oplus Mm_s(\mathbb{C}).$$

Hence $r = s$ and $n_i = m_i$.

$\square$

Proposition 2.9. *The algebras $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are two Morita-equivalent algebras if and only if the representation groups $R_{\mathbb{C}}(G)$ and $R_{\mathbb{C}}(H)$ are isomorphic.*

Proof. We assume that $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are Morita-equivalent.
Let T be the functor $Mod_{\mathbb{C}[G]} \to Mod_{\mathbb{C}[\mathbb{H}]}$ and T' be the functor $Mod_{\mathbb{C}[H]} \to Mod_{\mathbb{C}[\mathbb{G}]}$ such that: $T \circ T'$ is naturally isomorphic to $1_{Mod_{\mathbb{C}[H]}}$ and $T' \circ T$ is naturally isomorphic to $1_{Mod_{\mathbb{C}[G]}}$.

Let us show that T induces a group isomorphism t from $R_{\mathbb{C}}(G)$ to $R_{\mathbb{C}}(H)$.

$$t : R_{\mathbb{C}}(G) \longrightarrow R_{\mathbb{C}}(H)$$
$$\sum_i n_i[V_i, \pi_i] \longmapsto \sum_i n_i[T(V_i), \sigma_i]$$

The map t is well defined because:

(i) We consider (V_i, π_i) a $\mathbb{C}$-irreducible representation of G. Then V_i is an $\mathbb{C}[G]$-module, and $T(V_i)$ is an $\mathbb{C}[H]$-module. This means that $(T(V_i), \sigma_i)$ is a representation of H.

(ii) If $[V_1, \pi_1] = [V_1', \pi_1']$ in $R_{\mathbb{C}}(G)$, there exists an $\mathbb{C}[G]$-isomorphism α such that

$$\begin{array}{ccc} V_1 & \xrightarrow{\alpha} & V_1' \\ {\scriptstyle \pi_1(g)}\downarrow & & \downarrow{\scriptstyle \pi_1'(g)} \\ V_1 & \xrightarrow[\alpha]{} & V_1' \end{array}$$

for all $g \in G$. The map α satisfies $\alpha(g \cdot v_1) = g \cdot \alpha(v_1)$. By functoriality, $T(\alpha) : T(V_1) \longrightarrow T(V_1')$ is an $\mathbb{C}[H]$-isomorphism. Hence $[T(V_1), T(\pi_1)] = [T(V_1'), T(\pi_1')]$, i.e.

$$t([V_1, \pi_1]) = t([V_1', \pi_1']),$$

which shows that t is well defined.

In the same way we define

$$t' : R_{\mathbb{C}}(H) \longrightarrow R_{\mathbb{C}}(G)$$
$$\sum_i n_i[W_i, \sigma_i] \longmapsto \sum_i n_i[T'(W_i), \pi_i]$$

We have

$$[T' \circ T(V_i)] = [V_i]$$

because $T' \circ T$ is isomorphic to $1_{Mod_{\mathbb{C}[G]}}$. Hence,

$$t' \circ t = 1_{R_{\mathbb{C}}(G)} \quad \text{and} \quad t \circ t' = 1_{R_{\mathbb{C}}(H)}. \tag{2.1}$$

Conversely,we assume that the representation rings $R_{\mathbb{C}}(G)$ and $R_{\mathbb{C}}(H)$ are isomorphic that means there exist t and t' isomorphisms satisfying 2.1.
Let us show that $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are Morita-equivalent.
Let $[V_1], [V_2], ...[V_n]$ be the set of isomorphism classes of irreducible $\mathbb{C}$-representations of G, thus $t[V_1], t[V_2], ..., t[V_n]$ is the set of isomorphism classes of irreducible $\mathbb{C}$-representations of H, we denote $t([V_i]) = [W_i]$ $t([V_i \oplus V_j]) = t([V_i]) + t([V_j])$ The free $\mathbb{Z}$-module $R_{\mathbb{C}}(G)$ is generated by the equivalence classes $[V_1], ...[V_s]$ of the irreducible $\mathbb{C}$-representations of G. Let M be a $\mathbb{C}[G]$-module, M is a $\mathbb{C}$-representation of G with: $[M] = \sum_i n_i[V_i], n_i \in \mathbb{N}$

Lemma 2.10.

$$M \simeq \bigoplus_{i=1}^{s} V_i^{\oplus n_i}$$

Let T be the following functor from $Mod_{\mathbb{C}}[G]$ to $Mod_{\mathbb{C}}[H]$:

$$T(M) = W_1^{\oplus n_1} \oplus W_2^{\oplus n_2} \oplus \cdots \oplus W_s^{\oplus n_s}$$

Let M and P be two $\mathbb{C}[G]$-modules.

Lemma 2.11.

$$M \simeq V_1^{\oplus n_1} \oplus V_2^{\oplus n_2} \oplus \cdots \oplus V_s^{\oplus n_s}, \qquad P \simeq V_1^{\oplus m_1} \oplus V_2^{\oplus m_2} \oplus \cdots \oplus V_s^{\oplus m_s}.$$

and $\theta : M \longrightarrow P$.
Thus, for all $1 \leq k \leq s$, $1 \leq h \leq n_i$

$$\theta_h^k : V_i \longrightarrow M \xrightarrow{\theta} P \longrightarrow V_j$$
$$\theta_h^k = 0 \quad if \quad i \neq j$$
$$\theta_h^k \in End(V_i, V_i) \quad if \quad i = j$$

We define: $T(\theta_h^k) = \phi_i(\theta_h^k)$

Proof. It suffices to apply Schur's lemma.
and $T(\theta) = \bigoplus_i \bigoplus_h \bigoplus_k \phi_i(\theta_h^k)$ □

Lemma 2.12.

$$M \xrightarrow{\theta} P \xrightarrow{\beta} Q$$

Thus:

*) $(\beta \circ \theta)_h^k = \beta_h^k \circ \theta_h^k$
**) $T(\beta \circ \theta) = T(\beta) \circ T(\theta)$
***) $T(1_M) = 1_{T(M)}$

Proof. **) follows from *) □

Lemma 2.13. *T defines a natural equivalence between* $\mathrm{Mod}_{\mathbb{C}[G]}$ *and* $\mathrm{Mod}_{\mathbb{C}[H]}$

Indeed: Let $T' : \mathrm{Mod}_{\mathbb{C}[H]} \longrightarrow \mathrm{Mod}_{\mathbb{C}[G]}$ defined similarly to the above.
We have:

$$\begin{cases} T' \circ T([M]) = [M] \\ T \circ T'([M']) = [M'] \end{cases}$$

where M' a $\mathbb{C}[H]$-module

□

3. The representation ring of Q_8

3.1. **The quaternion group.** The quaternion group Q_8 is defined by :

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

with the following relations:

$$i^2 = j^2 = k^2 = ijk = -1$$

and the multiplication rules :

$$ij = k, \quad ji = -k, \quad jk = i, \quad kj = -i, \quad ki = j, \quad ik = -j.$$

Remark 3.1. Q_8 has 6 elements of order 4.

Lemma 3.2. *The conjugation classes of Q_8 are:*

$$\overline{1}, \overline{-1}, \overline{i}, \overline{j}, \overline{k}$$

Proof. We have :

$$i1i^{-1} = -i^2 = 1$$
$$j1j^{-1} = -j^2 = 1$$
$$k1k^{-1} = -k^2 = 1$$

Thus, the class of 1 is $\{1\}$.
The same for -1: the class of -1 is $\{-1\}$

- For i :

$$jij^{-1} = ji(-j) = -k \times -j = kj = -i$$
$$kik^{-1} = ki(-k) = -i$$

thus the class of i is $\{i, -i\}$.
- The same for: the class of j: $\{j, -j\}$
- And the class of k: $\{k, -k\}$

□

3.2. **The complex representation ring of Q_8.**

Proposition 3.3. *The group Q_8 admits 5 non-isomorphic irreducible representations of degree 1 and one of degree 2.*

Proof. Let ρ_i be the irreducible representations of Q_8 (See [10])
As:

$$\sum_{i=1}^{5} (\dim \rho_i)^2 = |Q_8| = 8.$$

And we know that there are 5 irreducible representations.
So the possible decomposition is:

$$\underbrace{1^2 + 1^2 + 1^2 + 1^2}_{\text{4 representations of dimension 1}} \quad + \quad \underbrace{2^2}_{\text{1 representation of dimension 2}} \quad = 4 + 4 = 8.$$

$\Rightarrow$ There are therefore 4 irreducible representations of dimension 1 and 1 irreducible representation of dimension 2.

□

Lemma 3.4. *The 4 representations of Q_8 of dimension 1 are:*

$$\sigma_1 : i \to 1 \ , \ j \to 1$$
$$\sigma_2 : i \to -1 \ , \ j \to 1$$
$$\sigma_3 : i \to 1 \ , \ j \to -1$$
$$\sigma_4 : i \to -1 \ , \ j \to -1$$

Proof. The commutator subgroup of Q_8 is given by:

$$[Q_8, Q_8] = \{-1, 1\}$$

Hence the abelianization is:

$$Q_8/[Q_8, Q_8] = \{\bar{1}, \bar{i}, \bar{j}, \bar{k}\}$$

Therefore:

$$Q_8/[Q_8, Q_8] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

We call this isomorphism g

$$\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{C}) = \{f_1, f_2, f_3, f_4\}$$

$$\begin{array}{ll} f_1(1,0) = 1, & f_1(0,1) = 1, \\ f_2(1,0) = -1, & f_2(0,1) = 1, \\ f_3(1,0) = 1, & f_3(0,1) = -1, \\ f_4(1,0) = -1, & f_4(0,1) = -1. \end{array}$$

Therefore,

$$\sigma_i = f_i \circ g \circ p$$

where

$$p : Q_8 \longrightarrow Q_8/[Q_8, Q_8], \text{ the canonical sujection}$$

□

We consider $H = \langle j \rangle$ the subgroup generated by j , and

$$\begin{array}{cccc} \epsilon_k : & H & \longrightarrow & Aut(\mathbb{C}) \\ & j & \longmapsto & \lambda = e^{\frac{2ik\pi}{4}} \end{array}$$

the representation of H. (H is a finite abelian group, so its representation is of dimension 1).

Proposition 3.5. *The representation π_k of Q_8 induced by ϵ_k is defined by:*

$$\pi_k(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \pi_k(-i) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pi_k(-1) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \pi_k(j) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix},$$

$$\pi_k(-j) = \begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \pi_k(k) = \begin{pmatrix} 0 & -\lambda \\ -\lambda & 0 \end{pmatrix}, \quad \pi_k(-k) = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$$

Proof.

- Q_8/H has two elements: the class of 1 and the class of i:

$$Q_8/H = \{\bar{1}, \bar{i}\}$$

with : $\bar{1} = \{1, j, -1, -j\}$ and $\bar{i} = \{i, k, -i, -k\}$

- Let $B_v = \{e_1\}$ be a basis of V. As $j^4 = 1$ therefore $\lambda = e^{\frac{2ik\pi}{4}}$
So we consider : for each $k \in \{0, 1, 2, 3\}$, we define a character (1-dimensional representation) of H :

$$\begin{aligned} \epsilon_k : \quad H &\longrightarrow Aut(\mathbb{C}) \\ j &\longmapsto \lambda = e^{\frac{2ik\pi}{4}} \end{aligned} \tag{3.1}$$

We then construct the induced representation of H to Q_8 :

$$\pi_k = \mathrm{Ind}_H^{Q_8}(\epsilon_k).$$

We know that $[Q_8 : H] = 2$, so the induced representation π_k acts on a 2-dimensional vector space. We define $W = V \oplus iV$

Let $z \in Q_8$, therefore $\bar{z} = \bar{1}$ where $\bar{z} = \bar{i}$

- **If** $z \in H$:

$$\pi(z) \cdot (v_1 + iv_2) = \epsilon(z)v_1 + i\,\epsilon(z)v_2$$

- **If** $z \in iH$
therefore $z = hi$ with $h \in H$ thus $h = zi^{-1} \in H$ so we can apply $\sigma(z)$

$$\pi(z) \cdot (v_1 + iv_2) = \epsilon(zi^{-1})v_1 + i\,\epsilon(zi^{-1})v_2$$

As $W = V \oplus iV$ therefore $B_W = \{e_1, ie_1\}$ is a basis of W
Thus, the representation π_k of Q_8 on W has the following matrix :

$$\pi_k(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \pi_k(-i) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pi_k(-1) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \pi_k(j) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix},$$

$$\pi_k(-j) = \begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \pi_k(k) = \begin{pmatrix} 0 & -\lambda \\ -\lambda & 0 \end{pmatrix}, \quad \pi_k(-k) = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$$

□

Proposition 3.6. *The only irreducible representations of Q_8 are: π_1 and π_3*

Proof. We will need the following lemma:

Lemma 3.7. *the character table $\chi_{\pi_k}(g)$:*

g	1	-1	$\pm i$	$\pm k$	$\pm j$
$\chi_{\pi_k}(g)$	2	$2(-1)^k$	0	0	$\frac{\lambda_k}{2}(1 + (-1)^k)$

the character of the induced representation χ_{π_k}

$$\chi_{\pi_k}(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \sigma_k(x^{-1}gx)$$

- For $g = 1$: we have $x^{-1}1x = 1 \in H$ for all x, thus:

$$\chi_{\pi_k}(1) = \frac{1}{4} \sum_{x \in G} \sigma_k(1) = \frac{1}{4} \cdot 8 = 2$$

- For $g = -1$: we have $\sigma_k(-1) = (-1)^k$

$$\chi_{\pi_k}(-1) = \frac{1}{4}\sum_{x\in G}\sigma_k(-1) = \frac{1}{4}\cdot 8\cdot(-1)^k = 2.(-1)^k$$

- $g = \pm i$ or $\pm k$:

$$\chi_{\pi_k}(g) = 0$$

- $g = \pm j$

$$\chi_{\pi_k}(j) = \frac{1}{4}\sum\sigma_k(j) + \sum\sigma_k(j) + \sigma_k(-j) + \sigma_k(-j) = \frac{1}{4}\sum 2\sigma_k(j) + 2\sigma_k(-j)$$

And we have $\sigma_k(j) = \lambda_k$ and $\sigma_k(-j) = (-1)^k\lambda_k$.
Thus:

$$\chi_{\pi_k}(j) = \frac{1}{4}[2\lambda_k + 2(-1)^k\lambda_k] = \frac{1}{4}[2\lambda_k(1+(-1)^k)]$$

Hence:

$$\chi_{\pi_k}(j) = \begin{cases} 0 & \text{If } k \text{ is odd} \\ \lambda_k & \text{if } k \text{ is even} \end{cases}$$

Hence the character table $\chi_{\pi_k}(g)$:

g	1	-1	$\pm i$	$\pm k$	$\pm j$
$\chi_{\pi_k}(g)$	2	$2(-1)^k$	0	0	$\frac{\lambda_k}{2}(1+(-1)^k)$

□

Corollary 3.8. *π_k is irreducible if and only if $k = 1$ or $k = 3$*

Proof. We have:

$$\langle\chi_{\pi_1},\chi_{\pi_1}\rangle = \langle\chi_{\pi_3},\chi_{\pi_3}\rangle = \frac{1}{8}\sum_{g\in G}|\chi_{\pi_1}(g)|^2 = \frac{1}{8}(2^2+(-2)^2+0+0+0+0) = \frac{8}{8} = 1$$

So, according to [5] π_1 and π_3 are irreducible.
On the other hand:

$$\langle\chi_{\pi_0},\chi_{\pi_0}\rangle = \frac{1}{8}(2^2+(-2)^2+0+0+0+0+\lambda_0+\lambda_0) = \frac{1}{8}(8+2) = \frac{10}{8} \neq 1$$

Hence π_0 is not irreducible.
Similarly :

$$\langle\chi_{\pi_2},\chi_{\pi_2}\rangle = \frac{1}{8}(2^2+(-2)^2+0+0+0+0+\lambda_2+\lambda_2) = \frac{1}{8}[8+(-2)] = \frac{6}{8} \neq 1$$

Hence π_2 is not irreducible
Thus, the only irreducible representations of dimension 2 are π_1 and π_3. □

Lemma 3.9. *π_1 and π_3 are equivalent.*

Proof. According to [5], it must be shown that

$$\langle\chi_{\pi_1},\chi_{\pi_3}\rangle = 1$$

We have:

$$\langle \chi_{\pi_1}, \chi_{\pi_3} \rangle = \frac{1}{8}\sum_{g\in G} \chi_{\pi_1}(g)\cdot\chi_{\pi_3}(g^{-1}) = \frac{1}{8}(2\cdot 2+(-2)\cdot(-2)+0+0+0+0) = \frac{1}{8}(4+4) = \frac{8}{8} = 1$$

Thus π_1 and π_3 are equivalent.

□

Hence, the only irreducible character of Q_8 of dimension 2 is that of π_1.

We denote by $1, c_1, c_2$, and c_3 the characters of the one-dimensional irreducible representations, and by c_4 the character of the two-dimensional irreducible representation.

Proposition 3.10. *The character table of the group Q_8:*

	1	-1	$\pm i$	$\pm j$	$\pm k$
1	1	1	1	1	1
c_1	1	1	-1	1	-1
c_2	1	1	1	-1	-1
c_3	1	1	-1	-1	1
c_4	2	-2	0	0	0

with the relations:

$$c_1^2 = c_2^2 = c_3^2 = 1$$

$$\{\, c_1c_2 = c_3, \quad c_2c_3 = c_1, \quad c_3c_1 = c_2, \quad c_4c_1 = c_4c_2 = c_4c_3 = c_4 \,\} \qquad (*)$$

Theorem 3.11. *As a $\mathbb{Z}$-module*

$$R(Q_8) = \mathbb{Z} \oplus \mathbb{Z}c_1 \oplus \mathbb{Z}c_2 \oplus \mathbb{Z}c_3 \oplus \mathbb{Z}c_4$$

the ring structure is given by the relation (*)

4. Representation Ring of D_8

4.1. **The dihedral group D_8.** The group D_8 has the following presentation:

$$D_8 = \langle x, y \mid x^4 = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle$$

D_8 contains 8 elements:

$$D_8 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$$

Remark 4.1. D_8 has 2 elements of order 4.

Lemma 4.2. *The conjugacy classes of D_8 :*

$$(\overline{1}, \overline{x}, \overline{x^2}, , \overline{y}, \overline{xy})$$

Proof. We have: $\overline{1} = \{1\}$, $\overline{x} = \{x, x^3\}$, $\overline{x^2} = \{x^2\}$, $\bar{y} = \{y, x^2y\}$, $\overline{xy} = \{xy, x^3y\}$

□

4.2. **The complex representation ring of D_8.**

Proposition 4.3. *The group D_8 has five non-isomorphic irreducible representation, 4 of degree 1 and one of degree 2.*

Proof. Let (ρ_i) be the irreducibles representations of D_8
As:

$$\sum_{i=1}^{5} (\dim \rho_i)^2 = |D_8| = 8.$$

And we know that there are 5 irreducible representations.
So the possible decomposition is:

$$\underbrace{1^2 + 1^2 + 1^2 + 1^2}_{\text{4 representations of dimension 1}} \quad + \quad \underbrace{2^2}_{\text{1 representation of dimension 2}} \quad = 4 + 4 = 8.$$

$\Rightarrow$ There are therefore 4 irreducible representations of dimension 1 and 1 irreducible representation of dimension 2.

$\square$

Lemma 4.4. *The 4 representations of D_8 of dimension 1 are:*

$$r_1 : x \to 1 \ , \ y \to 1$$
$$r_2 : x \to -1 \ , \ y \to 1$$
$$r_3 : x \to 1 \ , \ y \to -1$$
$$r_4 : x \to -1 \ , \ y \to -1$$

Proof. Indeed, the group's relations require:

$$r(y)^2 = 1,$$

thus

$$r(y) \in \{-1, 1\}.$$

and we have $r(x)^4 = 1$
therefore $r(x) \in \{1, -1, i, -i\}$ And as $r(yxy^-1) = r(x^{-1})$ and $r(y) \in \{1, -1\}$
Thus $r(x) = r(x^{-1})$
Hence $r(x) \in \{-1, 1\}$ This gives us the four possible combinations. $\square$

We consider $K = \langle x \rangle$ the subgroup of D_8 generated by x , and

$$\begin{array}{cccc} \phi_k : & K & \longrightarrow & Aut(\mathbb{C}) \\ & x & \longmapsto & \lambda = e^{\frac{2ik\pi}{4}} \end{array}$$

the representation of K. (K is a finite abelian group, so its representation is of dimension 1).

Proposition 4.5. *The representation ψ_k of D_8 induced by ϕ_k is defined by:*

$$\psi_k(x) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \psi_k(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Proof. We have: $D_8/K = \{\bar{1}, \bar{x}\}$ with $\bar{1} = \{1, x, x^2, x^3\}$ and $\bar{x} = \{y, xy, x^2y, x^3y\}$
$W = \mathbb{C} \oplus \mathbb{C}$
D_8 acts on W by:

- If $z = x$:

$$x.(a + yb) = x.a + x.yb = x.a + yx^{-1}b$$

- If $z = y$

$$y.(a + yb) = ya + y^2b = b + ya$$

Lemma 4.6. *the character table $\chi_{\psi_k}(g)$:*

g	1	x^2	x, x^3	y, x^2y	xy, x^3y
$\chi_{\psi_k}(g)$	2	$2(-1)^k$	$\frac{\lambda_k}{2}(1+(-1)^k)$	0	0

the character of the induced representation χ_{π_k}

$$\chi_{\psi_k}(g) = \frac{1}{|K|} \sum_{\substack{z \in D_8 \\ z^{-1}gz \in K}} \sigma_k(z^{-1}gz)$$

- For $g = 1$: we have $z^{-1}1z = 1 \in K$ for all z, thus:

$$\chi_{\psi_k}(1) = \frac{1}{4} \sum_{x \in D_8} \phi_k(1) = \frac{1}{4} \cdot 8 = 2$$

- For $g = x^2$:

$$\chi_{\psi_k}(x^2) = \frac{1}{4} \sum_{z \in D_8} \phi_k(x^2) = \frac{1}{4} \cdot 8 \cdot (-1)^k = 2.(-1)^k$$

- For $g \in \overline{y}, \overline{xy}$:

$$\chi_{\psi_k}(g) = 0$$

- $g \in \overline{x}$

$$\chi_{\psi_k}(x) = \frac{1}{4}[2\lambda_k + 2(-1)^k\lambda_k] = \frac{1}{4}[2\lambda_k(1 + (-1)^k)]$$

Hence:

$$\chi_{\psi_k}(x) = \begin{cases} 0 & \text{If } k \text{ is odd} \\ \lambda_k & \text{if } k \text{ is even} \end{cases}$$

Hence the character table $\chi_{\pi_k(g)}$:

g	1	x^2	y	x^3	x	xy	x^2y	x^3y
$\chi_{\psi_k}(g)$	2	$2(-1)^k$	0	0	$\frac{\lambda_k}{2}(1+(-1)^k)$	0	0	0

□

Proposition 4.7. *The only irreducible representations of D_8 of dimension 2 are: ψ_1 and ψ_3.*

Proof. We have:

$$\langle \chi_{\psi_1}, \chi_{\psi_1} \rangle = \langle \chi_{\psi_3}, \chi_{\psi_3} \rangle = \frac{1}{8} \sum_{g \in D_8} |\chi_{\psi_1}(g)|^2 = \frac{1}{8}(2^2 + (-2)^2 + 0 + 0 + 0 + 0) = \frac{8}{8} = 1$$

So, according to [5] ψ_1 and ψ_3 are irreducible.
On the other hand:

$$\langle \chi_{\psi_0}, \chi_{\psi_0} \rangle = \frac{1}{8}(2^2 + (-2)^2 + 0 + 0 + 0 + 0 + \lambda_0 + \lambda_0) = \frac{1}{8}(8 + 2) = \frac{10}{8} \neq 1$$

Hence ψ_0 is not irreducible.
Similarly :

$$\langle \chi_{\psi_2}, \chi_{\psi_2} \rangle = \frac{1}{8}(2^2 + (-2)^2 + 0 + 0 + 0 + 0 + \lambda_2 + \lambda_2) = \frac{1}{8}[8 + (-2)] = \frac{6}{8} \neq 1$$

Hence ψ_2 is not irreducible
Thus, the only irreducible representations of dimension 2 are ψ_1 and ψ_3. □

Proposition 4.8. *ψ_1 and ψ_3 are isomorphic.*

Proof. According to [5], it must be shown that

$$\langle \chi_{\psi_1}, \chi_{\psi_3} \rangle = 1$$

We have:

$$\langle \chi_{\psi_1}, \chi_{\psi_3} \rangle = \frac{1}{8} \sum_{g \in G} \chi_{\psi_1}(g) \cdot \chi_{\psi_3}(g^{-1}) = \frac{1}{8}(2 \cdot 2 + (-2) \cdot (-2) + 0 + 0 + 0 + 0) = \frac{1}{8}(4 + 4) = \frac{8}{8} = 1$$

Thus ψ_1 and ψ_3 are equivalent. □

We denote by $1, a, b$, and c the characters of the one-dimensional irreducible representations, and by d the character of the two-dimensional irreducible representation

Proposition 4.9. ***Character table of*** D_8

1	1	x^2	x, x^3	y, x^2y	xy, x^3y
1	1	1	1	1	1
a	1	1	1	-1	-1
b	1	1	-1	1	-1
c	1	1	-1	-1	1
d	2	-2	0	0	0

The two-dimensional representation:

$$d: \quad x \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$a^2 = b^2 = c^2 = 1$$

$$\{\, ab = c, \quad bc = a, \quad ca = b, \quad da = d, \quad db = d, \quad dc = d \,\} \qquad (**)$$

5. Conclusion

In general, the Proposition 2.9 does not give an isomorphism for the ring structure defined by:

$$[V]\cdot[W] = [V\otimes_{\mathbb{C}} W].$$

In the particular case of $C[Q_8]$ and $C[D_8]$ we have:

Theorem 5.1. *There exists ring isomorphism*

$$f : R_{\mathbb{C}}(Q_8) \longrightarrow R_{\mathbb{C}}(D_8)$$

defined on the irreducible characters by

$$f(1) = 1, \qquad f(c_1) = a, \qquad f(c_2) = b, \qquad f(c_3) = c, \qquad f(c_4) = d.$$

Proof. (i) As $R_{\mathbb{C}}(Q_8) = \mathbb{Z}\oplus c_1\mathbb{Z}\oplus c_2\mathbb{Z}\oplus c_3\mathbb{Z}\oplus c_4\mathbb{Z}$ and $R_{\mathbb{C}}(D_8) = \mathbb{Z}\oplus a\mathbb{Z}\oplus b\mathbb{Z}\oplus c\mathbb{Z}\oplus d\mathbb{Z}$, the map f defines a group isomorphism.

(ii) The group isomorphism f respects the relations $(*)$ and $(**)$, hence f is a ring isomorphism.

□

The groups Q_8 and D_8 are not isomorphic but:

Proposition 5.2. *The group algebras $\mathbb{C}[Q_8]$ and $\mathbb{C}[D_8]$ are isomorphic.*

Proof. As a consequence of Proposition 3.10, and by the Wedderburn decomposition, we have

$$\begin{aligned}\mathbb{C}[Q_8] &= \mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\oplus\left(\mathbb{C}^2\oplus\mathbb{C}^2\right)\\ &\simeq \mathbb{C}^4\oplus M_2(\mathbb{C}).\end{aligned}$$

And from Proposition 4.9 we get $\mathbb{C}[D_8]\simeq\mathbb{C}^4\oplus M_2(\mathbb{C})$. □

Corollary 5.3. *The algebraic K-theory groups of $\mathbb{C}[Q_8]$ and $\mathbb{C}[D_8]$ are isomorphic.*

Remark 5.4. In [8], the authors determine the structure of the completions of the representation rings and deduce the Atiyah topological K-theory of the classifying spaces BD_8 and BQ_8.

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References

[1] M.F.Atiyah, *Characters and cohomology of finite groups*, Inst. Hautes Études Sci. Publ. Math. **9** (1961), 23-64.
[2] Atanane, K., Azi, K., & Hamraoui, H. (2023). *Equivalence de Morita et applications*. Preprint.
[3] Azi, K., & Hamraoui, H. (2016). The representation ring of SL_d. *Gulf Journal of Mathematics*, 4(4), 48–53. https://doi.org/10.56947/gjom.v4i4.xxx
[4] Azi, K., & Hamraoui, H. (2016). Atiyah motivic theorem for the special group SL_d. *Gulf Journal of Mathematics*, 4(4), 54–59. https://doi.org/10.56947/gjom.v4i4.xxx
[5] Bouvier, A., & Richard, D. (1979). *Groupes : observations, théorie, pratique*. Hermann.
[6] Deblij, A., & Hamraoui, H. (2022). Cohomology ring of the finite split metacyclic group. *Gulf Journal of Mathematics*, 11(1), 43–56. https://doi.org/10.56947/gjom.v11i1.667

[7] Deblij, A., Haddar, N., & Hamraoui, H. (2022). Representations of SU(2) and some applications. *Annals of Mathematics and Computer Science*, 10, 65–70. https://annalsmcs.org/index.php/amcs/article/download/116/72

[8] Haddar, N., Azi, K., Deblij, A., & Hamraoui, H. (2025). *Atiyah K-theory and representation rings of the quaternion and the dihedral groups*. Preprint.

[9] Quillen, D. (1973). *Algebraic K-theory I: Higher K-theories* (Lecture Notes in Mathematics, Vol. 341, pp. 85–147). Springer, Berlin.

[10] Serre, J.-P. (1977). *Linear representations of finite groups* (Vol. 42). Springer.

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