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ON THE DIOPHANTINE EQUATIONS $J_n + J_m = L_k$ AND $L_n + L_m = J_k$

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Abstract. This study contains Lucas numbers that are sums of two Jacobsthal numbers and Jacobsthal numbers that are sums of two Lucas numbers. Our main conclusions are confirmed by a refinement of Baker's theorem for linear forms in logarithms and the reduction method of Dujella and Pethő's.

Keywords. Lucas sequence, Jacobsthal sequence, Linear forms in logarithms.
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1. Introduction

Lucas sequence is characterised by $L_0 = 2$, $L_1 = 1$ and

$$L_{n+2} = L_{n+1} + L_n \text{ for all } n \geq 0.$$

The history and applications of the Lucas sequence can be found in [14]. The Jacobsthal sequence is characterised by $J_0 = 0$, $J_1 = 1$ and

$$J_{n+2} = J_{n+1} + 2J_n \text{ for all } n \geq 0.$$

Horadam was among the first to investigate Jacobsthal representation numbers and polynomials, later examining their connections to Pell sequences. Building on this work, Djordjević and Yilmaz-Bozkurt introduced generalized Jacobsthal polynomials and order-k extensions. More recently, Aydin [1] provided further generalizations of the Jacobsthal sequence in number theory. These studies form a foundation for understanding the arithmetic properties of linear recurrence sequences [6, 11, 12, 13, 17, 18].

The representation of integers as sums, products, or differences of terms from classical sequences has attracted considerable attention in the context of Diophantine equations. For instance, Bravo and Luca [3, 4] completely characterized powers of two that can be expressed as sums of two Fibonacci numbers or as sums of two Lucas numbers, respectively. Later, Siar and Keskin [16] solved the Diophantine equation

$$F_n - F_m = 2^a,$$

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identifying all Fibonacci numbers whose differences are powers of two. Further results on sums of two Padovan or Fibonacci numbers were determined in [10]. In 2021, Erduvan and Keskin [8] addressed Diophantine equations involving products of Fibonacci and Jacobsthal numbers, resolving

$$F_k = J_n J_m \quad \text{and} \quad J_k = F_n F_m.$$

This line of inquiry continues to reveal intricate arithmetic relations among classical integer sequences. In 2023, Gaber [9] solved the equations

$$P_k = J_n + J_m \ \text{ and } Q_k = J_n + J_m,$$

where P_k and Q_k are the Pell and Pell-Lucas numbers.

In this paper, we solved the two Diophantine equations

$$J_n + J_m = L_k \tag{1.1}$$

$$L_n + L_m = J_k \tag{1.2}$$

in non-negative integers $n, m,$ and k. More precisely, we proved the following theorems.

Theorem 1.1. *The non-negative integer solutions (n, m, k), with $n \geq m$, of the Diophantine equation (1.1) are*

$$(1,0,1),(1,1,0),(2,0,1),(2,1,0),(2,2,0),(3,0,2),(3,1,3),(3,2,3),(5,0,5).$$

Theorem 1.2. *The non-negative integer solutions (n, m, k), with $n \geq m$, of the Diophantine equation (1.2) are*

$$(1,0,3),(2,0,4),(3,1,4),(4,3,5),(6,2,6),(15,1,12).$$

Our approach makes use of Matveev's Theorem, the most well-known variations [15] of Baker's theory [2], to limit each implicit variable to a single primary variable. Matveev's theorem allows us to obtain very large upper bounds for k and n. To reduce these bounds, we apply the reduction lemma due to Dujella and Pethő [7]. Finally, we perform Sage calculations to identify all solutions.

This document is arranged as follows: In Section 2, we go over some information that is essential to our proof. Section 3 presents the proof of Theorem 1.1. Section 4 presents the proof of Theorem 1.2.

2. Preliminary Results

2.1. **Lucas sequence.** The Binet formula for the Lucas sequence is

$$L_n = \theta^n + \eta^n, \ n \geq 0, \tag{2.1}$$

where

$$\theta = \frac{1+\sqrt{5}}{2}, \qquad \eta = \frac{1-\sqrt{5}}{2},$$

are the roots of the auxiliary equation $y^2 - y - 1 = 0$. The following inequality will assist us in determining the relationships among the three variables $n, m,$ and k.

$$\theta^{n-1} \leq L_n \leq 2\theta^n, \ n \geq 1. \tag{2.2}$$

2.2. **Jacobsthal sequence.** Jacobsthal sequence has the characteristic equation

$$y^2 - y - 2 = 0,$$

with roots

$$\Theta = 2, \quad \phi = -1,$$

and its Binet formula is given by

$$J_n = \frac{\Theta^n - \phi^n}{3} = \frac{2^n - (-1)^n}{3}, \ n \geq 0. \tag{2.3}$$

Forthermore, we have

$$2^{n-2} \leq J_n \leq 2^{n-1}, \ n \geq 1. \tag{2.4}$$

2.3. **Linear forms in logarithms.** Let α be an algebraic number of degree d with minimal polynomial

$$a_0 y^d + a_1 y^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} \left(y - \alpha^{(i)}\right) \ \in \mathbb{Z}[y],$$

where $(a_i, a_j) = 1$ for all $a_i \neq a_j$, $a_0 > 0$ and $\alpha^{(i)}$'s are the conjugates of α. The logarithmic height h of α is defined by

$$h(\alpha) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(max\left\{\left|\alpha^{(i)}\right|, 1\right\}\right)\right).$$

The function h has the following properties:

$$\begin{aligned} h\left(\alpha_1 \pm \alpha_2\right) &\leq h\left(\alpha_1\right) + h\left(\alpha_2\right) + \log 2; \\ h\left(\alpha_1 \alpha_2^{\pm 1}\right) &\leq h\left(\alpha_1\right) + h\left(\alpha_2\right); \\ h\left(\alpha^s\right) &= |s| \ h\left(\alpha\right) \ \ (s \in \mathbb{Z}). \end{aligned}$$

These properties will be used in the following sections without further reference.

We use the following theorem, due to Matveev, in getting upper bounds for the our variables.

Theorem 2.1. *(Matveev's Theorem)* [15] *Let L be a real algebraic number field of degree D, $\alpha_1, \ldots, \alpha_t$ be positive real algebraic numbers in L, $b_1, \ldots, b_t$ be non zero integers and*

$$\alpha = \alpha_1^{b_1} \cdots \alpha_t^{b_t} - 1 \neq 0.$$

Then

$$\log|\alpha| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2\,(1 + \log D)(1 + \log B)\, A_1\, A_2\, A_3, \tag{2.5}$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and $A_i \geq \max\{D\, h(\alpha_i), |\log \alpha_i|, 0.16\}$ for all $i = 1, \ldots, t$.

The following lemma, a variation of a result originally due to Baker–Davenport [2], was established by Dujella and Pethő [7]. We will apply this lemma to reduce the upper bounds obtained by Theorem 2.1.

Lemma 2.2 (Dujella–Pethő)**.** *Let M be a positive integer, and let $\alpha, \mu, A > 0$ and $B > 1$ be given real numbers. Suppose that $\frac{p}{q}$ is a convergent of α with $q > 6M$, and define*

$$\epsilon := \|\mu q\| - M\|\tau q\| > 0,$$

where, for a real number X, $\|X\| := \min\{|X - n| : n \in \mathbb{Z}\}$ denotes the distance from X to the nearest integer.

If positive integers $n, m, \omega \in \mathbb{Z}^+$ satisfy

$$0 < |n\alpha - m + \mu| < \frac{A}{B^{\omega}} \quad \textit{with} \quad n \leq M,$$

then

$$\omega < \frac{\log(Aq/\epsilon)}{\log B}.$$

The following Theorem is called Legendre Theorem of continued fractions. More details about this result can be found in [5].

Theorem 2.3. *Let $x \in \mathbb{R}$, $p, q \in \mathbb{Z}$ and let $x = [a_0, a_1, ...]$. If*

$$\left|\frac{p}{q} - x\right| < \frac{1}{2q^2},$$

then $\frac{p}{q}$ is a convergent of the continued fraction of x. Furthermore, let M and n be nonnegative integers such that $q_n > M$. Put $b = \max\{a_i : 0 \leq i \leq n\}$, then

$$\frac{1}{(b+2)\,s^2} < \left|\frac{r}{s} - x\right|$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

3. Proof of Theorem 1.1

Proof. Assume that $n \geq m$. With Sage, we discovered that the only answers to equation (1.1) in the range $0 \leq m \leq n \leq 200$ are those listed above in Theorem 1.1. We now assume that $n > 200$.

Applying the inequalities (2.2) and (2.4) and equation (1.1), we obtain

$$\theta^{k-1} \leq L_k \leq 2^n \text{ and } 2^{n-2} \leq L_k \leq 2\theta^k. \tag{3.1}$$

Then,

$$(n-3)\frac{\log 2}{\log \theta} \leq k \leq n\frac{\log 2}{\log \theta} + 1. \tag{3.2}$$

Hence,

$$n < k < 2n. \tag{3.3}$$

3.1. **Determining the upper boundaries of k and n.** Rewriting equation (1.1) as

$$\frac{2^n}{3} - \theta^k = -J_m + \eta^k + \frac{(-1)^n}{3},$$

we obtain

$$\left|\frac{2^n}{3} - \theta^k\right| < 2^m + \frac{11}{6}, \tag{3.4}$$

since $\frac{|\eta|^k}{\sqrt{5}} < \frac{1}{2}$ for all $k \geq 0$, and $J_m < 2^m$, for all $m \geq 0$. Dividing (3.4) by $2^n/3$, we obtain

$$\left|1 - 2^{-n}\,\theta^k\,\;3\right| < \frac{9}{2^{n-m}}. \tag{3.5}$$

Thus,

$$\log\left|1 - 2^{-n}\,\theta^k\,\;3\right| < \log 9 - (n-m)\log 2. \tag{3.6}$$

Define

$$\zeta_1 := 3 \cdot 2^{-n}\theta^k - 1.$$

If $\zeta_1 = 0$, then $3\theta^k = 2^n$. Conjugating this equation, we obtain $3\eta^k = 2^n$. Taking absolute values yields

$$2^n = |3\eta^k| < 5,$$

which is a contradiction. Hence, $\zeta_1 \neq 0$.

Assume that

$$\alpha_1 = 2, \quad \alpha_2 = \theta, \quad \alpha_3 = 3, \quad b_1 = -n, \quad b_2 = k, \quad b_3 = 1, \quad t = 3.$$

Since

$$\max\{|b_1|, |b_2|, |b_3|\} = k,$$

we consider $B = k$. The smallest field containing $\alpha_1, \alpha_2, \alpha_3$ is $\mathbb{Q}(\sqrt{5})$, so $D = 2$. The logarithmic heights are

$$h(\alpha_1) = \log 2, \quad h(\alpha_2) = \frac{\log\theta}{2}, \quad h(\alpha_3) = \log 3.$$

So, we take

$$A_1 = 1.4, \quad A_2 = 0.5, \quad A_3 = 2.2.$$

Applying Theorem 2.1, we get

$$\log|\zeta_1| > -C\,(1 + \log k), \tag{3.7}$$

where

$$C = 1.54 \times 1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2) < 14.938 \times 10^{11}.$$

Using (3.6) and (3.7), we get

$$\log 9 - (n-m)\log 2 > -14.938 \times 10^{11}(1 + \log k) > -2.99 \times 10^{12}\log k,$$

hence,

$$(n-m)\log 2 < 3 \times 10^{12}\,\log k. \tag{3.8}$$

We now write equation (1.1) as

$$\frac{2^n}{3} + \frac{2^m}{3} - \theta^k = \eta^k + \frac{(-1)^n}{3} + \frac{(-1)^m}{3}.$$

Consequently,

$$\left|\frac{2^n}{3}\left(1+2^{m-n}\right)-\theta^k\right|<\frac{11}{6}.$$

By dividing the above inequality by $\frac{2^n}{3}\left(1+2^{m-n}\right)$, we obtain

$$\left|1-\theta^k\,2^{-n}\,3\left(1+2^{m-n}\right)^{-1}\right|<\frac{11}{2}\cdot 2^{-n}. \tag{3.9}$$

Thus,

$$\log\left|1-\theta^k\,2^{-n}\,3\left(1+2^{m-n}\right)^{-1}\right|<\log\frac{11}{2}-n\log 2. \tag{3.10}$$

Define

$$\zeta_2:=\theta^k\,2^{-n}\,3\left(1+2^{m-n}\right)^{-1}-1.$$

If $\zeta_2=0$, then $3\theta^k=2^n+2^m$. Conjugating this equation gives

$$|3\eta^k|=2^n+2^m<5,$$

which is a contradiction. Hence, $\zeta_2\neq 0$.

Assume

$$\alpha_1=2,\quad \alpha_2=\theta,\quad \alpha_3=3\left(1+2^{m-n}\right)^{-1},\quad b_1=-n,\quad b_2=k,\quad b_3=1,\quad t=3.$$

We observe that α_1, α_2, and α_3 belong to $\mathbb{Q}(\sqrt{5})$, so we take $D=2$. Also, we choose $B=k$.

The logarithmic heights are

$$h(\alpha_1)=\log 2,\quad h(\alpha_2)=\frac{1}{2}\log\theta,$$

and

$$\begin{aligned} h(\alpha_3)&=h\Big(3\left(1+2^{m-n}\right)^{-1}\Big)\\ &\leq h(3)+h(2^{m-n})+\log 2\\ &=\log 3+|m-n|\,h(2)+\log 2\\ &=\log 6+(n-m)\log 2. \end{aligned}$$

Moreover,

$$|\log\alpha_3|=\left|\log 3-\log(1+2^{m-n})\right|<\log 3+\log 2<1.8.$$

Hence, we may take

$$A_1=1.4,\quad A_2=0.5,\quad A_3=4+2(n-m)\log 2.$$

By applying Theorem 2.1, we obtain

$$\log|\zeta_2|>-C\,(1+\log k)\,\big(4+2(n-m)\log 2\big),$$

where

$$C=(1.4)^2\times 0.5\times 30^6\times 3^{4.5}\times 2^2\times(1+\log 2)<6.79\times 10^{11}.$$

Therefore,

$$\log|\zeta_2|>-6.79\times 10^{11}\,(1+\log k)>-1.36\times 10^{12}\,(\log k)\,\big(4+2(n-m)\log 2\big). \tag{3.11}$$

Using inequalities (3.10) and (3.11), we derive

$$n \log 2 < 1.37 \times 10^{12} (\log k) \big(4 + 2(n-m) \log 2\big). \tag{3.12}$$

Substituting inequality (3.8) into (3.12) gives

$$n \log 2 < 1.37 \times 10^{12} (\log k) \big(4 + 6 \times 10^{12} \log k\big). \tag{3.13}$$

Using inequalities (3.2) and (3.13), we obtain

$$k \log 2 < 2.74 \times 10^{12} (\log k) \big(4 + 6 \times 10^{12} \log k\big). \tag{3.14}$$

Solving inequality (3.14) using Sage, we obtain

$$n < k < 1.1 \times 10^{29}. \tag{3.15}$$

3.2. **Lowering the limit on n.** We now apply Lemma 2.2 to reduce the upper bound in (3.15).
Let

$$Q = k \log \theta - n \log 2 + \log 3. \tag{3.16}$$

Rewrite equation (3.5) as

$$\left|e^Q - 1\right| < \frac{9}{2^{n-m}}. \tag{3.17}$$

As we proved that $\zeta_1 \neq 0$, we obtain that $Q \neq 0$.
Case 1: $Q > 0$

$$0 < Q < e^Q - 1 = \left|e^Q - 1\right| < \frac{9}{2^{n-m}}. \tag{3.18}$$

Case 2: $Q < 0$ and $n - m \geq 25$

$$\left|e^Q - 1\right| = 1 - e^Q < \frac{1}{2},$$

which implies

$$e^{|Q|} = e^{-Q} < 2.$$

Therefore, we obtain

$$0 < |Q| < e^{|Q|} - 1 = e^{|Q|}(1 - e^Q) = e^{|Q|} \left|e^Q - 1\right| < \frac{18}{2^{n-m}}. \tag{3.19}$$

In both cases ($Q > 0$ and $Q < 0$), inequality (3.19) holds. By substituting the formula (3.16) for Q in the inequality above and dividing the result by $\log 2$, we obtain

$$0 < \left|k \frac{\log \theta}{\log 2} - n + \frac{\log\ 3}{\log 2}\right| < \frac{26}{2^{n-m}}. \tag{3.20}$$

Let

$$\mu = \frac{\log\ 3}{\log 2}, \qquad \alpha = \frac{\log \theta}{\log 2}, \qquad \omega = n - m, \qquad A = 26, \quad \text{and} \quad B = 2.$$

Using inequality (3.15), we may take $M = 3 \times 10^{29}$ as an upper bound on k. For

$$q_{75} = 2523397903096531890297742113715934 42 > 6M,$$

the denominator of the 75^{th} convergent of the continued fraction of α, we find that

$$\epsilon = \|\mu\, q_{75}\| - M\, \|\alpha\, q_{75}\| > 0.140378035627.$$

Applying Lemma 2.2, we conclude that

$$n - m < 127. \tag{3.21}$$

To handle the previous upper bound of n inequality (3.15), we consider inequality (3.9).
Let

$$S = k \log \theta - n \log 2 + \log \left(3 \left(1 + 2^{m-n}\right)^{-1}\right). \tag{3.22}$$

Inequality (3.8), we have

$$\left|1 - e^{S}\right| < \frac{11}{2^{n+1}}. \tag{3.23}$$

Since $\zeta_2 \neq 0$, we have $S \neq 0$. If $S > 0$, we obtain that

$$0 < S < e^{S} - 1 = \left|e^{S} - 1\right| < \frac{11}{2^{n+1}}, \tag{3.24}$$

and if $S < 0$, we have

$$1 - e^{S} = \left|e^{S} - 1\right| < \frac{11}{2^{n+1}} < \frac{1}{2} \quad \text{for all } n > 200. \tag{3.25}$$

Therefore, the last two inequalities imply $e^{|S|} < 2$, and therefore

$$0 < |S| < e^{|S|} - 1 = e^{|S|}\left(1 - e^{S}\right) = e^{|S|}\left|e^{S} - 1\right| < \frac{11}{2^{n}}. \tag{3.26}$$

In both cases, ($S > 0$, or $S < 0$), inequality (3.26) holds. By substituting the formula (3.22) for S in the inequality above and dividing the result by $\log 2$, we obtain

$$0 < \left| k \frac{\log \theta}{\log 2} - n + \frac{\log\ 3(1 + 2^{m-n})^{-1}}{\log 2} \right| < \frac{1}{2^{n-4}}. \tag{3.27}$$

Let

$$\mu = \frac{\log 3\left(1 + 2^{-(n-m)}\right)^{-1}}{\log 2}, \quad \alpha = \frac{\log \theta}{\log 2}, \quad \omega = n - 4, \quad A = 1, \quad \text{and} \quad B = 2.$$

Using inequality (3.15), we can take $M = 3 \times 10^{29}$ as an upper bound on k. For

$$q_{75} = 2523397903096531890297742113715934442 > 6M,$$

the denominator of the 75^{th} convergent of the continued fraction of α is q_{75}, we find that for all integers $n - m \in [0, 128]$, the smallest ϵ occurs at $n - m = 121$, except when $n - m = 1$, giving

$$\epsilon = \|\mu\, q_{75}\| - M\, \|\alpha\, q_{75}\| > 0.00343788493.$$

By applying Lemma 2.2, we obtain

$$n < 200, \tag{3.28}$$

which contradicts the assumption $n > 200$. Therefore, equation (1.1) has no solution when $n > 200$. Thus, we obtain $n \leq 200$, so

$$m \leq n \leq 200, \tag{3.29}$$

and

$$k < 2n < 400. \tag{3.30}$$

If $n - m = 1$, we obtain ϵ negative. So if $n - m = 1$, equation (1.1) becomes $L_k = J_n + J_{n+1}$, which can be reduced to

$$L_k = 2^n. \tag{3.31}$$

Then $k < 2n$, and from equation (3.15) we obtain $n < 3 \times 10^{29}$. Using the same method of verification $\zeta_1 < \frac{9}{2^{m-n}}$, we obtain

$$\theta^k 2^{-n} - 1 < \frac{3}{2^n},$$

thus

$$\left| k \frac{\log \theta}{\log 2} - n \right| < \frac{18}{2^n}.$$

Since $\frac{18}{2^n} < \frac{1}{2k^2}$ for all $n > 10$ and $k < 2n$. Theorem 2.3 implies that $\frac{n}{k}$ is a convergent of $\frac{\log \theta}{\log 2}$. We know that $k < M$, then using computation with sage gives

$$q_{67} \leq M < q_{68} \quad \text{and} \quad b := \max\{a_i : i = 0, 2, ..., 69\} = 134.$$

Therefore, Theorem 2.3 implies

$$\frac{1}{(b+2)\,k} < \left| k \frac{\log \theta}{\log 2} - n \right| < \frac{18}{2^n}.$$

Thus

$$2^n < 1.944 \times 10^{32}.$$

Then $n \leq 110$. Hence, Theorem 1.1 is proved. □

4. Proof of Theorem 1.2

Proof. Using Sage, it was discovered that the aforementioned solutions in Theorem 1.2 are the only solutions of equation (1.2) inside of the range $0 \leq m \leq n \leq 200$. Thus we may assume that $n > 200$.

4.1. **Finding relationship between k and n.** .

By applying the inequalities (2.2) and (2.4) together with equation (1.2) to create a relation between k and n, we obtain

$$2^{k-2} \leq J_k \leq 2L_n \leq 4\theta^n, \tag{4.1}$$

and consequently,

$$k \leq n \frac{\log \theta}{\log 2} + 4. \tag{4.2}$$

Hence,

$$k < n. \tag{4.3}$$

4.2. **Finding upper bound of n.** .
By writing equation (1.2) as

$$\theta^n - \frac{2^k}{3} = -L_m - \frac{(-1)^k}{3} - \eta^n,$$

we obtain

$$\left|\theta^n - \frac{2^k}{3}\right| \leq |\ L_m| \ + |\eta|^n + \frac{1}{3} < 2\theta^m + \frac{11}{6}, \tag{4.4}$$

since $\frac{|\eta|^n}{\sqrt{5}} < \frac{1}{2}$ for all $n \geq 0$, and $L_m < 2\theta^m$ for every $m \geq 1$. Dividing (4.4) by θ^n, we obtain

$$\left|1 - 2^k\ \theta^{-n}\ \frac{1}{3}\right| < \frac{4}{\theta^{n-m}}. \tag{4.5}$$

As a result,

$$\log\left|1 - 2^k\ \theta^{-n}\ \frac{1}{3}\right| < \log 4 - \log\theta^{n-m}. \tag{4.6}$$

Define

$$\zeta_3 := 2^k\,\theta^{-n}\,\frac{1}{3} - 1.$$

If $\zeta_3 = 0$, then $3\theta^n = 2^k$. Conjugating this equation gives

$$2^k = 3|\eta^n| < 5,$$

which is a contradiction. Hence, $\zeta_3 \neq 0$.

Let

$$\alpha_1 = 2, \quad \alpha_2 = \theta, \quad \alpha_3 = \frac{1}{3}, \quad b_1 = k, \quad b_2 = -n, \quad b_3 = 1, \quad t = 3.$$

Since $n > k$, we take $B = n$, and let $L = \mathbb{Q}(\sqrt{5})$ with degree $D = 2$. The logarithmic heights are

$$h(\alpha_1) = \log 2, \quad h(\alpha_2) = \frac{\log\theta}{2}, \quad h(\alpha_3) = h\Big(\frac{1}{3}\Big) = \log 3.$$

Hence, we set

$$A_1 = 1.4, \quad A_2 = 0.5, \quad A_3 = 2.2.$$

Applying Theorem 2.1, we obtain

$$\log|\zeta_3| > -C\,(1 + \log n), \tag{4.7}$$

where

$$C = 1.54 \times 1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2) < 14.938 \times 10^{11}.$$

Using (4.6) and (4.7), we obtain

$$\log 4 - \log\theta^{n-m} > -14.938 \times 10^{11}(1 + \log n) > -2.987 \times 10^{12} \times \log n,$$

thus,

$$(n - m)\log\theta < 3 \times 10^{12}\log n. \tag{4.8}$$

We now rewrite equation (1.2) as

$$\theta^n + \theta^m - \frac{2^k}{3} = -\eta^n - \eta^m - \frac{(-1)^k}{3}.$$

Thus,

$$\left|\theta^n\left(1+\theta^{m-n}\right)-\frac{2^k}{3}\right| \leq |\eta|^n+|\eta|^m+\frac{1}{3}<4, \tag{4.9}$$

where $\frac{|\eta|^n}{\sqrt{5}}<\frac{1}{2}$ for all $n \geq 0$. Dividing (4.9) by $\theta^n\left(1+\theta^{m-n}\right)$, we obtain

$$\left|1-2^k\, \theta^{-n}\, \frac{1}{3}\left(1+\theta^{m-n}\right)^{-1}\right|<\frac{4}{\theta^n}. \tag{4.10}$$

Thus,

$$\log \left|1-2^k\, \theta^{-n}\, \frac{1}{3}\left(1+\theta^{m-n}\right)^{-1}\right|<\log 4-n \log \theta. \tag{4.11}$$

Define

$$\zeta_4:=2^k\, \theta^{-n}\, \frac{1}{3}\left(1+\theta^{m-n}\right)^{-1}-1.$$

If $\zeta_4=0$, then $3\left(\theta^n+\theta^m\right)=\ 2^k$. By conjugating this equation, we obtain

$$2^k=3\left|\eta^n+\eta^m\right|<9,$$

which is a contradiction. Hence $\zeta_4 \neq 0$. Let

$$\alpha_1=2, \quad \alpha_2=\theta, \quad \alpha_3=\frac{1}{3}\left(1+\theta^{m-n}\right)^{-1}, \quad b_1=k, \quad b_2=-n, \quad b_3=1, \quad t=3.$$

We take $B=n$, $L=\mathbb{Q}(\sqrt{5})$, and $D=2$. The logarithmic heights are

$$h(\alpha_1)=\log 2, \qquad h(\alpha_2)=\frac{1}{2} \log \theta,$$

and

$$\begin{aligned}
h\left(\frac{1}{3}\left(1+\theta^{m-n}\right)^{-1}\right) & \leq h\left(\frac{1}{3}\right)+h\big(1+\theta^{m-n}\big) \\
& \leq \log 3+h\big(\theta^{m-n}\big)+\log 2 \\
& =\log 6+|m-n|\, h(\theta) \\
& =\log 6+(n-m)\, \frac{\log \theta}{2}.
\end{aligned}$$

Since

$$\begin{aligned}
|\log \alpha_3| & = \left|\log \frac{1}{3}-\log \left(1+\theta^{m-n}\right)\right| \\
& < \left|\log \frac{1}{3}\right|+\left|\log \left(1+\theta^{m-n}\right)\right| \\
& < \left|\log \frac{1}{3}\right|+|\log 2|<1.1+0.7=1.8,
\end{aligned}$$

we can therefore take

$$A_1=1.4, \quad A_2=0.5, \quad \text{and} \quad A_3=4+(n-m) \log \theta>\max \left\{2 h(\alpha_3),|\log \alpha_3|, 0.16\right\}.$$

Applying Theorem 1.2, we obtain

$$\log |\zeta_4|>-C\,(1+\log n)\,((n-m) \log \theta+4), \tag{4.12}$$

where

$$C = (1.4)^2 \times 0.5 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2) < 6.79 \times 10^{11}.$$

Using inequalities (4.11) and (4.12), we deduce

$$\log 4 - n \log\theta > -6.79\times 10^{11}\,((n-m)\log\theta + 4)\,(1+\log n) > -1.36\times 10^{12}\,\log n\,((n-m)\log\theta+4),$$

which implies

$$n\log\theta < 1.37\times 10^{12}\,\log n\,((n-m)\log\theta + 4). \tag{4.13}$$

Using inequalities (4.8) and (4.13), we obtain

$$n\log\theta < 1.37\times 10^{12}\times\log n\times(4 + 3\times 10^{12}\times\log n). \tag{4.14}$$

By Sage, we obtain

$$n < 3.7\times 10^{28}. \tag{4.15}$$

4.3. **Reducing the bound on n.** .

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We now aim to sharpen the previously obtained upper bound in inequality (4.15) using Lemma 2.2. Let

$$Q = k\log 2 - n\log\theta + \log\frac{1}{3}. \tag{4.16}$$

Inequality (4.10) implies that

$$\left|1 - e^Q\right| < \frac{4}{\theta^{n-m}}.$$

Obsrve that $Q \neq 0$, since $\zeta_3 \neq 0$. If $Q > 0$, we obtain

$$0 < Q < e^Q - 1 = \left|e^Q - 1\right| < \frac{4}{\theta^{n-m}}. \tag{4.17}$$

If $Q < 0$, we assume that $n-m \geq 20$. Since $\frac{4}{\theta^{n-m}} < \frac{1}{2}$ for all $n-m \geq 20$, we get

$$\left|1 - e^Q\right| = 1 - e^Q < \frac{1}{2}.$$

Consequently,

$$e^{|Q|} < 2.$$

Therefore,

$$0 < |Q| < e^{|Q|} - 1 = e^{|Q|}(1 - e^Q) = e^{|Q|}\ \left|e^Q - 1\right| < \frac{8}{\theta^{n-m}}. \tag{4.18}$$

In both cases ($Q > 0$ and $Q < 0$), inequality (4.18) holds. Substituting the formula (4.16) for Q in the above inequality and dividing the result by $\log 2$, we obtain

$$0 < \left|n\frac{\log\theta}{\log\ 2} - k + \frac{\log\ 3}{\log\ 2}\right| < \frac{12}{\theta^{n-m}}. \tag{4.19}$$

Let

$$\mu = \frac{\log\ 3}{\log 2},\quad \alpha = \frac{\log\theta}{\log 2},\quad \omega = n-m,\quad A = 12,\quad \text{and}\quad B = \theta.$$

From inequality (4.15), we may take $M = 4\times 10^{28}$ as an upper bound for n. Let

$$q_{70} = 2286663434222676088439108961099 13 > 6M,$$

where q_{70} denotes the denominator of the 70th convergent in the continued fraction expansion of α. A direct computation yields

$$\epsilon = \|\mu q_{70}\| - M\|\alpha q_{70}\| > 0.0328403974748.$$

By applying Lemma 2.2, we obtain

$$n - m < 168. \tag{4.20}$$

Next, we consider the inequality

$$\left|1 - 2^k\theta^{-n}\frac{1}{3}\left(1+\theta^{m-n}\right)^{-1}\right| < \frac{4}{\theta^n}.$$

Let

$$S = k\log 2 - n\log\theta + \log\left(\frac{1}{3}\left(1+\theta^{m-n}\right)^{-1}\right). \tag{4.21}$$

Then

$$|1 - e^S| < \frac{4}{\theta^n}.$$

Since $\zeta_4 \neq 0$, we have $S \neq 0$. If $S > 0$, then

$$0 < S < e^S - 1 = |e^S - 1| < \frac{2}{\theta^n}.$$

As $n > 200$, it follows that $\frac{2}{\theta^n} < \frac{1}{2}$. Hence, if $S < 0$, we obtain

$$|1 - e^S| = 1 - e^S < \frac{1}{2},$$

which implies

$$e^{|S|} = e^{-S} < 2.$$

Consequently,

$$0 < |S| < e^{|S|} - 1 = e^{|S|}(1 - e^S) = e^{|S|}|e^S - 1| < \frac{8}{\theta^n}. \tag{4.22}$$

Therefore, in both cases ($S > 0$) and ($S < 0$), inequality (4.22) holds. Substituting the expression (4.21) for S into this inequality and dividing by $\log 2$, we obtain

$$0 < \left|k - n\frac{\log\theta}{\log 2} + \frac{\log\left(\frac{1}{3}(1+\theta^{m-n})^{-1}\right)}{\log 2}\right| < \frac{12}{\theta^n}. \tag{4.23}$$

Equivalently,

$$0 < \left|n\frac{\log\theta}{\log 2} - k + \frac{\log(3(1+\theta^{m-n}))}{\log 2}\right| < \frac{12}{\theta^n}. \tag{4.24}$$

Let

$$\mu = \frac{\log 3(1+\theta^{m-n})}{\log 2}, \quad \alpha = \frac{\log\theta}{\log 2}, \quad \omega = n, \quad A = 12, \quad \text{and} \quad B = \theta.$$

Using inequality (4.15), we may take

$$M = 4 \times 10^{28},$$

which satisfies $M > n > k$, as an upper bound for k. For the same value

$$q_{70} = 2286663434222676088439108961099913 > 6M,$$

where q_{70} denotes the denominator of the 70th convergent of the continued fraction of α, we verified that for all integers $n-m \in [0,168]$ the minimal value of

$$\epsilon = \|\mu q_{70}\| - M\|\alpha q_{70}\|$$

satisfies

$$\epsilon > 0.004.$$

Applying Lemma 2.2, we obtain

$$n < 172. \tag{4.25}$$

This contradicts our standing assumption that $n > 200$. Hence, equation (1.2) admits no solutions for $n > 200$. □

References

1. Aydin, F. T. (2018). On generalizations of the Jacobsthal sequence. *Notes on Number Theory and Discrete Mathematics*, 24(1), 120–135. https://doi.org/10.7546/nntdm.2018.24.1.120-135
2. Baker, A., & Davenport, H. (1969). The equations $3x^2-2=y^2$ and $8x^2-7=z^2$. *Quarterly Journal of Mathematics*, 20(1), 129–137. https://doi.org/10.1093/qmath/20.1.129
3. Bravo, J. J., & Luca, F. (2014). Powers of two as sums of two Lucas numbers. *Journal of Integer Sequences*, 17, Article 14.8.3. Retrieved from
4. Bravo, J. J., & Luca, F. (2016). On the Diophantine equation $F_n+F_m=2^a$. *Quaestiones Mathematicae*, 39(3), 391–400. https://doi.org/10.2989/16073606.2015.1070377
5. Ddamulira, M. (2020). Repdigits as sums of three Padovan numbers. *Boletín de la Sociedad Matemática Mexicana*, 26(2), 247–261. https://doi.org/10.1007/s40590-019-00269-9
6. Djordjević, G. B. (2000). Generalized Jacobsthal polynomials. *The Fibonacci Quarterly*, 38(3), 239–243. Retrieved from
7. Dujella, A., & Pethő, A. (1998). A generalization of a theorem of Baker and Davenport. *The Quarterly Journal of Mathematics*, 49(3), 291–306. https://doi.org/10.1093/qmathj/49.3.291
8. Erduvan, F., & Keskin, R. (2021). Fibonacci numbers which are products of two Jacobsthal numbers. *Tbilisi Mathematical Journal*, 14(2), 105–116. https://doi.org/10.32513/tmj/19322008126
9. Gaber, A. (2023). Intersections of Pell, Pell-Lucas numbers and sums of two Jacobsthal numbers. *Punjab University Journal of Mathematics*, 55(5-6), 241–252. https://doi.org/10.52280/pujm.2023.55(5-6)05
10. García Lomelí, A.C., Hernández, S., & Luca, F. (2022). Fibonacci numbers as sums of two Padovan numbers. *Afrika Matematika*, 33(1), Article 14. https://doi.org/10.1007/s13370-021-00943-0
11. Horadam, A. F. (1988). Jacobsthal and Pell curves. *The Fibonacci Quarterly*, 26(1), 77–83. https://doi.org/10.1080/00150517.1988.12429664
12. Horadam, A. F. (1996). Jacobsthal representation numbers. *The Fibonacci Quarterly*, 34(1), 40–54. Retrieved from
13. Horadam, A. F. (1997). Jacobsthal representation polynomials. *The Fibonacci Quarterly*, 35(2), 137–148. https://doi.org/10.1080/00150517.1997.12429009
14. Koshy, T. (2018). *Fibonacci and Lucas numbers with applications*. John Wiley & Sons. https://doi.org/10.1002/9781118742327
15. Matveev, E. M. (2000). An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers. II. *Izvestiya: Mathematics*, 64(6), 1217–1269. https://doi.org/10.1070/IM2000v064n06ABEH000314
16. Şiar, Z., & Keskin, R. (2020). On the Diophantine equation $F_n-F_m=2^a$. *Colloquium Mathematicum*, 159, 119–126. https://doi.org/10.4064/cm7485-12-2018

17. Tougma, C. W.-W., & Nikiéma, S. (2025). Repdigits as absolute value of difference between Lucas and Fibonacci numbers. *Gulf Journal of Mathematics*, 19(2), 1–19. https://doi.org/10.56947/gjom.v19i2.2680
18. Yilmaz, F., & Bozkurt, D. (2009). The generalized order-k Jacobsthal numbers. *International Journal of Contemporary Mathematical Sciences*, 4(33-36), 1685–1694. Retrieved from

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