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RATIONAL FUNCTION-BASED SHAPE CONTROL FOR ARBITRARY TOPOLOGY SURFACE SUBDIVISION

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Abstract. This paper proposes a new non-stationary surface subdivision scheme driven by a rational control function. In regular regions, the scheme is defined by a tensor-product extension of a five-point univariate rule; around extraordinary vertices, we specify local update rules with a block-circulant structure. We prove C^2 smoothness in regular regions via first-order asymptotic equivalence with with the well-known Catmull–Clark scheme and establish tangent-plane continuity at extraordinary points by bounding the deviation of local blocks and invoking the spectral gap of the stationary limit. A parametric study shows how (a, b, μ) influences curvature and tension, yielding practical shape control without increasing computational complexity. We illustrate the method on standard test meshes and report indicative comparisons surface for representative parameter choices.

Keywords. Non-stationary subdivision, Geometric modeling, Arbitrary topology, Shape control, Convergence analysis, Surface continuity.
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1. Introduction

Subdivision schemes provide a practical and visually pleasing means to transform rough polygonal meshes into smooth curves and surfaces. Through successive refinement steps, these methods can produce highly detailed geometric forms starting from simple control meshes. Because of this capability, they have become indispensable tools in areas such as computer-aided geometric design (CAGD), computer graphics, and industrial surface modeling. Since the pioneering works of Catmull and Clark [3] and Doo and Sabin [6], subdivision theory has matured into a broad mathematical framework that draws on concepts from approximation theory, numerical computation, and differential geometry. A subdivision scheme can be thought of as a discrete dynamical system that advances a control mesh through discrete local averaging techniques. Stationary methods are those that use a fixed subdivision mask at all subdivision levels, making it simpler to predict the smoothness properties. The uniform cubic B-spline and Catmull-Clark schemes are classic examples. They ensure C^2 continuity in regular regions and

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G^1 continuity at exceptional vertices [17]. However, the inability to adjust shape easily when applying a fixed mask is a limitation.

To overcome this issue, non-stationary schemes, which have refinement rules that change according to the subdivision level, have been developed to enable the reproduction of more complicated functional spaces including polynomial, exponential, trigonometric, and hyperbolic bases [2, 9, 12, 13, 14, 15, 16, 19]. Enhancements in the flexibility of local curvature control and shape modifications using these adaptive non-stationary techniques. Theoretical work has been done on 'Non-stationary schemes' as well. Dyn and Levin [7, 8] demonstrated that one can determine convergence and smoothness of a non-stationary scheme because of its closeness to a stationary scheme through the concept of asymptotic equivalence, thus equating control and smoothness. After that, Charina et al. [4] and Conti et al [5] expanded this approach using matrix techniques to demonstrate that the regularity of limit surfaces is governed by the spectral properties of local subdivision matrices. These works specified a certain set of criteria under which a non-stationary scheme can have (C^m) or (G^1) regularity, even in the presence of exceptional points. These theoretical contributions inspired numerous authors to design practical non-stationary subdivision rules that improve shape control. Beccari et al. [2] proposed a four-point interpolatory method that makes envelopes of conic sections using a tension parameter. Sabin and Dodgson [18] provided a version of the four-point method that preserves circles.

Barrera et al. [1] and Fakhar et al. [10, 11] developed hybrid trigonometric-hyperbolic and tension-controlled subdivision schemes applicable to meshes of any geometric form. Zhang et al. [19] proposed the first generalized exponential B-spline technique where the user has the ability to define the shape of a mesh using several adjustable parameters. Despite these advances, most current models still rely on heuristic adjustments of the form parameters. This indicates a lack of basic mathematical understanding of how the shape parameters correlate to the curvature of the generated shape. Motivated by these observations, this work introduces a rational control function, $\mathcal{H}(x) = \frac{x^a}{x^a+x^b}$, to define a new non-stationary subdivision scheme for arbitrary topology meshes, where the parameters $((a, b, \mu))$ allow intuitive control of the surface local curvature and tension. The proposed approach extends the univariate rational subdivision technique to two variables, maintaining C^2 continuity in regular parts and ensuring G^1 continuity around irregular vertices. The proposed design combines concepts of asymptotic equivalence [7, 5] with recent work in mixed-function subdivision geometry [1, 10] to provide a clear mathematical relationship between the parameters and the geometry of the surface. The remainder of this paper is organized in the following manner.

In Section 2, the focus is on the theoretical groundwork for understanding the smoothing and convergence aspects. Section 3 analyzes the working principles behind the univariate rational subdivision rule. Then, in Section 4, the discussion centers on the bivariate extension for meshes of any topology and includes techniques for refining extraordinary points. Section 5 provides proofs concerning convergence and smoothness, whereas Section 6 presents the numerical findings and analyzes the sensitivity of the parameters. Finally, Section 7 concludes the paper.

2. Preliminaries

This part sets up the strict theoretical basis for looking at our non-stationary subdivision scheme. We provide essential definitions, convergence theorems, and mathematical instruments that constitute the groundwork for demonstrating regularity features in the ensuing sections.

Definition 1 ([3, 17])**.** *A stationary subdivision scheme is characterized by a linear operator $S : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$ that operates on sequences of control points through a fixed mask $a = \{a_\alpha\}_{\alpha\in\mathbb{Z}^s}$. For an initial sequence $p^0 = \{p^0_\alpha\}$, the subdivided sequence is defined as follows:*

$$p^{k+1}_\alpha = (Sp^k)_\alpha = \sum_{\beta\in\mathbb{Z}^s} a_{\alpha-2\beta} p^k_\beta$$

Definition 2 ([5, 4])**.** *A non-stationary subdivision scheme consists of a succession of operators $\{S_{a^j}\}_{j\geq 0}$, with each operator S_{a^j} characterized by a mask $a^j = \{a^j_\alpha\}_{\alpha\in\mathbb{Z}^s}$ that varies according to the refinement level j. The subdivision rule is:*

$$p^{j+1}_\alpha = (S_{a^j} p^j)_\alpha = \sum_{\beta\in\mathbb{Z}^s} a^j_{\alpha-2\beta} p^j_\beta$$

Definition 3 ([7])**.** *The symbol of a mask We may define $a = \{a_\alpha\}$ using the following equation:*

$$a(z) = \sum_{\alpha\in\mathbb{Z}^s} a_\alpha z^\alpha$$

where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$ for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$.

Definition 4 ([5])**.** *The factor $(1 + z_1)(1 + z_2)$ is in the symbol $a(z)$. If:*

$$a(z) = (1 + z_1)(1 + z_2)b(z)$$

for a certain symbol $b(z)$. This factor is necessary for the C^1 convergence of surface schemes.

Definition 5 ([17])**.** *A subdivision scheme is considered C^m-convergent if, given any bounded initial data p^0, the series of piecewise linear functions f^j constructed on the mesh $\mathcal{M}^j$ uniformly converges to a limit function f^∞ of class C^m, with all derivatives up to order m also converging.*

Definition 6 ([5])**.** *For a bivariate subdivision scheme with mask c^j, the uniform norm is defined by:*

$$\|S_{c^j}\|_\infty := \max\left\{ \sum_{\theta\in\mathbb{Z}^2} \left|c^j_{\tau-2\theta}\right| : \tau \in \{(0,0),(0,1),(1,0),(1,1)\} \right\}$$

Definition 7 ([7]). *Two non-stationary schemes, $\{S_{a^j}\}_{j\geq 0}$ and $\{S_{b^j}\}_{j\geq 0}$, Asymptotic equivalence occurs when:*

$$\sum_{j\geq 0} \|S_{a^j} - S_{b^j}\|_\infty < \infty$$

Definition 8 ([5]). *The schemes are first-order asymptotically identical if:*

$$\sum_{j\geq 0} 2^j \|S_{a^j} - S_{b^j}\|_\infty < \infty$$

This stronger condition is needed to keep C^1 regularity.

Theorem 1 ([5]). *Let $S = \{S_{c^j}\}_{j\geq 0}$ be a non-stationary subdivision scheme and $\tilde{S} = \{\tilde{S}\}$ be a stationary reference scheme. Let $\mathcal{M}^{(0)}$ be an initial mesh of arbitrary topology, $\mathcal{R}^{(0)}$ be the neighborhood of a regular face, and $\mathcal{E}^{(0)}$ be the neighborhood of an extraordinary face. Let's say that:*

(i) $\tilde{S}$ converges to C^1 in $\mathcal{R}^{(0)}$ with the factor $(1+z_1)(1+z_2)$ in the symbol $c(z)$, and G^1-convergent in $\mathcal{E}^{(0)}$

(ii) In $\mathcal{R}^{(0)}$, S is defined. by the letters $c^j(z)$, $j \geq 1$, where each $c^j(z)$ has the factor $(1+z_1)(1+z_2)$

(iii) In $\mathcal{R}^{(0)}$, S is first-order asymptotically identical to $\tilde{S}$.

(iv) In $\mathcal{E}^{(0)}$, the matrices S_j and S (obtained by block-circulant transformation of the local matrices) satisfy:

$$\|S_j - S\|_\infty \leq \frac{C}{\sigma^j}, \quad \forall j \geq 1$$

with $C > 0$ a finite constant and $\sigma > \frac{1}{\lambda_1} > 1$, where $\lambda_1 \in \mathbb{R}^+$ is the subdominant eigenvalue of S, double and non-defective.

Then, for any bounded initial data, S converges in $\mathcal{E}^{(0)}$ and generates a tangent plane continuous surface at the limit points of extraordinary faces.

3. Univariate Rational Subdivision Scheme

This section introduces the univariate rational subdivision method utilizing the control function $\mathcal{H}(x) = \frac{x^a}{x^a+x^b}$. We formulate the subdivision rules, examine convergence and regularity characteristics, and create theoretical underpinnings for the bivariate extension in the following sections.

3.1. **Subdivision Rules and Control Function.**

Definition 9. *Let $P^0 = \{P_i^0\}_{i=0}^m \subset \mathbb{R}^d$ be an initial set of control points. The rational non-stationary subdivision scheme $\{S_{r_j}\}_{j\geq 0}$ generates recursively new control points $P^j = \{P_i^j\}_{i=0}^{2^j(m-1)+1}$ at level $j \in \mathbb{N}$ through the subdivision rules:*

$$\begin{cases} P_{2i+1}^{j+1} = \dfrac{1}{2}(P_i^j + P_{i+1}^j), \\ P_{2i}^{j+1} = \dfrac{\mathcal{H}(\mu^{j+1})}{4}(P_{i-1}^j + P_{i+1}^j) + \left(1 - \dfrac{\mathcal{H}(\mu^{j+1})}{2}\right) P_i^j, \end{cases} \tag{1}$$

where the rational control function $\mathcal{H}(x)$ is defined as:

$$\mathcal{H}(x) = \frac{x^a}{x^a + x^b}, \quad a, b \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad x \in \Omega_{a,b} = \{x : x^a + x^b \neq 0\}. \tag{2}$$

Definition 10. *The tension parameter is defined through the sequence:*

$$\mu^j = \frac{1}{2}(e^{t_j} + e^{-t_j}) = \cosh(t_j), \quad t_j = \frac{t}{2^j}, \tag{3}$$

where t is a non-negative real or pure imaginary constant. The sequence μ^j satisfies the recurrence relation:

$$\mu^{j+1} = \sqrt{\frac{1 + \mu^j}{2}}, \quad \textit{with } \mu^0 \in (-1, \infty). \tag{4}$$

Proposition 1. *The sequence $\{\mu^j\}_{j\geq 0}$ defined in (4) satisfies:*

$$\lim_{j\to\infty} \mu^j = 1. \tag{5}$$

Proof. The recurrence relation (4) represents the half-angle formula for hyperbolic cosine. For any initial value $\mu^0 \in (-1, \infty)$, the sequence converges to the fixed point of the equation $x = \sqrt{(1 + x)/2}$, which is $x = 1$. □

Theorem 2. *The rational control function $\mathcal{H}(x) = \frac{x^a}{x^a + x^b}$ possesses the following properties:*

(1) ***Smoothness****: $\mathcal{H} \in C^\infty(\mathbb{R}_*^+)$*
(2) ***Symmetry****: $\mathcal{H}_{a,b}(x) = 1 - \mathcal{H}_{b,a}(x)$*
(3) ***Normalization****: $\mathcal{H}(1) = \frac{1}{2}$ for all $a, b \in \mathbb{N}_0$*
(4) ***Asymptotic behavior****:*
   - *If $a > b$: $\lim_{x\to 0^+} \mathcal{H}(x) = 0$, $\lim_{x\to\infty} \mathcal{H}(x) = 1$*
   - *If $a < b$: $\lim_{x\to 0^+} \mathcal{H}(x) = 1$, $\lim_{x\to\infty} \mathcal{H}(x) = 0$*
   - *If $a = b$: $\mathcal{H}(x) = \frac{1}{2}$*
(5) ***Derivatives****:*

$$\mathcal{H}'(x) = \frac{(b - a)x^{a+b-1}}{(x^a + x^b)^2} \tag{6}$$

$$\mathcal{H}''(x) = \frac{(b - a)x^{a+b-2}[(a + b - 1)(x^a + x^b) - 2(ax^a + bx^b)]}{(x^a + x^b)^3} \tag{7}$$

Proof. Properties 1-4 follow directly from the definition and algebraic manipulation. For property 5, we apply the quotient rule:

$$\mathcal{H}'(x) = \frac{ax^{a-1}(x^a + x^b) - x^a(ax^{a-1} + bx^{b-1})}{(x^a + x^b)^2} = \frac{(b-a)x^{a+b-1}}{(x^a + x^b)^2}.$$

The second derivative is obtained by applying the quotient rule again to $\mathcal{H}'(x)$. □

Corollary 1. *For the sequence $\{\mu^j\}_{j\geq 0}$ defined in (4), we have:*

$$\lim_{j\to\infty} \mathcal{H}(\mu^{j+1}) = \mathcal{H}(1) = \frac{1}{2}, \quad \lim_{j\to\infty} (\mathcal{H}(\mu^{j+1}))^2 = \frac{1}{4}. \tag{8}$$

Proof. This follows directly from Proposition 3.1 and the continuity of $\mathcal{H}(x)$ and $x \mapsto x^2$ at $x = 1$. □

3.2. Mask and Symbol Analysis.

Definition 11. *The mask $\{c^j\}_{j\geq 0}$ corresponding to the univariate rational scheme is given by:*

$$c^j = \left(\frac{\mathcal{H}(\mu^{j+1})}{4}, \frac{1}{2}, 1 - \frac{\mathcal{H}(\mu^{j+1})}{2}, \frac{1}{2}, \frac{\mathcal{H}(\mu^{j+1})}{4} \right). \tag{9}$$

Definition 12. *The j-level symbol of the univariate rational scheme is:*

$$a^j(z) = \frac{\mathcal{H}(\mu^{j+1})}{4} z^{-2} + \frac{1}{2} z^{-1} + \left(1 - \frac{\mathcal{H}(\mu^{j+1})}{2}\right) + \frac{1}{2} z + \frac{\mathcal{H}(\mu^{j+1})}{4} z^2. \tag{10}$$

Lemma 1. *The symbol $a^j(z)$ can be factorized as:*

$$a^j(z) = \left(\frac{1+z}{2}\right)^2 \left[\mathcal{H}(\mu^{j+1}) + (2 - 2\mathcal{H}(\mu^{j+1}))z + \mathcal{H}(\mu^{j+1})z^2\right]. \tag{11}$$

Proof. Expanding the right-hand side:

$$\begin{aligned}
&\left(\frac{1+z}{2}\right)^2 \left[\mathcal{H}(\mu^{j+1}) + (2 - 2\mathcal{H}(\mu^{j+1}))z + \mathcal{H}(\mu^{j+1})z^2\right] \\
&= \frac{1 + 2z + z^2}{4} \cdot \left[\mathcal{H}(\mu^{j+1}) + (2 - 2\mathcal{H}(\mu^{j+1}))z + \mathcal{H}(\mu^{j+1})z^2\right] \\
&= \frac{\mathcal{H}(\mu^{j+1})}{4} + \frac{2 - 2\mathcal{H}(\mu^{j+1})}{4} z + \frac{\mathcal{H}(\mu^{j+1})}{4} z^2 \\
&\quad + \frac{2\mathcal{H}(\mu^{j+1})}{4} z + \frac{4 - 4\mathcal{H}(\mu^{j+1})}{4} z^2 + \frac{2\mathcal{H}(\mu^{j+1})}{4} z^3 \\
&\quad + \frac{\mathcal{H}(\mu^{j+1})}{4} z^2 + \frac{2 - 2\mathcal{H}(\mu^{j+1})}{4} z^3 + \frac{\mathcal{H}(\mu^{j+1})}{4} z^4.
\end{aligned}$$

Now collecting terms for each power of z we obtain the result:

$$a^j(z) = \frac{\mathcal{H}(\mu^{j+1})}{4} + \frac{1}{2} z + z^2 + \frac{1}{2} z^3 + \frac{\mathcal{H}(\mu^{j+1})}{4} z^4$$

. □

3.3. **Convergence and Regularity Analysis.**

Theorem 3. *For $a, b \in \mathbb{N}_0$, the univariate rational scheme $\{S_{c^j}\}_{j\geq 0}$ is C^2-convergent.*

Proof. From Lemma 1 The symbol $a^j(z)$ can be factorized as follow:

$$a^j(z) = \left(\frac{1+z}{2}\right) d^j(z). \tag{12}$$

Where

$$d^j(z) = \frac{\mathcal{H}(\mu^{j+1})}{2} + (1 - \frac{\mathcal{H}(\mu^{j+1})}{2})z + (1 - \frac{\mathcal{H}(\mu^{j+1})}{2})z^2 + \frac{\mathcal{H}(\mu^{j+1})}{2}z^3.$$

Now to prove the theorem it is sufficient to show that S_{d^j} corresponding to $d^j(z)$ is $\mathcal{C}^1$.

The Chaikin scheme has the stationary mask:

$$d = \left(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right)$$

and is known to be C^1-convergent [8].

From Corollary 1, we have:

$$\lim_{j\to\infty} \mathcal{H}(\mu^{j+1}) = \frac{1}{2}.$$

Moreover, by Taylor expansion around $x = 1$:

$$\mathcal{H}(x) = \frac{1}{2} + \frac{b-a}{4}(x-1) + \mathcal{O}((x-1)^2).$$

Since $|\mu^{j+1} - 1| \leq \frac{C}{2^{2j}}$ from the recurrence relation (4), we obtain:

$$|\mathcal{H}(\mu^{j+1}) - \tfrac{1}{2}| \leq \frac{K_1}{2^{2j}}, \quad |(\mathcal{H}(\mu^{j+1}))^2 - \tfrac{1}{4}| \leq \frac{K_2}{2^{2j}}.$$

The difference between the non-stationary mask c^j and the stationary mask c is:

$$d^j - d = \left(\frac{\mathcal{H}(\mu^{j+1}) - \frac{1}{2}}{2}, -\frac{\mathcal{H}(\mu^{j+1}) - \frac{1}{2}}{2}, -\frac{\mathcal{H}(\mu^{j+1}) - \frac{1}{2}}{2}, \frac{\mathcal{H}(\mu^{j+1}) - \frac{1}{2}}{2}\right).$$

Therefore:

$$\|S_{d^j} - S_d\|_\infty = \max_\tau \sum_{\theta\in\mathbb{Z}} |d^j_{\tau-2\theta} - d_{\tau-2\theta}| \leq \frac{M}{2^{2j}}.$$

We have:

$$\sum_{j\geq 0} 2^j \|S_{c^j} - S_c\|_\infty \leq M \sum_{j\geq 0} \frac{1}{2^j} < \infty,$$

which establishes first-order asymptotic equivalence.

hence, the rational scheme is C^2. □

Remark 1. *The parameters a and b do not impact the convergence order but modify the form of the limit curve. Various selections of a and b produce distinct families of curves while preserving C^2 regularity.*

Proposition 2. *The parameters a and b provide geometric control over the limit curve:*

- *$a > b$: The scheme system prioritizes elevated values of μ^j, resulting in more constricted curves.*
- *$a < b$: The scheme prioritizes reduced values of μ^j, resulting in more relaxed curves.*
- *$a = b$: The scheme reduces to a stationary subdivision scheme*
- *The ratio a/b controls the sensitivity to the tension parameter t*

Algorithm 1: Univariate Rational Subdivision Process

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1 **Input**: Initial control polygon $P^0 = \{P_i^0\}_{i=0}^m$, parameters a, b, t, iterations N

2 **Output**: Refined control polygon P^N

(1) Initialize $\mu^0 = \cosh(t)$ using (3)

(2) **For** $j = 0$ to $N - 1$:

  (a) Update μ^{j+1} using recurrence (4)

  (b) Compute $\mathcal{H}(\mu^{j+1})$ using control function (2)

  (c) Apply subdivision rules (1) to generate P^{j+1}

(3) **Return** P^N

One can use Algorithm 1 to generate flexible limit curves from the same initial control points for différent values of a, b and μ^{j+1}.

4. Bivariate Rational Subdivision Scheme for Arbitrary Topological Meshes

This section generalizes the univariate rational approach to bivariate surfaces of any topology. We outline the creation of the subdivision mask in regular regions by tensor products, formulate the subdivision rules for extraordinary points, and deliver the comprehensive geometric refinement rules for surface production. The proposed extension maintains C^2-continuity in regular parts of the mesh and guarantees G^1-smoothness at extraordinary vertices through carefully adjusted refinement rules.

4.1. Examination of the Catmull-Clark Subdivision Scheme.

We begin by revisiting the classical Catmull–Clark algorithm, which serves as the foundation for our rational formulation. This approach is noteworthy because it supports meshes of arbitrary topology while still ensuring an excellent level of surface smoothness.

Definition 13 ([3])**.** *In regular regions, the subdivision mask associated with the Catmull–Clark procedure can be expressed through a 5×5 matrix.*

$$c = \begin{pmatrix} \frac{1}{64} & \frac{1}{16} & \frac{3}{32} & \frac{1}{16} & \frac{1}{64} \\ \frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} \\ \frac{3}{32} & \frac{3}{8} & \frac{9}{16} & \frac{3}{8} & \frac{3}{32} \\ \frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} \\ \frac{1}{64} & \frac{1}{16} & \frac{3}{32} & \frac{1}{16} & \frac{1}{64} \end{pmatrix} \tag{13}$$

This mask delineates the weights employed to calculate new vertex positions during subdivision.

Definition 14 ([3]). *The corresponding subdivision symbol of the Catmull-Clark scheme is:*

$$c(z_1, z_2) = \frac{(z_1+1)^4(z_2+1)^4}{64} \tag{14}$$

This matrix representation, or symbol, provides an algebraic way to study the refinement process and plays a central role in evaluating both convergence and smoothness.

The Catmull–Clark model yields C^2-continuous surfaces in uniform areas and G^1-continuous transitions near irregular vertices, making it an ideal benchmark for our rational extension.

4.2. **Bivariate Rational Mask in Regular Domains.** Our method constructs a bivariate rational mask by extending the univariate case through the use of tensor-product operations. This ensures consistency with the one-dimensional formulation while enabling surface refinement in two dimensions.

Definition 15. *Applying the tensor-product operator to the univariate rational scheme $S_{r^j}{}_{j\geq 0}$ gives rise to the bivariate rational mask for regular regions.:*

$$r^j = \begin{pmatrix} \vec{a}_4^j & \vec{b}_4^j & \vec{c}_4^j & \vec{b}_4^j & \vec{a}_4^j \\ \vec{b}_4^j & \vec{d}^j & \vec{e}^j & \vec{d}^j & \vec{b}_4^j \\ \vec{c}_4^j & \vec{e}^j & \vec{f}^j & \vec{e}^j & \vec{c}_4^j \\ \vec{b}_4^j & \vec{d}^j & \vec{e}^j & \vec{d}^j & \vec{b}_4^j \\ \vec{a}_4^j & \vec{b}_4^j & \vec{c}_4^j & \vec{b}_4^j & \vec{a}_4^j \end{pmatrix}, \quad j \geq 0 \tag{15}$$

The coefficients evolve with the subdivision level j according to the rational control function $\mathcal{H}(\mu^{j+1})$, which provides adaptive shape regulation.

Definition 16. *The j-level symbol of the bivariate rational scheme is expressed as follows:*

$$\begin{aligned} r^j(z_1, z_2) = \frac{(z_1+1)^2(z_2+1)^2}{4} \times \\ \left(\frac{\mathcal{H}(\mu^{j+1})}{2} + (1-\mathcal{H}(\mu^{j+1}))z_1 + \frac{\mathcal{H}(\mu^{j+1})}{2} z_1^2\right) \times \\ \left(\frac{\mathcal{H}(\mu^{j+1})}{2} + (1-\mathcal{H}(\mu^{j+1}))z_2 + \frac{\mathcal{H}(\mu^{j+1})}{2} z_2^2\right) \end{aligned} \tag{16}$$

This symbol compactly represents the algebraic structure of the refinement process and facilitates the theoretical study of its smoothness behavior.

Proposition 3 (Factorization property). *The expression $r^j(z_1, z_2)$ contains the factor $(1+z_1)^2(1+z_2)^2$, which ensures that the resulting surfaces meet the conditions for C^1-convergence in standard regions.*

Proof. Equation (16) explicitly exhibits this factorization. The occurrence of $(1+z_1)^2(1+z_2)^2$ satisfies the standard criterion for C^1-continuity [6]. This component corresponds to the linear precision property required for maintaining surface regularity. □

Because of the tensor-product design, all vertices of valence 4 (regular configuration) preserve the C^2-continuity inherited from the univariate scheme while allowing the rational control term to influence the local surface shape.

4.3. **Rational Rules around Irregular Vertices.** For meshes of arbitrary topology, vertices whose valence differs from 4 require particular attention. Specific local rules are introduced to maintain surface smoothness and geometric quality.

Definition 17 (Rational local subdivision matrix)**.** *For a vertex of valence n, the local subdivision matrix at level j of the rational scheme is defined accordingly:*

$$\tilde{S}^j = \begin{pmatrix} \tilde{\alpha}^j & (\tilde{\beta}^j)^T & (\tilde{\beta}^j)^T & \cdots & (\tilde{\beta}^j)^T \\ \tilde{\gamma}^j & \tilde{B}_0^j & \tilde{B}_1^j & \cdots & \tilde{B}_{n-1}^j \\ \tilde{\gamma}^j & \tilde{B}_{n-1}^j & \tilde{B}_0^j & \cdots & \tilde{B}_{n-2}^j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{\gamma}^j & \tilde{B}_1^j & \cdots & \tilde{B}_{n-1}^j & \tilde{B}_0^j \end{pmatrix} \tag{17}$$

This block-circulant form captures the rotational symmetry surrounding extraordinary points while embedding the rational control parameters.

Definition 18. *The parameters for exceptional points in the rational framework are as follows:*

$$\tilde{\alpha}^j = 1 - \frac{4\mathcal{H}(\mu^{j+1}) - (\mathcal{H}(\mu^{j+1}))^2}{n}, \tag{18}$$

$$\tilde{\beta}^j = \left(\frac{4\mathcal{H}(\mu^{j+1}) - 2(\mathcal{H}(\mu^{j+1}))^2}{n^2}, \frac{(\mathcal{H}(\mu^{j+1}))^2}{n^2}, 0, 0, 0, 0\right)^T, \tag{19}$$

$$\tilde{\gamma}^j = \left(\vec{e}^j, \vec{d}^j, \vec{c}_4^j, \vec{b}_4^j, \vec{a}_4^j, \vec{b}_4^j\right)^T. \tag{20}$$

These parameters guarantee that the scheme adjusts to varying vertex valences while preserving the division of unity attribute crucial for convergence.

The extraordinary point rules are formulated to integrate seamlessly with the standard rules, guaranteeing that the transition between normal and unusual areas does not produce artifacts or discontinuities in the limit surface.

4.4. **Geometric Refinement Rules.** We offer geometric refinement rules that directly calculate new vertex positions based on local mesh connection for practical implementation.

The j-th level geometric refinement rules of the rational scheme $\{S_{r^j}\}_{j\geq 0}$ are expressed as follows:

$$V^{\text{new}} = \beta_n^j V^{\text{old}} + \gamma_n^j \sum_{d=1}^{n} V_E^d + \delta_n^j \sum_{d=1}^{n} V_F^d, \quad j \geq 0 \tag{21}$$

where:

- V^{old} Vertex prior to modification
- V^{new} Vertex subsequent to modification: V_E^d Vertices associated with V^{old} Vertices that are contiguous by an edge
- V_F^d Vertices associated with V^{old} Vertices next to a face

These criteria offer a clear geometric interpretation of the subdivision process (See Fig.1).

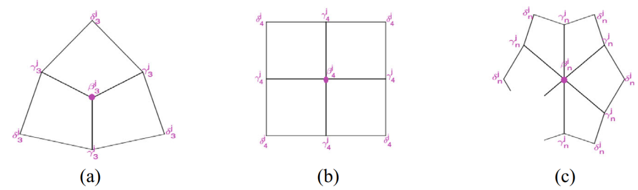


Figure 1. Graphical representation of the jth level geometric refinement rules for rational subdivision surfaces. The magenta point denotes the newly added point according to the geometric refinement guidelines for regular (b) and extraordinary points (a) and (c).

The geometric coefficients in the refinement rules are defined as follows:

$$\gamma_n^j = \frac{4\mathcal{H}(\mu^{j+1}) - 2(\mathcal{H}(\mu^{j+1}))^2}{n^2}, \tag{22}$$

$$\delta_n^j = \frac{(\mathcal{H}(\mu^{j+1}))^2}{n^2}, \tag{23}$$

$$\beta_n^j = 1 - n\gamma_n^j - n\delta_n^j. \tag{24}$$

The geometric coefficients adhere to the division of unity property.

$$\beta_n^j + n\gamma_n^j + n\delta_n^j = 1, \quad \forall j \geq 0. \tag{25}$$

Since $\lim_{j\to\infty} \mathcal{H}(\mu^{j+1}) = \frac{1}{2}$, the geometric coefficients converge to their Catmull-Clark equivalents:

$$\lim_{j\to\infty} \gamma_n^j = \frac{3}{2n^2}, \quad \lim_{j\to\infty} \delta_n^j = \frac{1}{4n^2}, \quad \lim_{j\to\infty} \beta_n^j = 1 - \frac{7}{4n}.$$

Algorithm 2: Bivariate Rational Subdivision

Input: Initial mesh $\mathcal{M}^0$, parameters a, b, t, and the number of iterations N

Output: Enhanced mesh $\mathcal{M}^N$

(1) Set $\mu^0 = \cosh(t)$.
(2) For $j = 0$ to $N-1$:
  (a) Calculate $\mu^{j+1} = \sqrt{\frac{1+\mu^j}{2}}$
  (b) Compute $\mathcal{H}^j = \mathcal{H}(\mu^{j+1}) = \frac{(\mu^{j+1})^a}{(\mu^{j+1})^a + (\mu^{j+1})^b}$.
  (c) Calculate mask coefficients.
  (d) For each standard face:
    - Implement tensor product mask (15)
  (e) for each exceptional vertex of valence n:
    - Calculate geometric coefficients utilizing (24)
    - Implement geometric refinement rule (21)
  (f) Revise mesh to $\mathcal{M}^{j+1}$
(3) Return $\mathcal{M}^N$

Remark 2. *When $a = b$, the rational process becomes stationary and coincides with the classical Catmull–Clark formulation.*

The parameters a and b control the overall surface shape in both regular and irregular regions:

- $a > b$: produces surfaces with stronger curvature and a closer adherence to the control mesh;
- $a < b$: yields gentler curvature and smoother global appearance

5. Analysis of Convergence and Smoothness

This section investigates the convergence and smoothness behavior of the proposed bivariate rational subdivision. We show that the scheme attains C^2-continuity in regular portions of the mesh and G^1-smoothness at extraordinary vertices, based on the theoretical results outlined earlier. The analysis is divided into two parts: smoothness over regular regions and convergence near irregular ones.

5.1. **Smoothness in Regular Regions.** We first examine regions composed exclusively of quadrilateral faces where each vertex has valence 4. In these zones, the refinement process can be expressed as a tensor product of the univariate rational rule, which greatly simplifies the smoothness study.

Theorem 4. *The sequence $S_{r^j\, j\geq 0}$ converges to a C^2-smooth limit surface in every regular part of the mesh.*

Proof. We establish this result by demonstrating first-order asymptotic equivalence with the Catmull–Clark algorithm.

According to the limit behavior demonstrated in Section 3, we obtain:

$$\lim_{j\to\infty} \mathcal{H}(\mu^{j+1}) = \frac{1}{2}, \quad \lim_{j\to\infty} (\mathcal{H}(\mu^{j+1}))^2 = \frac{1}{4}.$$

By employing Taylor expansion centered at $x = 1$ and the recurrence relation $|\mu^{j+1} - 1| \leq \frac{C}{2^{2j}}$, we derive the estimates for exponential decay:

$$|\mathcal{H}(\mu^{j+1}) - \tfrac{1}{2}| \leq \frac{K_1}{2^{2j}}, \quad |(\mathcal{H}(\mu^{j+1}))^2 - \tfrac{1}{4}| \leq \frac{K_2}{2^{2j}}.$$

We assess the discrepancies between rational mask coefficients and Catmull-Clark coefficients:

$$\begin{aligned}
|\vec{a}_4^j - \tfrac{1}{64}| &= \tfrac{1}{16}|(\mathcal{H}(\mu^{j+1}))^2 - \tfrac{1}{4}| \leq \tfrac{\mathcal{A}}{2^{2j}} \\
|\vec{b}_4^j - \tfrac{1}{16}| &= \tfrac{1}{8}|\mathcal{H}(\mu^{j+1}) - \tfrac{1}{2}| \leq \tfrac{\mathcal{B}}{2^{2j}} \\
|\vec{c}_4^j - \tfrac{3}{32}| &\leq \tfrac{1}{8}|(\mathcal{H}(\mu^{j+1}))^2 - \tfrac{1}{4}| + \tfrac{1}{4}|\mathcal{H}(\mu^{j+1}) - \tfrac{1}{2}| \leq \tfrac{\mathcal{C}}{2^{2j}} \\
|\vec{d}^j - \tfrac{1}{4}| &= 0 \\
|\vec{e}^j - \tfrac{3}{8}| &= \tfrac{1}{2}|\mathcal{H}(\mu^{j+1}) - \tfrac{1}{2}| \leq \tfrac{\mathcal{E}}{2^{2j}} \\
|\vec{f}^j - \tfrac{9}{16}| &\leq \tfrac{1}{8}|(\mathcal{H}(\mu^{j+1}))^2 - \tfrac{1}{4}| + |\mathcal{H}(\mu^{j+1}) - \tfrac{1}{2}| \leq \tfrac{\mathcal{F}}{2^{2j}}
\end{aligned}$$

The uniform norm of the discrepancy between subdivision operators is given by:

$$\begin{aligned}
\|S_{r^j} - S_c\|_\infty = \max\Big\{ & 4|\vec{a}_4^j - \tfrac{1}{64}| + 4|\vec{c}_4^j - \tfrac{3}{32}| + |\vec{f}^j - \tfrac{9}{16}|, \\
& 2|\vec{e}^j - \tfrac{3}{8}| + 4|\vec{b}_4^j - \tfrac{1}{16}|, \\
& 4|\vec{d}^j - \tfrac{1}{4}| \Big\}
\end{aligned}$$

By substituting the estimates, we obtain:

$$\|S_{r^j} - S_c\|_\infty \leq \frac{1}{2^{2j}} \max\{4\mathcal{A} + 4\mathcal{C} + \mathcal{F},\ 2\mathcal{E} + 4\mathcal{B},\ 0\}$$

$$\sum_{j=1}^{\infty} 2^j \|S_{r^j} - S_c\|_\infty \leq M \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty$$

This proves first-order asymptotic equivalence with the Catmull-Clark scheme, which is recognized as C^2-convergent in regular regions. Thus, the rational scheme possesses the C^2-convergence property in regular domains.

□

The aforementioned finding indicates that the rational scheme preserves high-order continuity in regular regions, which is essential for generating smooth surfaces across most of the mesh.

5.2. **Convergence at Extraordinary Points.** Next, we focus on vertices with valence other than 4, often called extraordinary points, and analyze the behavior of the scheme in their neighborhood. These points necessitate careful consideration since they may interrupt the continuity of the limit surface.

Theorem 5. *The non-stationary rational scheme $\{S_{r^j}\}_{j\geq 0}$ converges at extraordinary faces and generates limit surfaces that exhibit continuous tangent planes at the limit positions of extraordinary points.*

Proof. We establish this result by confirming all prerequisites of Theorem 1.

Condition (i): Properties of the reference scheme The Catmull-Clark algorithm $\tilde{S}$ is:

- C^2-convergent (thus C^1) in usual regions
- G^1-convergent at extraordinary spots [6]

The symbol is defined as $c(z_1, z_2) = \frac{(z_1+1)^4(z_2+1)^4}{64}$ contains $= (1+z_1)(1+z_2)$

Condition (ii): Preservation of factors The j-level symbol $r^j(z_1, z_2)$ of our system incorporates the factor $(1+z_1)^2(1+z_2)^2$, as demonstrated in equation (16), and therefore includes $(1+z_1)(1+z_2)$.

Condition (iii): First-order asymptotic equivalence in regular domains This was demonstrated in the proof of Theorem 4.

Condition (iv): Convergence of local matrices We examine the block matrices B_i^j and their discrepancies from the Catmull-Clark blocks B_i. □

Lemma 2. *The distinctions between rational and Catmull-Clark block matrices are as follows:*

$$\begin{aligned}
\|B_0^j - B_0\|_\infty &\leq \frac{O_1}{2^{2j}} \\
\|B_1^j - B_1\|_\infty &\leq \frac{O_2}{2^{2j}} \\
\|B_i^j - B_i\|_\infty &\leq \frac{O_3}{n2^{2j}}, \quad i = 2, \ldots, n-2 \\
\|B_{n-1}^j - B_{n-1}\|_\infty &\leq \frac{O_4}{2^{2j}}
\end{aligned}$$

for positive constants O_1, O_2, O_3, O_4 that are independent of j.

Proof. We present the comprehensive estimation for $\|B_0^j - B_0\|_\infty$:

$$\begin{aligned}
\|B_0^j - B_0\|_\infty = \max \Bigg\{ & \frac{1}{4n}|7-(16\mathcal{H}(\mu^{j+1})-4(\mathcal{H}(\mu^{j+1}))^2)| \\
& + \frac{1}{2n^2}|3-(8\mathcal{H}(\mu^{j+1})-4(\mathcal{H}(\mu^{j+1}))^2)| \\
& + \frac{1}{4n^2}|1-(4\mathcal{H}(\mu^{j+1}))^2|, \\
& \left(1+\frac{1}{n}\right)|\vec{e}^j - \tfrac{3}{8}| + |\vec{b}_4^j - \tfrac{1}{16}|, \\
& \left(2+\frac{1}{n}\right)|\vec{a}_4^j - \tfrac{1}{64}|, \\
& \left(2+\frac{1}{n}\right)|\vec{c}_4^j - \tfrac{3}{32}| + |\vec{f}^j - \tfrac{9}{16}| + |\vec{a}_4^j - \tfrac{1}{64}|, \\
& \left(2+\frac{1}{n}\right)|\vec{b}_4^j - \tfrac{1}{16}| + 2|\vec{e}^j - \tfrac{3}{8}|, \\
& \left(2+\frac{1}{n}\right)|\vec{a}_4^j - \tfrac{1}{64}| + 3|\vec{c}_4^j - \tfrac{3}{32}| + |\vec{f}^j - \tfrac{9}{16}|, \\
& \left(2+\frac{1}{n}\right)|\vec{b}_4^j - \tfrac{1}{16}| + |\vec{e}^j - \tfrac{3}{8}| \Bigg\}
\end{aligned}$$

The essential terminology includes phrases such as:

$$|7-(16\mathcal{H}(\mu^{j+1})-4(\mathcal{H}(\mu^{j+1}))^2)| \leq 16|\mathcal{H}(\mu^{j+1})-\tfrac{1}{2}| + 4|(\mathcal{H}(\mu^{j+1}))^2-\tfrac{1}{4}| \leq \frac{M_1}{2^{2j}}$$

$$|3-(8\mathcal{H}(\mu^{j+1})-4(\mathcal{H}(\mu^{j+1}))^2)| \leq 8|\mathcal{H}(\mu^{j+1})-\tfrac{1}{2}| + 4|(\mathcal{H}(\mu^{j+1}))^2-\tfrac{1}{4}| \leq \frac{M_2}{2^{2j}}$$

$$|1-(4\mathcal{H}(\mu^{j+1}))^2| \leq \frac{M_3}{2^{2j}}$$

Integrating them with the coefficient estimates from Theorem 4 yields the requisite constraint. The disparities in the opposite block adhere to analogous rationale. □

Lemma 3. *Subdivision matrix convergence The local subdivision matrices fulfill:*

$$\|S^j - S\|_\infty \leq \frac{O}{2^{2j}}$$

for a constant $O > 0$ *that is independent on* j*.*

Proof. Employing Lemma 2:

$$\|S^j - S\|_\infty \leq \sum_{i=0}^{n-1} \|B_i^j - B_i\|_\infty \leq \frac{O_1 + O_2 + (n-3)\frac{O_3}{n} + O_4}{2^{2j}} \leq \frac{O}{2^{2j}}$$

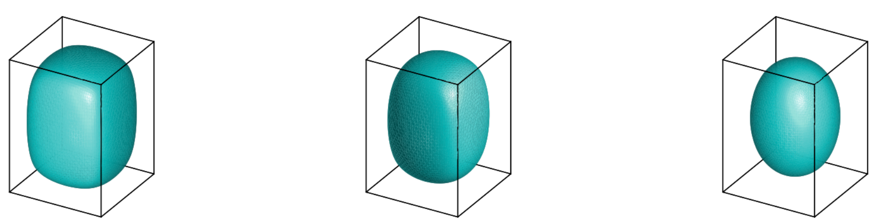
Verification of spectral conditions The Catmull-Clark subdivision matrix S possesses the following characteristics: The dominant eigenvalue is $\lambda_0 = 1$. The subdominant eigenvalue λ_1 satisfies the inequality $0.5 < \lambda_1 < 1$. Furthermore, λ_1 is a double and non-defective eigenvalue [6]. Consequently, $\sigma = 4 > \frac{1}{\lambda_1} > 1$ fulfills condition (iv).

Given that the prerequisites of Theorem 1 are fulfilled, the rational scheme converges at extraordinary points and yields G^1-continuous surfaces.

The demonstration of G^1-continuity at remarkable points guarantees that the scheme generates visually smooth surfaces, even amidst vertices with atypical valences, which is essential for practical modeling applications.

6. Numerical Experiments

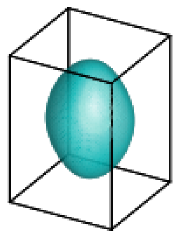
This section provides thorough numerical validation of the proposed subdivision scheme $\{S_{c^j}\}_{j\geq 0}$ via a series of meticulously constructed experiments and using Algorithm 2.



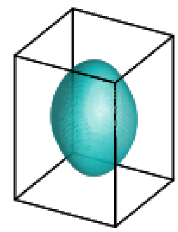
(a) $\nu^0 = 4$ (b) $\nu^0 = 1$ (c) $\nu^0 = \cos(\pi/4)$

Figure 2. Example surfaces generated through the proposed rational subdivision when using the parameters $a = 2$, $b = 1$, and several initial settings of μ^0 by using Algorithm 2.

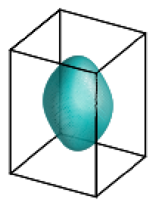
Figure 2 illustrates representative surfaces generated by our method. The experiments confirm that by adjusting parameters a and b, one can obtain surfaces with a wide range of morphological characteristics.



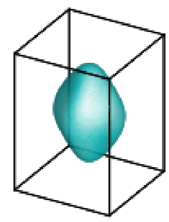
(a) $a = 6, b = 4$



(b) $a = 4, b = 3$



(c) $a = 3, b = 1$



(d) $a = 2, b = 1$

Figure 3. Sensitivity study of the parameters a and b, demonstrating how their variation affects the overall surface geometry using Algorithm 2.

In Figure 3, with $\mu^0 = -0.1$, we explore how different parameter combinations influence the resulting shapes. This parametric investigation offers valuable guidelines for users seeking to tune the parameters for specific applications.

7. Conclusion

We have proposed a rational, non-stationary subdivision framework governed by parameters (a, b, μ) that deliver controllable tension while maintaining the smoothness standards of classical schemes—C^2 in regular regions by asymptotic equivalence and G^1 near extraordinary vertices through bounded block deviations. The parameter study demonstrates that efficient shape control can be achieved without increasing computational cost. Future research will address reproduction theorems for selected function families and develop automated strategies for parameter estimation based on geometric descriptors.

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