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LAPLACIAN EIGENVALUES OF THE EXTENDED ZERO DIVISOR GRAPH OF SOME FINITE COMMUTATIVE RINGS

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Abstract. In this paper, we have evaluated the Laplacian eigenvalues of the extended zero divisor graph for certain finite commutative rings. Additionally, we present the conditions under which the Laplacian spectral radius of the extended zero divisor graph is equivalent to the order of the graph. Furthermore, we calculate the clique number, independent number, domination number, and connected domination number of the extended zero divisor graph for specific finite commutative rings.

Keywords. Laplacian Eigenvalues, Extended zero divisor graph, Commutative rings, Domination number, Clique number, Independent number.

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1. Introduction

Beck [6] introduced the concept of the zero divisor graph (ZDG) for a commutative ring R, denoted as $\Gamma(R)$. This concept was subsequently modified by Anderson and Livingston in [2]. They investigated the relationship between the ring-theoretic and graph-theoretic properties of ZDG. Thereafter, the ZDG has garnered attention from numerous researchers [22, 23, 5, 3]. Other algebraic graphs related to the ZDG, such as weakly ZDG, co-zero divisor graphs, and extensions of ZDG, have also garnered attention from various researchers. For more details, refer to [8, 16, 19].

Initially, the adjacency matrix and eigenvalues of the ZDG were discussed by Young in [24]. Chattopadhyay et al. [10], discussed the Laplacian eigenvalues (L-eigenvalues) of ZDG for the ring $\mathbb{Z}_{p^r}$. Later, Rather et al. [21] studied the L-eigenvalues of the ZDG for the ring $\mathbb{Z}_{p^rq^s}$. Subsequently, Rather et al.[20] evaluated the normalized L-eigenvalues of ZDG for some commutative rings. Motivated by the idea of evaluating various spectrum such as L-eigenvalues, signless Laplacian eigenvalues, and normalized L-eigenvalues of ZDG, many researchers have contributed to the evaluation of spectrum for other algebraic graphs [1, 4].

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Bennis et al. [7] extend the ZDG of a commutative ring by defining the vertex set as the non-zero zero divisors of R. Two vertices a and b are connected if there are positive integers m and l such that $a^m b^l = 0$, while ensuring $a^m \neq 0$ and $b^l \neq 0$. They termed this graph the extended ZDG, and we denote this graph by $\Gamma_{ex}(R)$. Since $\Gamma(R)$ is a subgraph of $\Gamma_{ex}(R)$. This poses an interesting problem of finding the conditions under which the extended ZDG is the same as the ZDG. Bennis et al. [7], gives the solution of this problem for the ring $\mathbb{Z}_n$. They demonstrated that $\Gamma_{ex}(\mathbb{Z}_n)$ equals $\Gamma(\mathbb{Z}_n)$ if and only if n is either a product of different prime numbers or the square of a single prime number. However, in this paper we consider the rings for which $\Gamma_{ex}(\mathbb{Z}_n) \neq \Gamma(\mathbb{Z}_n)$, that is $n = p^r$ where $r \geq 3$ or n is divisible by prime power and tackle the following problems

(1) How does an increase in the edge density of $\Gamma_{ex}(R)$ compared to $\Gamma(R)$, affect the Laplacian eigenvalues?
(2) How are other graph properties like Laplacian radius, clique number, independent number, etc. affected due to edge density in $\Gamma_{ex}(R)$?

By systematically analyzing these relationships, this study aims to provide a deeper understanding of the spectral properties of densely connected graphs and lay a foundation for future applications in complex networks and computational models. The eigenvalues of the Laplacian matrix not only reveal the fundamental characteristics of a graph, such as its connectivity and robustness, but also influence applications ranging from network design to machine learning.

The remainder of the paper is organized as follows: Section 2 addresses the fundamental concepts and definitions that underpin the subsequent sections. In Section 3, we focus on the first problem and represent the extended ZDG as a join of induced subgraphs. Furthermore, we assess the L-spectrum of $\Gamma_{ex}(\mathbb{Z}_n)$ for $n = p^r, p^r q, pq^s$ and $p^r q^s$. Finally, in Section 4, we analyse the second problem and discuss some graph properties of $\Gamma_{ex}(\mathbb{Z}_n)$.

2. Preliminaries

In this part, we introduce some graph theory terminology pertinent to this paper. A graph is represented as $H = H(V, E)$, where $V = \{x_1, x_2, \ldots, x_k\}$ denotes the set of vertices and E indicates the set of edges in H. The order of H is given by $|V| = k$, and its size is $|E|$. When two vertices, such as x_i and x_j, are adjacent, it is denoted as $x_i \sim x_j$, with the edge connecting them represented by the endpoints ij. Throughout this paper, H is considered a simple, undirected, and connected graph. The set of vertices in H that are adjacent to a vertex x is known as the neighborhood of vertex x, denoted as $\mathsf{N}_H(x)$. The degree of x is the number of vertices in $\mathsf{N}_H(x)$, represented by $deg(x)$. If every vertex has the same degree, r, then H is referred to as an r-regular graph. The adjacency matrix of H, denoted as $A(H)$, is a square matrix of order k, where the entry at (i, j) is 1 if $x_i \sim x_j$, and 0 otherwise. The matrix $D(H)$ is a diagonal matrix with diagonal entries corresponding to the degrees of the vertices. The matrix $L(H) = D(H) - A(H)$ is the Laplacian matrices of H.

The spectrum of a square matrix M is the multiset of all the eigenvalues of M. If $\lambda_1, \lambda_2, \ldots, \lambda_t$ be the eigenvalues of matrix M with respective multiplicities

$s_1, s_2, s_3, \ldots, s_t$, then we denote spectrum of M as,

$$\sigma(H) = \{[\lambda_1]^{s_1}, [\lambda_2]^{s_2}, [\lambda_3]^{s_3}, \ldots, [\lambda_t]^{s_t}\}.$$

If e is any number, $e + \sigma(H)$ denotes that e is added to every eigenvalue of M. The expression $\sigma(H)\backslash\{\lambda_i\}$ indicates that from the multiset of eigenvalues $\sigma(H)$, a single occurrence of the eigenvalue λ_i is removed. The spectrum of matrices $A(H)$ and $L(H)$ are denoted as $\sigma_A(H)$ and $\sigma_L(H)$, respectively.

Consider a graph H with vertex set $V = \{x_1, \ldots, x_k\}$, and let D_i represent disjoint graphs of order r_i for $1 \leq i \leq k$. The graph $H[D_1, \ldots, D_k]$ is constructed by taking the graphs $D_1, \ldots, D_k$ and connecting each vertex of D_i to every vertex of D_j whenever x_i is adjacent to x_j in H. This construction is referred to as the generalized join operation [9]. The graph $H[D_1, \ldots, D_k]$ is called as H-generalized join graph. If H has only two adjacent vertices or P_2 graph, then the generalized join graph coincides with the usual join $D_1 \vee D_2$ of D_1 and D_2. The adjacency and Laplacian spectrum (L-spectrum) of H-generalized join graph were analysed in [9]. For additional information on the join spectra and Laplacian matrix, refer to [14, 17, 21]. For the matrix $L(H)$, we denote the Laplacian characteristic polynomial (L_{ch}-polynomial) as $\Phi(H, x)$.

3. Laplacian spectrum of $\Gamma_{ex}(\mathbb{Z}_n)$

An element a in $\mathbb{Z}_n$ is considered a zero divisor if and only if the greatest common divisor of a and n, denoted as (a, n), is not equal to 1. Consequently, the total number of vertices in $\Gamma_{ex}(\mathbb{Z}_n)$ is given by $n - \phi(n) - 1$, where $\phi(n)$ represents the count of positive integers up to n that are coprime with n. Consider the set $\mathcal{T} = \{d \in \mathbb{Z} : d \mid n, 1 < d < n\}$. For each $d \in \mathcal{T}$, define the set $V_d = \{a \in \mathbb{Z}_n : (a, n) = d\}$. It is evident that the sets V_d constitute a partition of the vertex set of $\Gamma_{ex}(\mathbb{Z}_n)$ into disjoint subsets, each containing $\phi(n/d)$ vertices. The subsequent lemmas elucidate the adjacency relations among the vertices of $\Gamma_{ex}(\mathbb{Z}_n)$.

Lemma 3.1. *For any $d \in \mathcal{T}$, any two distinct vertices of V_d are adjacent in $\Gamma_{ex}(\mathbb{Z}_n)$ if and only if d is divisible by every prime factor of n.*

Proof. Suppose that any two distinct vertices of V_d are adjacent in $\Gamma_{ex}(\mathbb{Z}_n)$. Assume that there exists a prime p such that $p \mid n$ but $p \nmid d$. Then p cannot be a prime divisor of any a and b in V_d. Consequently, there are no positive integers m and l such that $p \mid a^m$ and $p \mid b^l$. Hence, $p \nmid a^m b^l$ and $n \nmid a^m b^l$. This contradicts the assumption that a and b are adjacent. Hence, d must be divisible by each prime factor of n.

Conversely, let d is divisible by each prime factor of n, where $n = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_t}$ and $d = p_1^{s_1} p_2^{s_2} \ldots p_k^{s_t}$ where $1 \leq s_i \leq r_i$. Let $a, b \in V_d$ with $a \neq b$ and $a = dk_1$, $b = dk_2$, with $(k_1, n/d) = 1 = (k_2, n/d)$ for some integers k_1 and k_2. Note that for any positive integer x,

$$a^x \equiv 0 \pmod{n} \iff n \mid a^x \iff n \mid d^x k_1^x \iff n \mid d^x.$$

Since $(k_1^x, n) = 1$, there is no common prime factor of n and k_1. This means that the divisibility of a^x depends only on the power of d. That is, we must have

$xs_i \geq r_i$ for $i = 1, \ldots, t$. Let α be the smallest multiple of s_i such that $xs_i \geq r_i$ for all $i = 1, \ldots, t$, that is

$$\alpha = \min \left\{ x \in \mathbb{Z}^+ : xs_i \geq r_i \text{ for all } i \right\}.$$

Clearly $a^\alpha \equiv 0 \pmod{n}$. Since $1 < d < n$, there must be at least one prime p_j such that $s_j < r_j$, and therefore $\alpha \geq 2$. By definition of α, $a^{\alpha-1} \not\equiv 0 \pmod{n}$. Take $m = \alpha - 1$ and $l = 1$, we have $a^m \not\equiv 0 \pmod{n}$ and $b^l \not\equiv 0 \pmod{n}$, however

$$a^m b^l = a^{\alpha-1} b = (dk_1)^{\alpha-1}(dk_2) = d^\alpha (k_1^{\alpha-1} k_2) \equiv 0 \pmod{n}.$$

□

For each $d \in \mathcal{T}$, consider the induced subgraphs D_d of $\Gamma_{ex}(\mathbb{Z}_n)$ with vertex set V_d containing $\phi(n/d)$ vertices. By the above lemma, the induced subgraph D_d is either a complete graph or the complement of the complete graph, depending on whether d is divisible by each prime factor of n.

Lemma 3.2. *For any two distinct $d_i, d_j \in \mathcal{T}$, every vertex of V_{d_i} is adjacent to each vertex of V_{d_j} in $\Gamma_{ex}(\mathbb{Z}_n)$ if and only if there exist positive integers m and l such that n divides $d_i^m d_j^l$, with $d_i^m \neq 0$ and $d_j^l \neq 0$.*

Proof. Suppose that every vertex of V_{d_i} is adjacent to every vertex of V_{d_j} in $\Gamma_{ex}(\mathbb{Z}_n)$. Since $d_i \in V_{d_i}$ and $d_j \in V_{d_j}$, by our assumption d_i is adjacent to d_j in $\Gamma_{ex}(\mathbb{Z}_n)$. Therefore, there exist positive integers m and l such that $n \mid d_i^m d_j^l$ with $d_i^m \neq 0$, $d_j^l \neq 0$.

Conversely, let there exist positive integers m and l such that $d_i^m \neq 0$ and $d_j^l \neq 0$, however $n \mid d_i^m d_j^l$ that is $d_i^m d_j^l \equiv 0 \pmod{n}$. If $a \in V_{d_i}$ and $b \in V_{d_j}$ then $(a, n) = d_i$ and $(b, n) = d_j$. Write $a = d_i k$ and $b = d_j k'$, where $k, k' \in \mathbb{Z}$, which gives

$$a^m b^l = (d_i k)^m (d_j k')^l = d_i^m d_j^l k^m k'^l \equiv 0 \pmod{n}.$$

We also need to show that $a^m \not\equiv 0 \pmod{n}$ and $b^l \not\equiv 0 \pmod{n}$. If possible, let $a^m \equiv 0 \pmod{n}$ that is $n \mid a^m = d_i^m k^m$. This is equivalent to say,

$$\frac{n}{d_i} \mid d_i^{m-1} k^m.$$

Since $(k, n/d_i) = 1$, it follows that $(k^m, n/d_i) = 1$. This gives $\frac{n}{d_i} \mid d_i^{m-1}$ or $n \mid d_i^m$, which contradict the assumption that $d_i^m \neq 0$ in $\mathbb{Z}_n$. This completes the proof of the lemma. □

In the next lemma, we represent $\Gamma_{ex}(\mathbb{Z}_n)$ as a generalized join of its induced subgraphs. Let $n = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$ and $\mathcal{T} = \{d_1, d_2, \ldots, d_k\}$. Consider a graph ψ_n where the vertex set is denoted by $\mathcal{T}$. Two distinct vertices, d_i and d_j, are connected by an edge if and only if $d_i^m d_j^l = 0$ holds true, with the conditions that $d_i^m \neq 0$ and $d_j^l \neq 0$ in $\mathbb{Z}_n$ for some positive integers m and l. Clearly, the graph ψ_n is a simple and connected graph with $|\psi_n| = \prod_{i=1}^{t}(r_i + 1) - 2$ [9]. Assigning the induced subgraph D_{d_i} of $\Gamma_{ex}(\mathbb{Z}_n)$ corresponding to each vertex d_i of ψ_n, the following result is a direct implication of Lemma 3.1 and Lemma 3.2.

Lemma 3.3. *Let D_{d_i} be the induced subgraphs of $\Gamma_{ex}(\mathbb{Z}_n)$ with vertex set V_{d_i}, for $1 \leq i \leq k$. Then, $\Gamma_{ex}(\mathbb{Z}_n) = \psi_n [D_{d_1}, D_{d_2} \ldots, D_{d_k}]$.*

The above lemma implies that $\Gamma_{ex}(\mathbb{Z}_n)$ is a generalized join graph of some complete and edgeless graphs. We utilise this structure to evaluate the L-eigenvalues of $\Gamma_{ex}(\mathbb{Z}_n)$ in light of the result [10, Theorem 3.3]. The L-spectrum of the edgeless graph $\overline{K}_n$ and complete graph K_n are given by $\{[0]^n\}$ and $\{0,[n]^{n-1}\}$, respectively. In the subsequent theorems we evaluate the L- eigenvalues of $\Gamma_{ex}(\mathbb{Z}_n)$, where $n = p^r$ and p^rq. We begin by analysing the structure of $\Gamma_{ex}(\mathbb{Z}_{p^r})$.

Theorem 3.4. *If $r \geq 3$, then $\Gamma_{ex}(\mathbb{Z}_{p^r})$ is isomorphic to a complete graph and hence L-spectrum of $\Gamma_{ex}(\mathbb{Z}_{p^r})$ is $\left\{0,[p^{r-1}-1]^{p^{r-1}-2}\right\}$.*

Proof. For $n = p^r$, the set $\mathcal{T}$ is $\{p^i \mid 1 \leq i \leq r-1\}$ and $V_{p^i} = \{a \in \mathbb{Z}_{p^r} : (a,p^r) = p^i\}$ with $|V_{p^i}| = \phi(p^{r-i})$. The vertex set of $\Gamma_{ex}(\mathbb{Z}_{p^r})$ is the union of subsets V_{p^i}, where $1 \leq i \leq r-1$. According to Lemma 3.1, for every $i = 1,\ldots,r-1$, any pair of distinct vertices in V_{p^i} are connected. When $i \neq j$, we demonstrate that each vertex in V_{p^i} is adjacent to every vertex in V_{p^j} by applying Lemma 3.2. To establish this, it suffices to find positive integers m and l such that p^{im+jl} is divisible by p^r, while neither p^{im} nor p^{jl} is divisible by p^r. If $i+j \geq r$, take $m = l = 1$; if $i+j < r$, take m as the largest multiple of i less than r that is, $m = \lfloor \frac{r-1}{i} \rfloor$ and l as the largest multiple of j less than r that is, $l = \lfloor \frac{r-1}{j} \rfloor$. These values of m and l satisfied the required conditions. Thus, any two distinct vertices in $\Gamma_{ex}(\mathbb{Z}_{p^r})$ are adjacent and hence realized as $K_{p^{r-1}-1}$ with L-spectrum $\left\{0,[p^{r-1}-1]^{p^{r-1}-2}\right\}$. □

Note that the L-spectrum of a graph H, where $H = \cup_{i=1}^{k} D_i$, is equal to the union of the L-spectrums of D_is. Also, the L_{ch}-polynomial of the join of two graphs H and G with n_1 and n_2 vertices, respectively [15, Corollary 3.7] is given by,

$$\Phi(H \vee G, x) = \frac{x(x-n_1-n_2)}{(x-n_1)(x-n_2)}\Phi(H, x-n_2)\Phi(G, x-n_1). \tag{3.1}$$

Furthermore, we evaluate the L-spectrum of $\Gamma_{ex}(\mathbb{Z}_n)$, for $n = p^rq$ by representing it as a generalized join of its induced subgraphs to find two graphs such that $\Gamma_{ex}(\mathbb{Z}_n)$ is the join of these two graphs. We consider the case when $r \geq 2$ because $\Gamma_{ex}(\mathbb{Z}_{pq}) = \Gamma(\mathbb{Z}_{pq})$.

Theorem 3.5. *The L-spectrum of $\Gamma_{ex}(\mathbb{Z}_n)$ for $n = p^rq$, where $r \geq 2$ and $p \neq q$ is,*

$$\left\{0,\left[(q-1)p^{r-1}+p^r-1\right],\left[(q-1)p^{r-1}\right]^{p^r-p^{r-1}},[p^r-1]^{(q-1)p^{r-1}-1},\left[qp^{r-1}-1\right]^{p^{r-1}-2}\right\}.$$

Proof. The set $\mathcal{T}$ contains the elements, $\{p^i \mid 1 \leq i \leq r\} \cup \{p^kq \mid 0 \leq k \leq r-1\}$. Partition the vertex set of $\Gamma_{ex}(\mathbb{Z}_n)$ into subsets

$$V_{p^i} = \{b \in \mathbb{Z}_n : (b,n) = p^i\},$$

with $|V_{p^i}| = \phi(p^{r-i}q)$ for $1 \leq i \leq r$ and

$$V_{p^kq} = \{b \in \mathbb{Z}_n : (b,n) = p^kq\},$$

with $|V_{p^kq}| = \phi(p^{r-k})$ for $0 \leq k \leq r-1$. Corresponding to each i and k, consider the induced subgraphs D_{p^i} and D_{p^kq} of $\Gamma_{ex}(\mathbb{Z}_n)$ with vertex subsets V_{p^i} and V_{p^kq} respectively. From Lemma 3.1, for $1 \leq i \leq r$, $D_{p^i} \cong \overline{K}_{\phi(p^{r-i}q)}$, $D_q \cong \overline{K}_{\phi(p^r)}$ and for $1 \leq k \leq r-1$, $D_{p^kq} \cong K_{\phi(p^{r-k})}$. By Lemma 3.3,

$$\Gamma_{ex}(\mathbb{Z}_{p^rq}) = \psi_{p^rq}\left[D_p, D_{p^2} \ldots, D_{p^r}, D_q, D_{pq}, \ldots D_{p^{r-1}q}\right].$$

Now, consider the graph $H = \cup_{i=1}^{r} D_{p^i}$. Further, consider $G_1 = D_{pq}$ and find the graph G_{r-1} recursively by the relation $G_j = D_{p^jq} \vee G_{j-1}$. Let $G = D_q \cup G_{r-1}$. From the adjacency relations in ψ_{p^rq}, it is clear that $\Gamma_{ex}(\mathbb{Z}_{p^rq}) = H \vee G$.

The L_{ch}-polynomial of D_{p^i} is $x^{\phi(p^{r-i}q)}$. Since the L_{ch}-polynomial of the union of disjoint graphs is equal to the product of L_{ch}-polynomial of those graphs. Thus, L_{ch}-polynomial of H is

$$\begin{aligned}\Phi(H,x) &= x^{\phi(p^{r-1}q)}x^{\phi(p^{r-2}q)}\cdots x^{\phi(pq)}x^{\phi(q)}\\ &= x^{\phi(p^{r-1}q)+\phi(p^{r-2}q)+\cdots+\phi(pq)+\phi(q)}\\ &= x^{\phi(q)p^{r-1}}\\ &= x^{(q-1)p^{r-1}}.\end{aligned}$$

Also the L_{ch}-polynomial of D_{p^iq} is $x\times(x-\phi(p^{r-i}))^{\phi(p^{r-i})-1}$, in particular $\Phi(G_1,x) = \Phi(D_{pq},x) = x\times(x-\phi(p^{r-1}))^{\phi(p^{r-1})-1}$. Since $G_2 = D_{p^2q}\vee G_1$ with $n_1 = |V(D_{p^2q})| = \phi(p^{r-2})$ and $n_2 = |V(G_1)| = \phi(p^{r-1})$, by equation (3.1), the L_{ch}-polynomial of G_2 is

$$x \times (x - (\phi(p^{r-2}) + \phi(p^{r-1})))^{\phi(p^{r-2})+\phi(p^{r-1})-1}.$$

Likewise, for $2 \leq j \leq r-1$, when equation (3.1) is applied to each G_j, the L_{ch}-polynomial of G_{r-1}, is given by,

$$\Phi(G_{r-1},x) = x \times \big(x - \big(\phi(p^{r-1}) + \phi(p^{r-2}) + \cdots + \phi(p^2) + \phi(p)\big)\big)^{(\phi(p^{r-1})+\phi(p^{r-2})+\cdots+\phi(p)-1)}.$$

As $\phi(p^{r-1}) + \phi(p^{r-2}) + \phi(p^{r-3}) + \cdots + \phi(p^3) + \phi(p^2) + \phi(p) = p^{r-1} - 1$. Therefore, the L_{ch}-polynomial of G_{r-1} is

$$\Phi(G_{r-1},x) = x \times \big(x - (p^{r-1}-1)\big)^{p^{r-1}-2}.$$

Also the L_{ch}-polynomial of $\overline{K}_{\phi(p^r)}$ is $x^{p^r-p^{r-1}}$, so L_{ch}-polynomial of $G = D_q \cup G_{r-1}$ is

$$\Phi(G,x) = x^{p^r-p^{r-1}+1} \times \big(x - (p^{r-1}-1)\big)^{p^{r-1}-2}.$$

Now, we apply equation (3.1) with H and G to find the L_{ch}-polynomial of $\Gamma_{ex}(\mathbb{Z}_{p^rq})$. Note that here $n_1 = |V(H)| = (q-1)p^{r-1}$ and $n_2 = |V(G)| = p^r - 1$.

Therefore

$$\begin{aligned}\Phi(\Gamma_{ex}(\mathbb{Z}_{p^rq}, x) &= \Phi(H \vee G, x)\\ &= x \times \left(x-(q-1)p^{r-1}-p^r+1\right)\\ &\quad \times \left(x-(q-1)p^{r-1}\right)^{p^r-p^{r-1}} \times (x-p^r+1)^{(q-1)p^{r-1}-1}\\ &\quad \times \left(x-(q-1)p^{r-1}-p^{r-1}+1\right)^{p^{r-1}-2}.\end{aligned}$$

The result is obtained by evaluating $\Phi(\Gamma_{ex}(\mathbb{Z}_{p^rq}, x) = 0$. □

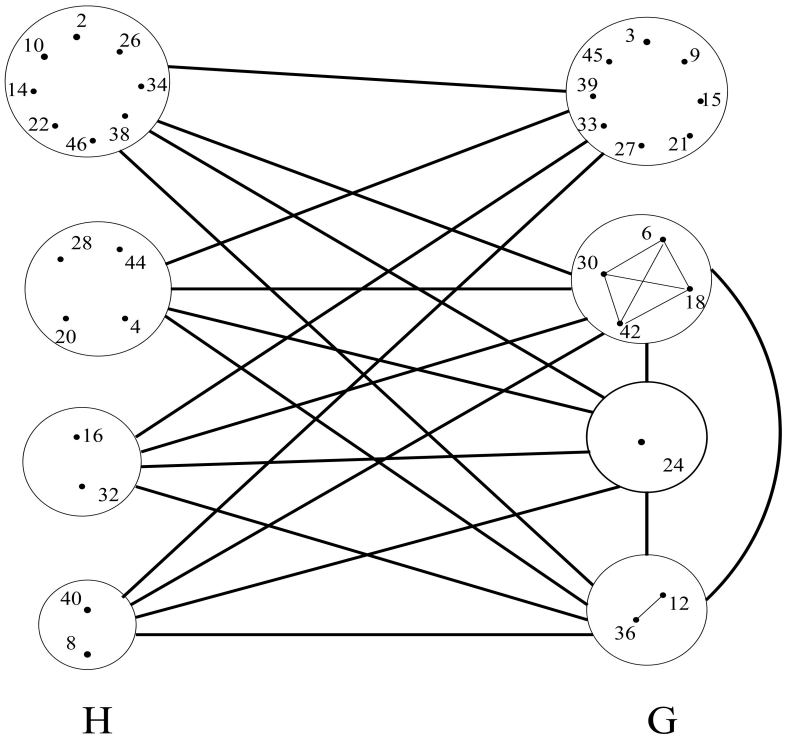


FIGURE 1. $\Gamma_{ex}(\mathbb{Z}_{48})$

Example 3.6. For $R = \mathbb{Z}_{48}$, the set $\mathcal{T}$ contains the elements $\{2^i | 1 \le i \le 4\} \cup \{2^k 3 | 0 \le k \le 3\}$. The vertex set of $\Gamma_{ex}(\mathbb{Z}_{48})$ can be partitioned into subsets

$$\begin{array}{ll} V_2 = \{2, 10, 14, 22, 26, 34, 38, 46\} & V_3 = \{3, 9, 15, 21, 27, 33, 39, 45\}\\ V_{2^2} = \{4, 20, 28, 44\} & V_{2\cdot 3} = \{6, 18, 30, 42\}\\ V_{2^3} = \{8, 40\} & V_{2^2 3} = \{12, 36\}\\ V_{2^4} = \{16, 32\} & V_{2^3 3} = \{24\}.\end{array}$$

For, $1 \le i \le 4$ consider the induced subgraph $D_{2^i} \cong \overline{K}_{2^{4-i}3}$, $D_3 = \overline{K}_8$ and for $0 \le k \le 3$, $D_{2^k 3} \cong K_{\phi(2^{4-k})}$. In Figure 1, the dark line indicates that each vertex of a vertex subset is adjacent to every vertex of the corresponding vertex subset. By Lemma 3.3, $\Gamma_{ex}(\mathbb{Z}_{48}) = \psi_{2^4 3}[D_2, D_{2^2}, D_{2^3}, D_{2^4}, D_3, D_{2.3}, D_{2^2 3}, D_{2^3 3}]$.

The L_{ch}-polynomial of $H = \cup_{i=1}^{4}$ is x^{16}. Also the L_{ch}-polynomial of $G_1 = D_{2\cdot 3} = x(x-4)^3$. Since $G_2 = D_{2^2 3} \vee G_1$ with $n_1 = |V(D_{2^2 3})| = 2$ and $n_2 =$

$|V(G_1)| = 4$, by equation (3.1), the L_{ch}-polynomial of G_2 is $x(x-6)^5$. Similarly, L_{ch}-polynomial of $G_3 = D_{2^3 3} \vee G_2$ with $n_1 = |V(D_{2^3 3})| = 1$ and $n_2 = |V(G_2)| = 6$, by equation (3.1), the L_{ch}-polynomial of G_3 is $x(x-7)^6$.

Further, the L_{ch}-polynomial of $G = D_q \cup G_3$ is product of L_{ch}-polynomial of G_3 and L_{ch}-polynomial of $D_q = \overline{K}_8$ that is $x^9(x-7)^6$. Again using the equation (3.1) with H and G to find the L_{ch}-polynomial of $\Gamma_{ex}(\mathbb{Z}_{48})$. Note that here $n_1 = |V(H)| = 16$ and $n_2 = |V(G)| = 15$, Therefore,

$$\begin{aligned}\Phi(\Gamma_{ex}(\mathbb{Z}_{48}), x) &= \Phi(H \vee G, x)\\ &= \frac{x(x-15-16)}{(x-15)(x-16)}(x-16)^9(x-7-16)^6(x-15)^{16}\\ &= x(x-31)(x-23)^6(x-16)^8(x-15)^{15}.\end{aligned}$$

Therefore, the result is obtained by evaluating $\Phi(\Gamma_{ex}(\mathbb{Z}_{48}), x) = 0$, that is,

$$\left\{0, 31, [16]^8, [23]^6, [15]^{15}\right\}.$$

Recall that the graph ψ_n has vertex set $\mathcal{T} = \{d \in \mathbb{Z} : d \mid n, 1 < d < n\}$, and two distinct vertices d_i and d_j are adjacent if and only if $d_i^m d_j^l = 0$ with $d_i^m \neq 0$ and $d_j^l \neq 0$ in $\mathbb{Z}_n$ for some positive integers m and l. For $d \in \mathcal{T}$, define $\mathsf{N}_d = \sum_{d' \in \mathsf{N}_{\psi_n}(d)} |V_{d'}|$. Following the approach of Chattopadhyay et al. [10, Theorem 3.3], we evaluate L-eigenvalues of $\Gamma_{ex}(\mathbb{Z}_n)$ as union of sets

$$\mathsf{N}_d + (\sigma_L(D_d) \backslash \{0\}),$$

and the spectrum of the square matrix $L(\psi_n)$. Here, $L(\psi_n)$ is the matrix associated with the graph ψ_n, whose vertex set is $\mathcal{T} = \{d_1, d_2, \ldots, d_k\}$. The matrix $L(\psi_n)$ is given as

$$L(\psi_n) = \begin{bmatrix} \mathsf{N}_{d_1} & -\alpha_{12} & \cdots & -\alpha_{1k} \\ -\alpha_{21} & \mathsf{N}_{d_2} & \cdots & \alpha_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{k1} & -\alpha_{k2} & \cdots & \mathsf{N}_{d_k} \end{bmatrix} \tag{3.2}$$

where,

$$\alpha_{ij} = \begin{cases} \phi(\frac{n}{d_j}), & \text{if } d_i \sim d_j \text{ in } \psi_n, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.7. *The L-spectrum of $\Gamma_{ex}(\mathbb{Z}_n)$ where $n = p^r q^s$ is given by the union of the spectrum of square matrix defined in equation (3.2) and the following multiset*

$$\begin{aligned}\Big\{ &\left[\phi(p^r)q^{s-1} + a + m\right]^{\phi(q^s p^{r-i})-1}, \left[\phi(q^s)p^{r-1} + b + m'\right]^{\phi(p^r q^{s-j})-1},\\ &\left[\phi(q^s)p^{r-1} + ab + a + b\right]^{\phi(p^{r-k})-1}, \left[\phi(p^r)q^{s-1} + ab + a + b\right]^{\phi(q^{s-l})-1},\\ &\left[m_{t_1t_2} + m'_{t_1t_2} + ab + a + b\right]^{\phi(p^{r-t_1}q^{s-t_2})-1}\Big\},\end{aligned}$$

where $1 \le i \le r$, $1 \le j \le s$, $1 \le k \le r-1$, $1 \le l \le s-1$, $1 \le t_1 \le r-1, 1 \le t_2 \le s-1$ and $a = p^{r-1} - 1$, $b = q^{s-1} - 1$. The values of $m, m', m_{t_1t_2}$ and $m'_{t_1t_2}$ are

$m = \sum_{t_1}^{r-1}\sum_{t_2}^{s-1}\phi(p^{r-t_1}q^{s-t_2})$ *for* $t_1\lceil\frac{s}{t_2}\rceil < r$,
$m' = \sum_{t_1}^{r-1}\sum_{t_2}^{s-1}\phi(p^{r-t_1}q^{s-t_2})$ *for* $t_2\lceil\frac{r}{t_1}\rceil < s$,

$$m_{t_1t_2} = \begin{cases}\phi(q^s)p^{r-1}, & \text{if } t_1\lceil\frac{s}{t_2}\rceil < r\\ 0, & \text{otherwise.}\end{cases}, \text{ and}$$

$$m'_{t_1t_2} = \begin{cases}\phi(p^r)q^{s-1}, & \text{if } t_2\lceil\frac{r}{t_1}\rceil < s,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. The set $\mathcal{T}$ contains the elements $\{p^i \mid 1 \le i \le r\}\cup\{q^j \mid 1 \le j \le s\}\cup\{p^kq^s \mid 1 \le k \le r-1\}\cup\{p^rq^l \mid 1 \le l \le s-1\}\cup\{p^{t_1}q^{t_2} \mid 1 \le t_1 \le r-1, 1 \le t_2 \le s-1\}$. By Lemma 3.3, $\Gamma_{ex}(\mathbb{Z}_{p^rq^s})$ is $\psi_{p^rq^s}$ generalized join of D_{p^i}, D_{q^j}, $D_{p^kq^s}$, $D_{p^rq^l}$, and $D_{p^{t_1}q^{t_2}}$, where $1 \le i \le r$, $1 \le j \le s$, $1 \le k \le r-1$, $1 \le l \le s-1$, and $1 \le t_1 \le r-1, 1 \le t_2 \le s-1$. From 3.2, by examining the adjacency relation in $\psi_{p^rq^s}$, we obtain the followings; Take $a = p^{r-1}-1$ and $b = q^{s-1}-1$ then for $1 \le i \le r$

$$\begin{aligned}\mathsf{N}_{p^i} &= \sum_{j=1}^{s}\phi(p^rq^{s-j}) + \sum_{k=1}^{r-1}\phi(p^{r-k}) + \sum_{t_1}^{r-1}\sum_{t_2}^{s-1}\phi(p^{r-t_1}q^{s-t_2})\\ &= \phi(p^r)q^{s-1} + a + m,\end{aligned}$$

where $m = \sum_{t_1}^{r-1}\sum_{t_2}^{s-1}\phi(p^{r-t_1}q^{s-t_2})$ for $t_1\lceil\frac{s}{t_2}\rceil < r$. For $1 \le j \le s$,

$$\begin{aligned}\mathsf{N}_{q^j} &= \sum_{i=1}^{r}\phi(p^{r-i}q^s) + \sum_{l=1}^{s-1}\phi(q^{s-l}) + \sum_{t_1}^{r-1}\sum_{t_2}^{s-1}\phi(p^{r-t_1}q^{s-t_2})\\ &= \phi(q^s)p^{r-1} + b + m',\end{aligned}$$

where $m' = \sum_{t_1}^{r-1}\sum_{t_2}^{s-1}\phi(p^{r-t_1}q^{s-t_2})$ for $t_2\lceil\frac{r}{t_1}\rceil < s$. For $1 \le k \le r-1$,

$$\begin{aligned}\mathsf{N}_{p^kq^s} &= \sum_{i=1}^{r}\phi(p^{r-i}q^s) + \sum_{t_1=1}^{r-1}\sum_{t_2=1}^{s-1}\phi(p^{r-t_1}q^{s-t_2}) + \sum_{k=1}^{r-1}\phi(p^{r-k}) + \sum_{l=1}^{s-1}\phi(q^{s-l}) - \phi(p^{r-k})\\ &= \phi(q^s)p^{r-1} + ab + b + a - \phi(p^{r-k}).\end{aligned}$$

For $1 \le l \le s-1$,

$$\begin{aligned}\mathsf{N}_{p^rq^l} &= \sum_{j=1}^{s}\phi(p^rq^{s-j}) + \sum_{t_1=1}^{r-1}\sum_{t_2=1}^{s-1}\phi(p^{r-t_1}q^{s-t_2}) + \sum_{k=1}^{r-1}\phi(p^{r-k}) + \sum_{l=1}^{s-1}\phi(q^{s-l}) - \phi(q^{s-l})\\ &= \phi(p^r)q^{s-1} + ab + b + a - \phi(q^{s-l}).\end{aligned}$$

For $1 \le t_1 \le r-1$ and $1 \le t_2 \le s-1$,

$$\begin{aligned}\mathsf{N}_{p^{t_1}q^{t_2}} &= \overline{M}_1 + \overline{M}_1 + \sum_{t_1=1}^{r-1}\sum_{t_2=1}^{s-1}\phi(p^{r-t_1}q^{s-t_2}) + \sum_{l=1}^{s-1}\phi(q^{s-l}) + \sum_{k=1}^{r-1}\phi(p^{r-k}) - \phi(p^{t_1}q^{t_2})\\ &= m_{t_1t_2} + m'_{t_1t_2} + ab + b + a - \phi(p^{r-t_1}q^{s-t_2}).\end{aligned}$$

where,

$$m_{t_1t_2} = \begin{cases} \phi(q^s)p^{r-1}, & \text{if } t_1\lceil \frac{s}{t_2} \rceil < r \\ 0, & \text{otherwise.} \end{cases} \text{ and}$$

$$m'_{t_1t_2} = \begin{cases} \phi(p^r)q^{s-1}, & \text{if } t_2\lceil \frac{r}{t_1} \rceil < s, \\ 0, & \text{otherwise.} \end{cases}$$

Further from Lemma 3.1, we have $D_{p^i} \cong \overline{K}_{\phi(p^{r-i}q^s)}$, $D_{q^j} \cong \overline{K}_{\phi(p^rq^{s-j})}$, $D_{p^kq^s} \cong K_{\phi(r-k)}$, $D_{p^rq^l} \cong K_{\phi(s-l)}$, and $D_{p^{t_1}q^{t_2}} \cong K_{\phi(p^{r-t_1}q^{s-t_2})}$. Thus, some L-eigenvalues of $\Gamma_{ex}(\mathbb{Z}_{p^rq^s})$ are given by

- $\mathsf{N}_{p^i} + \big(\sigma_L(D_{p^i})\backslash\{0\}\big) = \mathsf{N}_{p^i} + \Big\{[0]^{\phi(p^{r-i}q)-1}\Big\}$, where $1 \le i \le r$,
- $\mathsf{N}_{q^j} + \big(\sigma_L(D_{q^j})\backslash\{0\}\big) = \mathsf{N}_{q^j} + \Big\{[0]^{\phi(p^rq^{s-j})-1}\Big\}$, where $1 \le j \le s$,
- $\mathsf{N}_{p^kq^s} + \big(\sigma_L(D_{p^kq^s})\backslash\{0\}\big) = \mathsf{N}_{p^kq^s} + \Big\{[\phi(p^{r-k})]^{\phi(p^{r-k})-1}\Big\}$, where $1 \le k \le r-1$,
- $\mathsf{N}_{p^rq^l} + \big(\sigma_L(D_{p^rq^l})\backslash\{0\}\big) = \mathsf{N}_{p^rq^l} + \Big\{[\phi(q^{s-l})]^{\phi(q^{s-l})-1}\Big\}$, where $1 \le l \le s-1$,
- $\mathsf{N}_{p^{t_1}q^{t_2}} + \big(\sigma_L(D_{p^{t_1}q^{t_2}})\backslash\{0\}\big) = \mathsf{N}_{p^{t_1}q^{t_2}} + \Big\{[\phi(p^{r-t_1}q^{s-t_2})]^{\phi(p^{r-t_1}q^{s-t_2})-1}\Big\}$, where $1 \le t_1 \le r-1$ and $1 \le t_2 \le s-1$

and remaining L-eigenvalues are given by the spectrum of square matrix given in equation (3.2). Substituting the values of $\mathsf{N}_{p^i}, \mathsf{N}_{q^j}, \mathsf{N}_{p^kq^s}, \mathsf{N}_{p^rq^l}$, $\mathsf{N}_{p^{t_1}q^{t_2}}$ gives the desired result. □

4. Laplacian spectral radius, clique, domination and independent domination number in $\Gamma_{ex}(\mathbb{Z}_n)$

In this part, we examine certain graph characteristics of $\Gamma_{ex}(\mathbb{Z}_n)$, where $n = p^rq$ and p^rq^s. The algebraic connectivity of a graph H is represented by the second smallest L-eigenvalue of $L(H)$, symbolized as $\mu(H)$. The Laplacian spectral radius of a graph H is defined as the largest L-eigenvalue of the matrix $L(H)$, indicated by $\lambda(H)$. If n is the order of a graph H, then the spectral radius $\lambda(H)$ satisfies $\lambda(H) \le n$. Furthermore, equality holds precisely when the complement graph $\overline{H}$ is disconnected, which occurs if and only if H is expressed as the join of two graphs [11]. Also by Fiedler, $\lambda(H) = n - \mu(\overline{H})$, where $\overline{H}$ denotes the complement graph of H [11, 3.7 (1^o)]. Following these results, we provide the value of n, for which $\overline{\Gamma_{ex}}(\mathbb{Z}_n)$ is disconnected.

Theorem 4.1. *The $\overline{\Gamma_{ex}}(\mathbb{Z}_n)$ is disconnected if and only if $n = p^rq, r \ge 1$, p and q are distinct primes or $n = p^r$ with $r \ge 2$ and $n \ne 4$.*

Proof. For $n = pq$ or $n = p^2, p \ne 2$, $\Gamma_{ex}(\mathbb{Z}_n) = \Gamma(\mathbb{Z}_n)$. Therefore, $\overline{\Gamma_{ex}}(\mathbb{Z}_n)$ is a disconnected graph in these cases [10, Proposition 5.3]. Further, if $n = p^rq$ where $r \ge 2$, by Theorem 3.5, $\Gamma_{ex}(\mathbb{Z}_n)$ is equivalent to graph $H \vee G$ and hence $\Gamma_{ex}(\mathbb{Z}_n)$ is disconnected graph [11]. Moreover, when $n = p^r, r \ge 3$, from Theorem 3.4 $\Gamma_{ex}(\mathbb{Z}_n)$ is compete graph, thus $\overline{\Gamma_{ex}}(\mathbb{Z}_n)$ is disconnected graph.

For converse part, let n is expressed as $p_1^{r_1}p_2^{r_2}\dots p_t^{r_t}$, where $t, r_1, r_2, \dots, r_t$ are positive integers and $p_1, p_2, \dots, p_t$ are distinct primes. Now, we aim to show that $\overline{\Gamma_{ex}}(\mathbb{Z}_n)$ is connected when $n = 4$ or n is divisible by three distinct primes or when $t = 2$ with $r_1, r_2 \geq 2$. If $n = 4$, $\Gamma_{ex}(\mathbb{Z}_n) = \overline{\Gamma_{ex}}(\mathbb{Z}_n)$ is trivial graph, so it is connected. Assume that $t = 2$ with $r_1, r_2 \geq 2$. Then from Theorem 3.7 multiples of types p_1^i are not adjacent in ψ_n and p_2^j are not adjacent in ψ_n and hence form a clique in $\overline{\psi_n}$ for all $i \geq 1$ and $j \geq 1$. Let $d = p_1^{t_1}p_2^{t_2}$ a proper divisor of n, such that $d^l \neq 0$, and not satisfies condition of adjacency that is $t_2\lceil \frac{r}{t_1} \rceil < s$ and $t_1\lceil \frac{s}{t_2} \rceil < r$. Then $n \nmid p_i^{r_i}d^l$, implies that $p_i^{r_i}$ and d^l are not adjacent in ψ_n, and hence adjacent in $\overline{\psi_n}$. Also from Theorem 3.7, p_1^i are adjacent to $p_1^{r_1}p_2$ and p_2^j are adjacent to $p_1p_2^{r_2}$ in $\overline{\psi_n}$. Thus, $\overline{\Gamma_{ex}}(\mathbb{Z}_n)$ is connected.
Further assume that $t \geq 3$. Then for any $i, j \in \{1, 2, \dots, k\}$ there is $p_i^{t_1}$ and $p_j^{t_2}$ in ψ_n, then for any m and l positive integers, $(p_i)^{t_1}, (p_j)^{t_2} \neq 0$ and $n \nmid (p_i)^{t_1}(p_j)^{t_2}$. Therefore, the vertices p_i are not adjacent to p_j in ψ_n, for $1 \leq i \neq j \leq k$, and form a clique in $\overline{\psi_n}$, with vertices $p_1^{r_1}, p_2^{r_2}, \dots, p_k^{r_k}$. Now, claim that any vertex of $\overline{\psi_n}$, other than the vertices corresponding to the $p_i^{r_i}$, is adjacent to at least one of the p_i's, for $1 \leq i \leq k$. To prove this claim, let d be any vertex of ψ_n, such that d is not divisible by each prime factor of n, then there exists i, such that $1 \leq i \leq k$, such that $p_i^{r_i} \nmid d^m$, such that $d^t \neq 0$. Then for any $j \in \{1, 2, \dots, k\}\backslash\{i\}$, $n \nmid d^m(p_j)^l$. This condition implies that d and p^j are not adjacent in ψ_n and hence are adjacent in $\overline{\psi_n}$. Thus, $\overline{\psi_n}$ is connected. □

The following corollary provides the value of n, for which the equality holds for $\lambda(H)$:

Corollary 4.2. *$\lambda(\Gamma_{ex}(\mathbb{Z}_n)) = |\Gamma_{ex}(\mathbb{Z}_n)|$ if and only if $n = p^rq, r \geq 1$, p and q are distinct primes or $n = p^r, r \geq 2$ with $n \neq 4$.*

Proof. To show if part, consider the statement of Theorem 4.1, which implies that $\overline{\Gamma_{ex}}(\mathbb{Z}_n)$ is connected graph if $n = p_1^{r_1}p_2^{r_2}\dots p_t^{r_t}$, where $p < q$, with $t \geq 3$ or $t = 2$, $r_i \geq 2$, $i = 1, 2$. Therefore, $\lambda(\Gamma_{ex}(\mathbb{Z}_n)) < |\Gamma_{ex}(\mathbb{Z}_n)|$. Also, if $n = 4$ then $\lambda(\Gamma_{ex}(\mathbb{Z}_4)) = 0 < 1 = |\Gamma_{ex}(\mathbb{Z}_4)|$.
For only if part, consider $n = p^r$ where $r \geq 2$ with $n \neq 4$, if $r = 2$, then $\Gamma_{ex}(\mathbb{Z}_{p^2}) = \Gamma(\mathbb{Z}_{p^2})$. Thus, $\Gamma_{ex}(\mathbb{Z}_{p^2})$ is a complete graph and $\lambda(\Gamma_{ex}(\mathbb{Z}_n)) = |\Gamma_{ex}(\mathbb{Z}_n)|$. Further, if $r \geq 3$, then $\Gamma_{ex}(\mathbb{Z}_n)$ is a complete graph (see Theorem 3.4), hence $\lambda(\Gamma_{ex}(\mathbb{Z}_n)) = |\Gamma_{ex}(\mathbb{Z}_n)|$. Also, if $n = pq$, then $\Gamma_{ex}(\mathbb{Z}_{pq}) = \Gamma(\mathbb{Z}_{pq})$ and hence $\lambda(\Gamma_{ex}(\mathbb{Z}_{pq})) = |\Gamma_{ex}(\mathbb{Z}_{pq})|$ [10, Proposition 5.4]. Further, if $n = p^rq$ with $r \geq 2$, $|\Gamma_{ex}(\mathbb{Z}_n)| = \phi(q)p^{r-1} + p^r - 1$, therefore from Theorem 3.5, $\lambda(\Gamma_{ex}(\mathbb{Z}_n)) = |\Gamma_{ex}(\mathbb{Z}_n)|$. □

The clique number of a graph H refers to the cardinality of the largest subset of vertices Y, such that every pair of vertices within Y is adjacent. Alternatively, it can be defined as the order of the maximum complete subgraph of H. If $n = p^r, r \geq 2$, then $\Gamma_{ex}(\mathbb{Z}_{p^r})$ is a complete graph. Hence the clique number of $\Gamma_{ex}(\mathbb{Z}_{p^r})$ is $p^{r-1} - 1$. Also, if $n = pq$, then $\Gamma_{ex}(\mathbb{Z}_{pq})$ is complete bipartite graph. Thus, clique number of $\Gamma_{ex}(\mathbb{Z}_{pq})$ is 2. The subsequent theorem specifies the clique number of $\Gamma_{ex}(\mathbb{Z}_n)$ where $n = p^rq^s$.

Theorem 4.3. *The clique number of* $\Gamma_{ex}(\mathbb{Z}_n)$ *for* $n = p^r q^s$ *with* $r, s \geq 2$, *is* $p^{r-1}q^{s-1} - 1$.

Proof. From the definition of $\psi_{p^r q^s}$, the vertices of the clique are of the type $p^k q^s$ where $1 \leq k \leq r-1$, $p^r q^l$ where $1 \leq l \leq s-1$ and $p^{t_1} q^{t_2}$ where $1 \leq t_1 \leq r-1$ and $1 \leq t_2 \leq s-1$ (see Theorem 3.7) and total number of these vertices are, $rs-1$. Also corresponding subgraphs $D_{p^k q^s}$, $D_{p^r q^l}$ and $D_{p^{t_1} q^{t_2}}$ are complete graphs. Therefore the clique number of $\Gamma_{ex}(\mathbb{Z}_n)$ is sum of vertices in induced subgraphs, that is $|D_{pq}| + |D_{p^2q}| + |D_{p^3q}| + \cdots + |D_{p^k q^s}| + \cdots + |D_{p^r q^l}| + \cdots + |D_{p^{r-1}q^{s-1}}|$, that is,

$$
\begin{aligned}
&= \phi(p^{r-1}) + \phi(p^{r-2}) + \cdots + \phi(p^2) + \phi(p) + \phi(q^{s-1}) + \phi(q^{s-2}) + \cdots + \phi(q^2) + \phi(q) \\
&\quad + \phi(p^{r-1}q^{s-1}) + \phi(p^{r-2}q^{s-1}) + \cdots + \phi(p^2q^{s-1}) + \phi(pq^{s-1}) + \phi(p^{r-1}q^{s-2}) \\
&\quad + \phi(p^{r-2}q^{s-2}) + \cdots + \phi(p^2q^{s-2}) + \phi(pq^{s-2}) + \phi(p^{r-1}q^2) + \phi(p^{r-2}q^2) \\
&\quad + \cdots + \phi(p^2q^2) + \phi(pq^2) + \phi(p^{r-1}q) + \phi(p^{r-2}q) + \cdots + \phi(p^3q) \\
&\quad + \phi(p^2q) + \phi(pq).
\end{aligned}
$$

On simplifying, we get

$$
\begin{aligned}
\big(\phi(p^{r-1}) + \phi(p^{r-2}) + \cdots + \phi(p^2) + \phi(p)\big) \times \big(\phi(q^{s-1}) + \phi(q^{s-2}) + \ldots \\
+\phi(q^2) + \phi(q)\big) + (p^{r-1} - 1) + (q^{s-1} - 1) \\
= p^{r-1}q^{s-1} - 1.
\end{aligned}
$$

□

If $s = 1$, then the clique number of $\Gamma_{ex}(\mathbb{Z}_{p^r q})$ is $p^{r-1} - 1$. In a graph H, a subset $\tau \subseteq V$ is termed a dominating set if every vertex in $V \backslash \tau$ is connected to at least one vertex in τ. The smallest size of all possible dominating sets in H is known as the domination number of H, represented by $\gamma_\tau(H)$. A subset $\nu \subset V$ in a graph H is called an independent set if no two vertices within ν are connected in H. The largest size of an independent set is referred to as the independent number of H, denoted by $\nu(H)$. For additional terms like set domination number $\gamma_\tau(H)$ and connected domination number $\gamma_c(H)$, refer to [13]. If $n = p^r$, then $\Gamma_{ex}(\mathbb{Z}_{p^r})$ forms a complete graph, resulting in a domination number of 1. Similarly, if $n = pq$, then $\Gamma_{ex}(\mathbb{Z}_n)$ is a complete bipartite graph, leading to a domination number of 2. In the following theorems, we determine the domination and independent numbers in $\Gamma_{ex}(\mathbb{Z}_n)$ for $n = p^r q^s$.

Theorem 4.4. *The domination number* $\gamma(\Gamma_{ex}(\mathbb{Z}_n))$ *of* $\Gamma_{ex}(\mathbb{Z}_n)$, *where* $n = p^r q^s$, $r, s \geq 2$, *is equal to* 2.

Proof. From the definition of $\psi_{p^r q^s}$, we observe that vertices of the type $p^k q^s$ where $1 \leq k \leq r-1$ and $p^r q^l$ where $1 \leq l \leq s-1$, are adjacent to all other vertices (see Theorem 3.7). Clearly, every vertex which belongs to $V(\Gamma_{ex}(\mathbb{Z}_n)) \backslash \{p^k q^s, p^r q^l\}$ is adjacent to $S = \{p^k q^s, p^r q^l\}$. Now, take one representative of the different vertices from S say, $\{pq^s, p^r q\}$. Since every vertices in $V(\Gamma_{ex}(\mathbb{Z}_n)) \backslash \{pq^s, p^r q\}$, is adjacent to $\{pq^s, p^r q\}$. Therefore, $\{pq^s, p^r q\}$ form the domination set of $\Gamma_{ex}(\mathbb{Z}_n)$ and hence the domination number $\gamma(\Gamma_{ex}(\mathbb{Z}_{p^r q^s}))$ is 2. □

Also, from Theorem 3.7, we see that each vertex of $V_{p^kq^s}$ is adjacent to every vertex of $V_{p^rq^l}$, for $1 \leq k \leq r-1$ and $1 \leq l \leq s-1$. Thus, the connected domination number and domination number of $\Gamma_{ex}(\mathbb{Z}_{p^rq^s})$ are equal.

Theorem 4.5. *The independent number of $\Gamma_{ex}(\mathbb{Z}_n)$, where $n = p^rq^s$ with $p < q$, is equal to $\phi(q^s)p^{r-1}$.*

Proof. From Theorem 3.7, we observe that each D_{p^i} corresponding to V_{p^i} where $1 \leq i \leq r$ and each D_{q^j} corresponding to V_{q^j} where $1 \leq j \leq s$ are edgeless graphs. Therefore, the vertices from V_{p^i} or V_{q^j} form the independent set of $\Gamma_{ex}(\mathbb{Z}_n)$. Thus, the independent number is, $\max\{(|D_p| + |D_{p^2}| + \cdots + |D_{p^k}| + \cdots + |D_{p^r}|), (|D_q| + |D_{q^2}| + \cdots + |D_{q^l}| + \cdots + |D_{q^s}|)\}$. On evaluating,

$$\begin{aligned}|D_p| + |D_{p^2}| + \cdots + |D_{p^k}| + \cdots + |D_{p^r}| =& \phi(p^{r-1}q^s) + \phi(p^{r-2}q^s) + \cdots + \\ & \phi(p^{r-k}q^s) + \cdots + \phi(pq^s) + \phi(q^s) \\ =& \phi(q^s)p^{r-1}.\end{aligned}$$

Similarly,

$$\begin{aligned}|D_q| + |D_{q^2}| + \cdots + |D_{q^l}| + \cdots + |D_{q^s}| =& \phi(p^rq^{s-1}) + \phi(p^rq^{s-2}) + \cdots + \\ & \phi(p^rq^{s-l}) + \cdots + \phi(p^rq) + \phi(p^r). \\ =& \phi(p^r)\, q^{s-1}.\end{aligned}$$

Since $\phi(q^s)p^{r-1} > \phi(p^r)q^{s-1}$. Hence, the independent number is $\phi(q^s)\, p^{r-1}$. □

If $s = 1$, the independent number of p^rq is $\phi(q)p^{r-1}$.

Discussion and Conclusion

In this study, the L-eigenvalues of the extended ZDG $\Gamma_{ex}(\mathbb{Z}_n)$, where $n = p^rq, pq^s$ and p^rq^s are computed and compared with those of the classical ZDG $\Gamma(\mathbb{Z}_n)$. Notably, we observed that the spectral radius of $\Gamma_{ex}(\mathbb{Z}_n)$ is consistently larger than that of the classical ZDG. In Section 3, we demonstrate that if $n = p^r, r \geq 3$, $\Gamma_{ex}(\mathbb{Z}_n)$ is a complete graph, whereas $\Gamma(\mathbb{Z}_n)$ is a connected but not a complete graph. In addition, the algebraic connectivity, which is the second smallest L-eigenvalue of $\Gamma(\mathbb{Z}_n)$ for $n = p^rq$, is more than the algebraic connectivity of $\Gamma(\mathbb{Z}_n)$. Spectral comparison reveals that an increase in the edge density in the extended ZDG leads to higher Laplacian eigenvalues, signifying an increase in algebraic connectivity and overall graph energy.

In Section 4, we observed that the domination number and independent number of $\Gamma_{ex}(\mathbb{Z}_n)$ is less than that of $\Gamma(\mathbb{Z}_n)$, for $n = p^r, p^rq^s$. Also, Clique number of $\Gamma_{ex}(\mathbb{Z}_n)$ where $n = p^rq, p^rq^s$ is greater then the clique number of the $\Gamma_{ex}(\mathbb{Z}_n)$ [21]. Clearly, an increase in the edge density in $\Gamma_{ex}(\mathbb{Z}_n)$ decreases the domination and independence number while increase the clique number of $\Gamma_{ex}(\mathbb{Z}_n)$, relative to the classical $\Gamma(\mathbb{Z}_n)$. These findings emphasise that the extended ZDG provides a more intricate spectral and topological representation of the ring $\mathbb{Z}_n$, offering deeper insights into the relationship between its algebraic and graph-theoretic

properties. Furthermore, the L-eigenvalues are utilised to find the Laplacian energy of graphs. In [12], the Laplacian energy of H is defined as,

$$LE(H) = \sum_{i=1}^{k} \left| \mu_i - \frac{2|E(H)|}{n} \right|. \tag{4.1}$$

In this context, $\sum_{i=1}^{k} \mu_i$ represents the aggregate of the kth L-eigenvalues, while $|E(H)|$ denotes the total count of edges in the graph H. According to [18], the Laplacian energy is crucial in numerous chemical applications, extending its significance beyond the molecular orbital theory associated with conjugated molecules. Thus is case of $n = p^r$, and $p = pq^s$, one can find the $\Gamma_{ex}(\mathbb{Z}_n)$, using equation (4.1). For further study, one can evaluate the L-eigenvalues of extended ZDG for commutative rings like $\mathbb{Z}_{p_1^r p_2 p_3}$ or $\mathbb{Z}_{p_1^r p_2^s p_3}$ or $\mathbb{Z}_{p^r} \times \mathbb{Z}_q$ or $\mathbb{Z}_p \times \mathbb{Z}_{q^s}$, etc, where p_1, p_2, p_3, q are distinct prime numbers.

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