Gulf Journal of Mathematics
Vol 21, Issue 2 (2025) 222-118

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v21i2.3753

GENERALIZED STRONG ZERO-DIVISOR GRAPH OF RINGS WITH INVOLUTION

ANIL KHAIRNAR[1] and ANITA LANDE[2*]

Abstract. We introduce and examine the generalized strong zero-divisor graph $\Gamma'_s(R)$ of a $*$-ring, focusing on generalized p.q.-Baer $*$-rings. We show that $\Gamma'_s(R)$ is connected with diameter at most three and determine its girth. Beck's conjecture holds for this graph in the considered setting. We identify conditions under which $\Gamma'_s(R)$ is uniquely complemented or contains a cut vertex. For p.q.-Baer $*$-rings, connectedness of the complement of $\Gamma'_s(R)$ ensures the existence of at least six central projections. We also describe the diameter and girth of the complement and provide examples where the generalized strong zero-divisor graph is complemented.

1. Introduction

A $*$-ring is an associative ring equipped with an involution $*$ satisfying $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, and $(x^*)^* = x$ for all $x, y \in R$. An element $e \in R$ is a *projection* if $e^2 = e = e^*$ and is central if it lies in the center of R. For a subset $S \subseteq R$, the sets $r_R(S) = \{x \in R \mid sx = 0 \text{ for all } s \in S\}$ and $l_R(S) = \{x \in R \mid xs = 0 \text{ for all } s \in S\}$ denote its *right and left annihilators*. Kaplansky introduced Baer and Baer $*$-rings, and later generalizations such as quasi-Baer, p.q.-Baer $*$-rings, semi-baer modules were studied in [8, 9, 12, 24].

The concept of zero-divisor graphs began with Beck [5] and was later refined for commutative and non-commutative rings in [2, 21]. Further extensions include the strong zero-divisor graph [6], the zero-divisor graph of a $*$-ring [20], and its strong variant [13]. A generalized zero-divisor graph of a $*$-ring was recently introduced in [15]. The study of zero-divisor graphs has been extensively explored in works such as [10, 16, 17, 18].

In this paper, we define a new graph associated with a $*$-ring R, called the generalized strong zero-divisor graph (GSZDG), denoted $\Gamma'_s(R)$. Its vertex set is $V(\Gamma'_s(R)) = \{0 \neq x \in R \mid r_R(xR) \neq \{0\}\}$ and distinct vertices x, y are adjacent when $x^nRy^* = 0$ or $y^nRx^* = 0$ for some $n \geq 1$.

Section 2 provides preliminaries and examples showing that generalized p.q.-Baer $*$-rings form a broad class. Section 3 compares adjacency in $\Gamma'_s(R)$ with earlier graph constructions and studies connectedness and diameter. Section 4

Date: Received: Oct 31, 2025; Revised: Nov 29, 2025; Accepted: Dec 10, 2025.

* Corresponding author.

2020 *Mathematics Subject Classification.* Primary 16W10; Secondary 05C25, 13A70.

Key words and phrases. p.q.-Baer $*$-ring, generalized p.q.-Baer $*$-ring, central projection, generalized strong zero-divisor graph (GSZDG)..

analyzes $\Gamma'_s(R)$ for generalized p.q.-Baer $*$-rings, including girth and a verification of Beck's conjecture. Section 5 explores cut vertices, extending results of [4]. Section 6 shows that $\Gamma'_s(R)$ is uniquely complemented for reduced generalized p.q.-Baer $*$-rings and characterizes the structure of its complement.

2. Preliminaries

In this section, we summarize the foundational definitions and notions used throughout the paper. A $*$-ring R is called a *Baer $*$-ring* if the right annihilator of every subset of R is generated by a projection; that is, for any $B \subseteq R$, one has $r_R(B) = eR$ for some projection $e \in R$ [7]. Birkenmeier et al. later introduced *principally quasi-Baer (p.q.-Baer)* $*$-rings, defined as $*$-rings in which the right annihilator of each principal right ideal aR is of the form eR for a projection $e \in R$. A $*$-ring R is called a generalized p.q.-Baer $*$-ring if, for every principal ideal I, the annihilator of I^n is generated by a projection for some positive integer n depending on I [1]. Every generalized p.q.-Baer $*$-ring necessarily has a unity [1].

Next we provide examples of generalized p.q.-Baer $*$-rings that are not p.q.-Baer $*$-rings.

Example 2.1 ([1], Example 2.37)**.** Let $T = \{a_0+a_1i+a_2j+a_3k \mid a_n \in \mathbb{Z}_2 \text{ for } n = 0,1,2,3\}$ be the Hamilton quaternions over $\mathbb{Z}_2$. For $\alpha = a_0 + a_1i + a_2j + a_3k$ with $a_0, a_1, a_2, a_3 \in \mathbb{Z}_2$, define $\alpha^* = a_0 - a_1i - a_2j - a_3k$. Then T is not a p.q.-Baer $*$-ring. But T is a generalized p.q.-Baer $*$-ring.

A ring R is called *π-regular* if for each $a \in R$, there exists a positive integer n and an element $x \in R$ such that $a^n = a^nxa^n$. A ring R is called *semicommutative* if, for any $a, b \in R$, $ab = 0$ implies $aRb = 0$.

Example 2.2 ([1], Example 2.37)**.** Let T be a commutative π-regular ring. Then T with the identity involution is a generalized p.q.-Baer $*$-ring. Let $R = \left\langle \oplus_{\lambda\in\Lambda} T_\lambda, 1_{\prod_{\lambda\in\Lambda} T_\lambda} \right\rangle$. Then R is a generalized p.q.-Baer $*$-ring, but not a p.q.-Baer $*$-ring.

An element e of a $*$-ring R is called the *central cover* of an element $x \in R$ if e is the smallest central projection satisfying $ex = x$; this projection is denoted by $C(x)$ [7]. It is known that every p.q.-Baer $*$-ring has a unity, and every element in such a ring admits a central cover [11]. Moreover, generalized p.q.-Baer $*$-rings also possess unity [1]. However, unlike in the p.q.-Baer setting, elements of a generalized p.q.-Baer $*$-ring need not have central covers. To address this limitation, More and Khairnar [19] introduced the notion of generalized central covers for elements in a $*$-ring.

Definition 2.3 ([19, Definition 3.6])**.** Let R be a $*$-ring and $x \in R$. Then the smallest central projection $e \in R$ is called the *generalized central cover* of x if $x^ne = x^n$ for some $n \in \mathbb{N}$. We write it as $GC(x) = e$.

Proposition 2.4 ([19, Proposition 3.4])**.** *Let R be a generalized p.q.-Baer $*$-ring, and $x \in R$. Then there exists a central projection $e \in R$ such that*

(1) *$x^ne = x^n$ for some $n \in \mathbb{N}$;*

(2) *For some* $n \in \mathbb{N}$, $(xR)^n y = \{0\}$ *if and only if* $ey = 0$.

By Proposition 2.4, in a generalized p.q.-Baer $*$-ring R, the generalized central cover $GC(x)$ exists for every central element $x \in R$. Moreover, if x is non-central, then $GC(x) = 1$ when x is not nilpotent, and $GC(x) = 0$ when x is nilpotent. Thus, $GC(x)$ exists for all elements $x \in R$.

An element $a \in R$ is called a *strong zero-divisor* if $\langle a\rangle\langle b\rangle = 0$ or $\langle b\rangle\langle a\rangle = 0$ for some nonzero $b \in R$ [6], and the set of all nonzero strong zero-divisors is denoted by $S(R)$. The *strong zero-divisor graph* $\tilde{\Gamma}(R)$ has vertex set $S(R)^* = S(R) \setminus 0$, with distinct vertices a and b adjacent if $\langle a\rangle\langle b\rangle = 0$ or $\langle b\rangle\langle a\rangle = 0$ [6].

For a $*$-ring R, the *zero-divisor graph* $\Gamma^{(}R)$ is a simple undirected graph whose vertices are all nonzero left zero-divisors, i.e., $\{0 \neq a \in R \mid ab = 0 \text{ for some } 0 \neq b \in R\}$, with adjacency defined by $ab^* = 0$ [20]. The *generalized zero-divisor graph* $\Gamma'(R)$ has the same vertex set, with x and y adjacent if $x^n y^* = 0$ or $y^n x^* = 0$ for some positive integer n [15]. Finally, the *strong zero-divisor graph* $\Gamma_s^*(R)$ is defined by $V(\Gamma_s^*(R)) = \{0 \neq a \in R \mid r_R(aR) \neq 0\}$, with adjacency given by $aRb^* = 0$ [13].

Definition 2.5. A subset S of the vertex set $V(G)$ of a graph G is called a *clique* if every pair of distinct vertices in S is adjacent. The *clique number* $\omega(G)$ is defined as the size of the largest clique in G.

3. Generalized Strong zero-divisor graph of a $*$-ring

In this section, we introduce the generalized strong zero-divisor graph (GSZDG) of rings with involution. We also discuss the connectedness and diameter of GSZDG.

Definition 3.1. Let R be a $*$-ring. The *generalized strong zero-divisor graph* of a ring R, denoted by $\Gamma_s'(R)$, is a simple (undirected) graph with the vertex set $V(\Gamma_s'(R)) = \{0 \neq x \in R \mid r_R(xR) \neq \{0\}\}$ and two distinct vertices x and y are adjacent if and only if $x^n Ry^* = \{0\}$ or $y^n Rx^* = \{0\}$ for some positive integer n.

The following examples demonstrate that our adjacency is different from the adjacency defined in [13, 15, 20].

Example 3.2. Let $R = M_2(\mathbb{Z}_2)$ and $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in R$. Let $*$ be the transpose involution on R. Since $ef^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, e and f are adjacent in $\Gamma^*(R)$ with adjacency as in [20]. Let $r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Note that $e^n = e$, for any positive integer n and $e^n r f^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Hence $e^n Rf^* \neq \{0\}$.

Also, for $r = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $f^n re^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq 0.$

Therefore $e^nRf^* \neq \{0\}$ and $f^nRe^* \neq \{0\}$ for any positive integer n. Hence e and f are not adjacent in $\Gamma'_s(R)$.

Example 3.3. Let $A = \mathbb{Z}_8$ and $B = \langle 2\rangle$. Then $R = M_2(B)$ is a $*$-ring with transpose involution. Let $a = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in R$. Observe that $a^3 = 0$. Therefore $a^3Rb^* = \{0\}$. Hence a is adjacent to b in $\Gamma'_s(R)$. Since $ab^* = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \neq 0$, a is not adjacent to b in $\Gamma^*(R)$.

Example 3.4. Let $A = \mathbb{Z}_{16}$, $B = \langle 2\rangle$ and $R = M_2(B)$. Let $a = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in R$. Here $a^4Rb^* = \{0\}$. Hence a is adjacent to b in $\Gamma'_s(R)$.
If $r = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, then $arb^* = \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix} \neq 0$. Therefore $aRb^* \neq \{0\}$. Hence a is not adjacent to b in $\Gamma^*_s(R)$.

Example 3.5. Let $R = M_2(\mathbb{Z}_4)$ and $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in R$. Let $*$ be the transpose involution on R. Since $e^nf^* = 0$, e is adjacent to f in $\Gamma'(R)$. But $e^nRf^* \neq \{0\}$ and $f^nRe^* \neq \{0\}$ for any positive integer n. Hence e is not adjacent to f in $\Gamma'_s(R)$.

Remark 3.6.
(1) If $xRy^* = \{0\}$, then $x^nRy^* = \{0\}$. That is, if x is adjacent to y in $\Gamma^*_s(R)$, then x is adjacent to y in $\Gamma'_s(R)$. Therefore $\Gamma^*_s(R)$ is a subgraph of $\Gamma'_s(R)$.
(2) If R is a reduced ring, then $\Gamma^*_s(R) \simeq \Gamma'_s(R)$.
(3) Let R be a commutative $*$-ring with unity. Then $a^nb^* = 0$ if and only if $a^nRb^* = \{0\}$. That is, a is adjacent to b in $\Gamma'(R)$ if and only if a is adjacent to b in $\Gamma'_s(R)$. Hence for a commutative $*$-ring R with unity, $\Gamma'(R) \simeq \Gamma'_s(R)$.
(4) If $a \in R$ is a nilpotent element, then a is adjacent to every $b \in V(\Gamma'_s(R)) \setminus \{a\}$. Hence, if R contains a nilpotent element, then $\Gamma'_s(R)$ is connected.

A ring R is called *reflexive*, if for all $a, b \in R$, $aRb = \{0\}$ implies $bRa = \{0\}$.

Lemma 3.7. *Let R be a reflexive $*$-ring with unity and $x \in R$. Then $x \in V(\Gamma'_s(R))$ if and only if $x^* \in V(\Gamma'_s(R))$.*

Proof. Let $x \in V(\Gamma'_s(R))$. Then $r_R(xR) \neq \{0\}$. Therefore there exists a nonzero $y \in r_R(xR)$ such that $xRy = \{0\}$. This implies $yRx = \{0\}$, so $y \in V(\Gamma'_s(R))$. Since $x^*Ry^* = \{0\}$ and $y^* \neq 0$, we conclude that $x^* \in V(\Gamma'_s(R))$. Conversely, let $x^* \in V(\Gamma'_s(R))$, then by the above part $(x^*)^* = x \in V(\Gamma'_s(R))$. □

Lemma 3.8. *Let R be a $*$-ring with unity and e be a vertex in $\Gamma'_s(R)$ which is adjacent to all the other vertices in $\Gamma'_s(R)$. Then the following statements hold.*

(1) *If R is reflexive, then e is either a projection or a nilpotent element in R,*

(2) *If R is a generalized p.q.-Baer $*$-ring and e is not nilpotent, then e is a central projection and there is no element $a \in R$ other than e such that $GC(a) = e$.*

Proof. (1) Let R be reflexive and e be a vertex adjacent to all other vertices in $\Gamma'_s(R)$. If e is a nilpotent element, we are through, as this case directly satisfies the conditions. Suppose that e is not a nilpotent element. By Lemma 3.7, $e^* \in V(\Gamma'_s(R))$. If $e \neq e^*$, then e and e^* are adjacent, that is $e^n R(e^*)^* = \{0\}$, or $(e^*)^n Re^* = \{0\}$, for some positive integer n. Therefore $e^n Re = \{0\}$ or $eRe^n = \{0\}$ for some positive integer n. Since R contains unity this gives $e^{n+1} = 0$. Therefore, e is a nilpotent element, a contradiction. Hence $e = e^*$.
We claim that $e^2 = e$. If $e^2 \neq e$, then e and e^2 are adjacent. Therefore $e^n R(e^2)^* = \{0\}$ or $(e^2)^n Re^* = \{0\}$ for some positive integer n. That is $e^n Re^2 = \{0\}$ or $e^{2n} Re = \{0\}$, which gives $e^{n+2} = 0$ or $e^{2n+1} = 0$. Which implies e is nilpotent, a contradiction. Therefore $e^2 = e$. Hence e is a projection in R.

(2) Suppose R is a generalized p.q.-Baer $*$-ring, then every element in R has generalized central cover. Also, e and $GC(e)$ are adjacent to the same vertices. If $GC(e) \neq e$, then e and $GC(e)$ are adjacent, a contradiction. Hence $GC(e) = e$. Therefore, e is a central projection. Let $a \in R$ with $a \neq e$ such that $GC(a) = e$. Since e is adjacent to all other vertices, this implies that a is adjacent to e, but a and $GC(a)$ cannot be adjacent. Therefore, there is no element $a \in R$ other than e such that $GC(a) = e$. □

The following result gives an involution on the ring R such that $\Gamma'_s(R)$ is disconnected.

Proposition 3.9. *Let A be a commutative ring and $R = A \times A$ be a $*$-ring with the involution $(a, b)^* = (b, a)$. Then $\Gamma'_s(R)$ is connected if and only if A is not an integral domain.*

Proof. Assume that $\Gamma'_s(R)$ is connected. On the contrary, suppose that A is an integral domain. Then every vertex of $\Gamma'_s(R)$ is of the form $(x, 0)$ or $(0, y)$, where $x \neq 0$, $y \neq 0$, and no vertex of the type $(x, 0)$ is connected to the vertex of the type $(0, y)$. Hence $\Gamma'_s(R)$ is disconnected, a contradiction. Hence, A is not an integral domain. Conversely, suppose that A is not an integral domain, then $ab = 0$ for some nonzero $a, b \in A$. Let $(x, y) \in V(\Gamma'_s(R))$. Then there exists $(0, 0) \neq (z, w)^* \in r_R((x, y)R)$ such that $(x, y)R(z, w)^* = \{(0, 0)\}$, i.e., $(x, y)R(w, z) = \{(0, 0)\}$. Then $xRw = \{0\}$ and $yRz = \{0\}$, which gives (x, y) is adjacent to $(0, w)$ and $(0, w)$ is adjacent to $(0, b)$. Therefore, $(0, b)$ is connected by a path to every vertex in $\Gamma'_s(R)$. Hence $\Gamma'_s(R)$ is connected. □

Example 3.10. Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$ with involution $(x, y)^* = (y, x)$. Then $\Gamma'_s(R)$ is disconnected as depicted in Figure 1.

Recall that a ring R is said to be *decomposable* if it can be written in the form $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$. A ring is said to be *indecomposable* if it is not

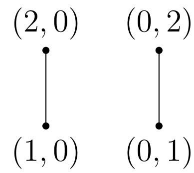

FIGURE 1. The generalized strong zero-divisor graph $\Gamma'_s(R)$

decomposable. The following result shows that the GSZDG of decomposable rings with componentwise involution is always connected.

Theorem 3.11. *Let R_1, R_2 be two $*$-rings with $V(\Gamma'_s(R_1)) \neq \varnothing$ and $V(\Gamma'_s(R_2)) \neq \varnothing$. Then $R = R_1 \oplus R_2$ is a $*$-ring with componentwise involution such that $\Gamma'_s(R)$ is connected and $diam(\Gamma'_s(R)) \leq 4$.*

Proof. Clearly, $(x, y) \in V(\Gamma'_s(R))$ if and only if $x \in V(\Gamma'_s(R_1))$ or $y \in V(\Gamma'_s(R_2))$. Let $(x_1, x_2), (y_1, y_2) \in V(\Gamma'_s(R))$ be any two vertices. Suppose that $x_1 \in V(\Gamma'_s(R_1))$ and $y_1 \in V(\Gamma'_s(R_1))$, i.e., $x_1 R_1 z_1 = \{0\}$ and $y_1 R_1 w_1 = \{0\}$, for some $z_1, w_1 \in R_1$. This gives a path $(x_1, x_2) \leftrightarrow (z_1^*, 0) \leftrightarrow (0, x_2) \leftrightarrow (w_1^*, 0) \leftrightarrow (y_1, y_2)$ of length four joining (x_1, x_2) and (y_1, y_2). If $x_1 \in V(\Gamma'_s(R_1))$ and $y_2 \in V(\Gamma'_s(R_2))$, i.e., $x_1 R_1 z_1 = \{0\}$ and $y_2 R_2 w_2 = \{0\}$, for some $z_1 \in R_1$ and $w_2 \in R_2$. Then $(x_1, x_2) \leftrightarrow (z_1^*, 0) \leftrightarrow (0, w_2^*) \leftrightarrow (y_1, y_2)$ is a path of length three joining (x_1, x_2) and (y_1, y_2). Similarly, we have a path joining (x_1, x_2) and (y_1, y_2) in the remaining cases. Therefore $\Gamma'_s(R)$ is connected and $diam(\Gamma'_s(R)) \leq 4$. □

The following corollary follows from the above theorem.

Corollary 3.12. *If $R_1, R_2, \ldots, R_n$ are $*$-subrings of a $*$-ring R such that $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$ with componentwise involution, then $\Gamma'_s(R)$ is connected.*

Corollary 3.13. *Let R be a $*$-ring with unity. If R contains a nontrivial central projection, then $\Gamma'_s(R)$ is connected and $diam(\Gamma'_s(R)) \leq 4$.*

Proof. Suppose R contains a nontrivial central projection, say e. Then $R_1 = eR$ and $R_2 = (1-e)R$ are $*$-rings such that $R = R_1 \oplus R_2$. By Theorem 3.11, $\Gamma'_s(R)$ is connected and $diam(\Gamma'_s(R)) \leq 4$. □

4. GSZDG of generalized p.q.-Baer $*$-rings

In this section, we examine the properties of the GSZDG of generalized p.q.-Baer $*$-rings. We characterize the girth of GSZDG and prove Beck's conjecture for GSZDG of a generalized p.q.-Baer $*$-rings.

The following lemma provides a connection between vertices in $\Gamma'_s(R)$ and generalized central covers of elements in R.

Lemma 4.1. *Let R be a generalized p.q.-Baer $*$-ring and $x \in R$ be a non-nilpotent element. Then $x \in V(\Gamma'_s(R))$ if and only if $GC(x)$ is nontrivial.*

Proof. Suppose that $e = GC(x)$, and $e \notin \{0, 1\}$, then $1 - e \notin \{0, 1\}$ with $x^n R(1 - e)^* = (x^n - x^n e)R = \{0\}$ for some positive integer n. This implies that x and $1 - e$ are adjacent in $\Gamma'_s(R)$. Hence $x \in V(\Gamma'_s(R))$. Conversely, let $x \in V(\Gamma'_s(R))$

and $e = GC(x)$. If $xRy^* = \{0\}$ (implies $x^nRy^* = \{0\}$), then $y^* \in r_R(x^nR)$. Since R is generalized p.q.-Baer $*$-ring, $r_R(x^nR) = (1-e)R$. If $e = 0$, then $x^n = 0$, a contradiction. Hence $e \neq 0$. If $e = 1$, then $r_R(x^nR) = \{0\}$, which implies that $x^n \notin V(\Gamma'_s(R))$, and consequently, $x \notin V(\Gamma'_s(R))$. For if $x \in V(\Gamma'_s(R))$, then $r_R(xR) \neq \{0\}$. That is there exist some nonzero $y \in R$ such that $xRy = \{0\}$, which yields $x^nRy = \{0\}$. This implies $x^n \in V(\Gamma'_s(R))$. Therefore $x \notin V(\Gamma'_s(R))$, which gives a contradiction. Therefore, $e \neq 1$, and it follows that $GC(x)$ is nontrivial for some positive integer n. $\square$

The set of all central projections of a generalized p.q.-Baer $*$-ring forms a lattice under the partial order defined as follows:

$$\text{“}e \leq f \;\; if \; and \; only \; if \; e = ef = fe\text{”}, \;\; \text{and}\; e \vee f = e+f-ef \;\text{and}\; e \wedge f = ef. \quad (4.1)$$

This lattice is denoted by $L(CP(R))$. An element $e \neq 0$ in a lattice is called an *atom* if $0 \leq f \leq e$ implies either $f = 0$ or $f = e$.

Corollary 4.2. *Let R be a finite generalized p.q.-Baer $*$-ring such that $V(\Gamma'_s(R)) \neq \varnothing$, and R contains nontrivial, non-nilpotent element. Then the lattice of central projections $L(CP(R))$ contains at least two atoms.*

Proof. Assume $x \in V(\Gamma'_s(R))$ is not a nilpotent element. Then $GC(x)$ and $1 - GC(x)$ both are nontrivial and distinct central projections in R such that $GC(x)(1 - GC(x)) = 0$. Therefore $GC(x)$ and $1 - GC(x)$ are incomparable. Consequently, $GC(x)$ and $1 - GC(x)$ are covered by two different atoms. Hence, the lattice $L(CP(R))$ contains at least two atoms. $\square$

In the following result, we establish a relationship between the adjacency of vertices and the adjacency in their central covers in a generalized strong zero-divisor graph of p.q. Baer $*$-rings.

Proposition 4.3. *Let R be a generalized p.q.-Baer $*$-ring. Then the vertices x and y are adjacent in $\Gamma'_s(R)$ if and only if $C(x^n)RC(y) = \{0\}$ or $C(x)RC(y^n) = \{0\}$ for some positive integer n. Moreover, if the vertex x is adjacent to y, then x is adjacent to $C(y^n)$ or y is adjacent to $C(x^n)$ for some positive integer n.*

Proof. Let $x, y \in V(\Gamma'_s(R))$, and suppose x is adjacent to y, then $x^nRy^* = \{0\}$ or $y^nRx^* = \{0\}$ for some positive integer n. Assume $x^nRy^* = \{0\}$. This implies, $y^* \in r_R(x^nR) = (1 - C(x^n))R$. Thus, $y^* = (1 - C(x^n))y^*$, which implies $y = y(1-C(x^n))$. Consequently, $yC(x^n) = 0$, which gives $yRC(x^n) = \{0\}$. Therefore, $C(x^n) \in r_R(yR) = (1-C(y))R$. Hence, $C(x^n) = (1-C(y))C(x^n)$, which implies $C(y)C(x^n) = 0$, i.e., $C(x^n)C(y) = 0$. This shows that $C(x^n)R(C(y))^* = \{0\}$. Hence, $C(x^n)$ is adjacent to $C(y)$. Similarly, if $y^nRx^* = \{0\}$, then $C(y^n)$ is adjacent to $C(x)$. Conversely, suppose $C(x^n)RC(y)^* = \{0\}$ or $C(y)RC(x^n)^* = \{0\}$. Suppose $C(x^n)RC(y)^* = \{0\}$, then $x^nRy^* = x^nC(x^n)R(yC(y))^* = x^nC(x^n)RC(y)y^* = \{0\}$. Therefore, x is adjacent to y. Suppose $C(y)RC(x^n)^* = \{0\}$. Note that $C(y)RC(x^n)^* = \{0\}$ if and only if $C(x^n)RC(y)^* = \{0\}$. Hence x is adjacent to y.

Note that $r_R(xR) = r_R(C(x)R)$. Suppose x is adjacent to y, then $x^nRy^* = \{0\}$ or $y^nRx^* = \{0\}$ for some positive integer n. If $x^nRy^* = \{0\}$, then $y^* \in r_R(x^nR) =$

$r_R(C(x^n)R)$. Hence $C(x^n)Ry^* = \{0\}$, which implies y is adjacent to $C(x^n)$. Similarly, if $y^nRx^* = \{0\}$, then x is adjacent to $C(y^n)$, for some positive integer n. □

The following result establishes the relation between generalized central cover of vertices in $\Gamma'_s(R)$ for a generalized p.q.-Baer $*$-ring R.

Lemma 4.4. *Let R be a generalized p.q.-Baer $*$-ring. If x and y are adjacent in $\Gamma'_s(R)$, then $GC(x)RGC(y) = \{0\}$.*

Proof. Suppose x is adjacent to y in $\Gamma'_s(R)$, then $x^nRy^* = \{0\}$ or $y^mRx^* = \{0\}$ for some positive integers n, m. If $x^nRy^* = \{0\}$, then $x^nR(y^m)^* = \{0\}$, which implies $(y^m)^* \in r_R(x^nR) = (1 - GC(x))R$. Therefore $(y^m)^* = (1 - GC(x))(y^m)^*$, which gives $y^mGC(x) = 0$. Thus $y^mRGC(x) = \{0\}$, and hence $GC(x) \in r_R(y^mR) = (1 - GC(y))R$. This implies $GC(x) = (1 - GC(y))GC(x)$, leading to $GC(x)GC(y) = 0$. Therefore $GC(x)RGC(y) = \{0\}$. Similarly, if $y^mRx^* = \{0\}$, then we show $GC(x)RGC(y) = \{0\}$. □

The converse of the above lemma is not true.

Example 4.5. Let $R = \mathbb{Z}_{72}$ and $x = \overline{2}$, $y = \overline{3} \in \mathbb{Z}_{72}$. Here $(x^2)\overline{64} = x^2$ and $(y^2)\overline{9} = y^2$, so $GC(x) = \overline{64}$ and $GC(y) = \overline{9}$. Observe that $GC(x)GC(y) = 0$, but $x^nRy^* \neq \{0\}$ and $y^nRx^* \neq \{0\}$ for any positive integer n. Hence x is not adjacent to y in $\Gamma'_s(R)$.

Proposition 4.6. *Let R be a generalized p.q.-Baer $*$-ring. If $\Gamma'_s(R)$ is triangle-free, and a and b are adjacent in $\Gamma'_s(R)$, then $GC(a) = 1 - GC(b)$.*

Proof. Suppose $\Gamma'_s(R)$ is triangle-free and a is adjacent to b in $\Gamma'_s(R)$. By Lemma 4.4, $GC(a)GC(b) = 0$. Therefore, $GC(a) + GC(b)$ is a central projection in R. If $GC(a) \neq 1 - GC(b)$, and let $e = 1 - (GC(a) + GC(b))$. Then, e is a nontrivial central projection, and hence $GC(a) \leftrightarrow GC(b) \leftrightarrow e \leftrightarrow GC(a)$ is a 3-cycle, a contradiction. Thus, $GC(a) = 1 - GC(b)$. □

Note that in a reduced ring R, $r_R(xR) = r_R(xR)^n$. Hence a reduced generalized p.q.-Baer $*$-ring is equivalent to p.q.-Baer $*$-ring. Recall the following result from [13].

Proposition 4.7 ([13])**.** *Let R be a p.q.-Baer $*$-ring. Then $\Gamma^*_s(R)$ is connected and $diam(\Gamma^*_s(R)) \leq 3$.*

Proposition 4.8. *Let R be a generalized p.q.-Baer $*$-ring. Then $\Gamma'_s(R)$ is connected and $diam(\Gamma'_s(R)) \leq 3$.*

Proof. If R contains a nilpotent element, then by Remark 3.6, $\Gamma'_s(R)$ is connected. Suppose R does not contain any nilpotent element, then R is a reduced generalized p.q.-Baer $*$-ring, hence a p.q.-Baer $*$-ring. Therefore, $\Gamma'_s(R) \simeq \Gamma^*_s(R)$. By Proposition 4.7, $\Gamma'_s(R)$ is connected, and $diam(\Gamma'_s(R)) \leq 3$. □

Recall that a ring is called *abelian* if all its idempotents are central. Since abelian Rickart $*$-rings are reduced p.q.-Baer $*$-rings, it follows that $\Gamma^*(R) \simeq \Gamma^*_s(R) \simeq \Gamma'_s(R)$ for abelian Rickart $*$-ring R. Consequently, as a corollary to the above proposition, we obtain the following result by Patil and Waphare.

Corollary 4.9 ([20], Proposition 3.3)**.** *Let R be an abelian Rickart $*$-ring. Then $\Gamma^*(R)$ is connected and diam$(\Gamma^*(R)) \leq 3$.*

In the following results, we determine the girth of a GSZDG of generalized p.q.-Baer $*$-rings.

Lemma 4.10. *Let R be a generalized p.q.-Baer $*$-ring and x be a non-projection element in $V(\Gamma'_s(R))$. Then $\Gamma'_s(R)$ contains a cycle and $gr(\Gamma'_s(R)) \leq 4$.*

Proof. Let $x \in V(\Gamma'_s(R))$. Then there exists $y \in R$ such that x is adjacent to y in $\Gamma'_s(R)$. Therefore $GC(x)GC(y) = 0$. Then $x \leftrightarrow y \leftrightarrow GC(x) \leftrightarrow GC(y) \leftrightarrow x$ is a cycle of length four. Hence $gr(\Gamma'_s(R)) \leq 4$. □

The following result follows immediately from the above lemma.

Corollary 4.11. *Let R be a generalized p.q.-Baer $*$-ring. Then $gr(\Gamma'_s(R)) \in \{3, 4, \infty\}$.*

Corollary 4.12. *Let C_n be a cycle on n vertices, $(n \geq 5)$. Then C_n cannot be a generalized strong zero-divisor graph of a generalized p.q.-Baer $*$-ring.*

Proposition 4.13. *Let R be a non-reduced generalized p.q.-Baer $*$-ring and $\Gamma'_s(R)$ is not a star graph. Then $gr(\Gamma'_s(R)) = 3$.*

Proof. Since $\Gamma'_s(R)$ is not a star graph, we have $|V(\Gamma'_s(R))| \geq 3$. Suppose $a \in R$ be a nilpotent element and $a, x_1, x_2, \ldots, x_n$ represents the vertices of $\Gamma'_s(R)$. Then a is adjacent to x_i for every $i \in \{1, 2, \ldots, n\}$. Suppose x_i and x_j are not adjacent for any $i \neq j$. Then $\Gamma'_s(R)$ is a star graph, contradicting the assumption. Therefore x_i is adjacent to x_j for some $i, j \in \{1, 2, \ldots, n\}$. This implies that $a \leftrightarrow x_i \leftrightarrow x_j \leftrightarrow a$ is a 3-cycle in $\Gamma'_s(R)$. Hence $gr(\Gamma'_s(R)) = 3$. □

Recall the following result.

Theorem 4.14. [13] *Let R be a p.q.-Baer $*$-ring. If $\Gamma^*_s(R)$ is complete r-partite graph, then $r = 2$. Moreover, $\Gamma^*_s(R)$ is complete bipartite if and only if R contains exactly 4 central projections.*

The following is an analogues result to the above theorem.

Corollary 4.15. *Let R be a generalized p.q.-Baer $*$-ring. If $\Gamma'_s(R)$ is complete r-partite graph, then $r = 2$. Moreover, if $\Gamma'_s(R)$ is complete bipartite, then R contains exactly 4 central projections.*

Proof. Suppose R contains a nilpotent element, then $\Gamma'_s(R)$ is not an r-partite graph. Therefore, R must be reduced. Hence, R is a p.q.-Baer $*$-ring and $\Gamma'_s(R) = \Gamma^*_s(R)$. The result then follows from Theorem 4.14. □

The converse of the above corollary is not true.

Example 4.16. Let $R = \mathbb{Z}_{12}$, then R is a generalized p.q.-Baer $*$-ring. The projections in R are $\overline{0}, \overline{1}, \overline{4}, \overline{9}$. But $\Gamma'_s(R)$ is not bipartite (see Figure 3).

To prove Beck's conjecture for $\Gamma'_s(R)$, the following results are required.

Lemma 4.17. *Let R be a generalized p.q.-Baer $*$-ring such that $\Gamma'_s(R)$ does not contain an infinite clique. Then the lattice $L(CP(R))$ of central projections of R satisfies the descending chain condition (DCC).*

Proof. Suppose $\Gamma'_s(R)$ does not contain an infinite clique. Let $f_1 \geq f_2 \geq f_3 \geq \cdots$ be an infinite descending chain in $L(CP(R))$. Define $e_i = f_i - f_{i+1}$. Since $f_{i+1} \leq f_i$. Therefore $f_i f_{i+1} = f_{i+1}$ by (4.1) and $f_i^2 = f_i$ for each i as f_i is a central projection. Hence $e_i^2 = f_i - 2f_i f_{i+1} + f_{i+1} = f_i - 2f_{i+1} + f_{i+1} = e_i$ and $e_i^* = e_i$. Thus e_i is a central projection for each i. Also, if $i \neq j$, then either $i < j$ or $j < i$. If $i < j$, then $e_j < e_i$, implying $e_i R e_j = R e_i e_j = R(f_i - f_{i+1})(f_j - f_{j+1}) = R(f_j - f_{j+1} - f_j + f_{j+1}) = \{0\}$. Similiarly, if $j < i$, we have $e_i R e_j = \{0\}$. Therefore $\{e_1, e_2, e_3, \ldots\}$ form an infinite clique in $\Gamma'_s(R)$, a contradiction. Hence, the descending chain $f_1 \geq f_2 \geq f_3 \geq \cdots$ must terminate, proving that $L(CP(R))$ satisfies the DCC. □

The following are immediate consequences of the above result.

Corollary 4.18. *Let R be a generalized p.q.-Baer $*$-ring such that $\Gamma'_s(R)$ does not contain an infinite clique. Then every nonzero central projection in R contains an atom in $L(CP(R))$.*

Corollary 4.19. *Let R be a generalized p.q.-Baer $*$-ring such that $\Gamma'_s(R)$ does not contain an infinite clique. Then every nonzero central projection of R contains a finite number of pairwise orthogonal atoms in $L(CP(R))$. Moreover, the number of pairwise orthogonal atoms in $L(CP(R))$ is finite.*

Proof. By Lemma 4.18, every nonzero central projection of R contains a finite number of pairwise orthogonal atoms in $L(CP(R))$. Suppose $\{f_1, f_2, f_3, \ldots\}$ is an infinite set of pairwise orthogonal atoms in $L(CP(R))$, then $\{f_1, f_2, f_3, \ldots\}$ would forms an infinite clique in $\Gamma'_s(R)$, which contradicts the assumption. Hence, the number of pairwise orthogonal atoms in $\Gamma'_s(R)$ is finite. □

Lemma 4.20. *If $a, b \in V(\Gamma'_s(R))$ such that $GC(a)$ and $GC(b)$ contain some common atoms in $L(CP(R))$, then a and b cannot be adjacent in $\Gamma'_s(R)$.*

Proof. Let e be an atom in $L(CP(R))$ such that $e \leq GC(a)$, $GC(b)$. If a and b are adjacent, then $GC(a)GC(b) = 0$. Since $e = eGC(a) = eGC(b)$, we get $e = eGC(b) = eGC(a)GC(b) = 0$, a contradiction. Hence, a and b are not adjacent. □

In the following result, we prove Beck's conjecture for GSZDG of a generalized p.q.-Baer $*$-ring.

Theorem 4.21. *Let R be a generalized p.q.-Baer $*$-ring such that $\Gamma'_s(R)$ does not contain an infinite clique. Then $\chi(\Gamma'_s(R)) = \omega(\Gamma'_s(R)) = n + k$, where n is the cardinality of the maximal set of pairwise orthogonal atoms in $L(CP(R))$, and k is the number of nilpotent elements in R.*

Proof. Let N denote the set of nonzero nilpotent elements in R. We partition the vertex set $V(\Gamma'_s(R))$ into two subsets $N \cup (V(\Gamma'_s(R)) \setminus N)$. Since every element in N is adjacent to every other element in $V(\Gamma'_s(R))$, the elements in N form a clique of size $|N|$ in $\Gamma'_s(R)$. Thus, $|N|$ colors are required to color the vertices in N. Now, let $x \in V(\Gamma'_s(R)) \setminus N$. Since x and $GC(x)$ are always nonadjacent, the same color can be assigned to x and $GC(x)$. Thus, it suffices to color the central projections. As R does not contain an infinite clique, by Corollary 4.19, $L(CP(R))$ contains a finite number of pairwise orthogonal atoms. Let $T = \{e_1, e_2, \dots, e_n\}$ be a maximal set of orthogonal atoms, and $e = e_1 + e_2 + \cdots + e_n$. We claim that $e = 1$. Suppose, on the contrary, that $e \neq 1$. Then $1 - e = 1 - (e_1 + e_2 + \cdots + e_n)$ is a nonzero central projection such that $(1-e)Re^* = \{0\}$. Also, $(1-e)$ is orthogonal to each e_i. By Corollary 4.18, there exists an atom $f \leq 1 - e$ with $fe_i = 0$, for each i, which contradicts the maximality of T. Hence $e_1 + e_2 + \cdots + e_n = 1$. For any central projection f, there must exist e_i such that $e_i f \neq 0$. Let f, g be two central projections in R that are adjacent in $\Gamma'_s(R)$. Therefore $fRg = \{0\}$. Without loss of generality, assume $e, f \neq 0$. We show that there exists $j \neq 1$ such that $e_j g \neq 0$. Suppose, on the contrary, that $e_1 g \neq 0$ and $e_i g = 0$, for $i \in \{2, 3, \dots, n\}$. Which, together with $e_1 + e_2 + \cdots + e_n = 1$ gives $g = e_1 g$. Thus $g \leq e_1$. Since e_1 is an atom, $g = e_1$. But $gf = 0$ implies that $g = e_1 f = gf = 0$, which is a contradiction. Hence there exists $j \neq 1$ such that $e_j g \neq 0$. We have $\omega(\Gamma'_s(R)) \leq \chi(\Gamma'_s(R)) = n + k$. Observe that elements of T form a clique of size n in $\Gamma'_s(R)$. Since N is the set of nilpotent elements, the elements of $T \cup N$ form a clique of size $n + |N| = n + k$ in $\Gamma'_s(R)$. Thus, $\omega(\Gamma'_s(R)) = \chi(\Gamma'_s(R)) = n + k$. □

5. Cut vertices in GSZDG

A vertex x of graph G is called a *cut-vertex* if the removal of x and all edges incident to x creates a graph with more connected components than G. The graph G is said to *split* into two subgraphs X and Y via a if (i) $|V(X)| \geq 2$ and $|V(Y)| \geq 2$, (ii) $X \cup Y = G$, (iii) $V(X) \cap V(Y) = \{a\}$, and (iv) $x \in V(X)\backslash\{a\}$ and $y \in V(Y)\backslash\{a\}$, then x and y are not adjacent. Axtell et al. [3] examined the cut-vertices in zero-divisor graphs and determined when the zero-divisors form an ideal in a finite commutative ring with identity. Further, Axtell et al. [4] and Redmond [22] studied the existence of cut-vertices in the zero-divisor graphs of commutative rings. In this section, we study cut vertices in GSZDG of generalized p.q.-Baer $*$-rings.

In the following theorem, we establish a condition for the existence of an ideal of cardinality 2.

Theorem 5.1. *Let R be a $*$-ring and $\Gamma'_s(R)$ splits into two subgraphs X and Y via a. Then $\{0, a^n\}$ is an ideal of R for some positive integer n.*

Proof. Suppose a is nilpotent, then $\{0, a^n\} = \{0\}$ is an ideal of R, and the result holds. Suppose a is not nilpotent. Therefore, $a^n \neq a^n + a^n$, as otherwise a would be nilpotent, and $a^n Rx^* = \{0\}$ implies $(a^n + a^n)Rx^* = \{0\}$. We claim that $a^n + a^n = 0$. If $a^n + a^n \neq 0$, then $a^n + a^n \in V(\Gamma'_s(R))$. Without loss of generality, assume $a^n + a^n \in V(X)\backslash\{a\}$. Let $b \in V(Y)\backslash\{a\}$ be adjacent to a. Therefore either $a^n Rb^* = \{0\}$ or $b^n Ra^* = \{0\}$ for some positive integer n. Suppose $a^n Rb^* = \{0\}$, then $(a^n + a^n)Rb^* = \{0\}$, which contradicts the assumption that a is a cut-vertex. If $b^n Ra^* = \{0\}$, then $aR(b^n)^* = \{0\}$, implying $a^n R(b^n)^* = \{0\}$. Therefore $(a^n + a^n)R(b^n)^* = \{0\}$, which again gives $b^n R(a^n + a^n)^* = \{0\}$, which yields b is

adjacent to $a^n + a^n$. Therefore, a can not be a cut-vertex, a contradiction. Hence $a^n + a^n \in \{0, a^n\}$.
Now, let $r \in R$. If b is adjacent to a, then $a^n Rb^* = \{0\}$ or $b^n Ra^* = \{0\}$ for some positive integer n. If $a^n Rb^* = \{0\}$, then $ra^n Rb^* = \{0\}$ and $a^n rRb^* = \{0\}$. If $ra^n \neq 0$ and $ra^n \neq a^n$, then each vertex adjacent to a is also adjacent to ra^n, contradicting the fact that a is a cut-vertex. Hence $ra^n \in \{0, a^n\}$. Similarly, $a^n r \in \{0, a^n\}$. If $b^n Ra^* = \{0\}$, then $b^n R(a^n r)^* = b^n(Rr^*)(a^n)^* = \{0\}$ and $b^n R(ra^n)^* = b^n R(a^n)^* r^* = \{0\}$. Therefore, in this case also, each vertex adjacent to a is adjacent to $a^n r$. Hence $a^n r \in \{0, a^n\}$. Similarly, $ra^n \in \{0, a^n\}$. Therefore $\{0, a^n\}$ is an ideal of R. ☐

In the following result, we give a sufficient condition on a $*$-ring R which provides the existence of an ideal in terms of vertices of $\Gamma'_s(R)$.

Theorem 5.2. *Let R be a commutative $*$-ring and $\Gamma'_s(R)$ splits into two subgraphs X and Y via a such that $X \setminus \{a\}$ is complete subgraph. Then $V(X) \cup \{0\}$ is an ideal of R.*

Proof. For a commutative $*$-ring, identity is the only involution. Let $b \in X \setminus \{a\}$ such that a and b are adjacent. Since $X \setminus \{a\}$ is complete, therefore x and y are adjacent to b for any $x, y \in V(X)$. Hence, $x^n Rb = \{0\}$ or $b^n Rx = \{0\}$ and $y^m Rb = \{0\}$ or $b^m Ry = \{0\}$ for some positive integers m, n.
Case (i): Suppose $x^n Rb = \{0\}$ and $y^m Rb = \{0\}$, then $(x + y)^{m+n} Rb = \{0\}$. Hence $x + y \in V(X) \cup \{0\}$.
Case (ii): Suppose $x^n Rb = \{0\}$ and $b^m Ry = \{0\}$, then $x^n R(b^m) = \{0\}$ and $yR(b^m) = \{0\}$, which implies $(x + y)^{n+1} Rb^m = \{0\}$. Consequently, $x + y \in V(X) \cup \{0\}$.
Case (iii): Suppose $b^n Rx = \{0\}$ and $y^m Rb = \{0\}$, the argument follows similarly to Case (ii), and we conclude $x + y \in V(X) \cup \{0\}$.
Case (iv): Suppose $b^n Rx = \{0\}$ and $b^m Ry = \{0\}$, let $k = \max\{n, m\}$. Then $b^k Rx = \{0\}$ and $b^k Ry = \{0\}$, which implies $b^k R(x + y) = \{0\}$. Hence $x + y \in V(X) \cup \{0\}$. In all the cases $x + y \in V(X) \cup \{0\}$.
Now, let $r \in R$ and $x \in V(X) \cup \{0\}$. Since $X \setminus \{a\}$ is a complete graph, x is adjacent to $b \in V(X) \setminus \{a\}$. Therefore $x^n Rb = \{0\}$ or $b^n Rx = \{0\}$ for some positive integer n. This implies $(rx)^n Rb = r^n x^n Rb = \{0\}$ or $b^n R(xr) = \{0\}$, and hence $rx = xr \in V(X) \cup \{0\}$. Therefore $V(X) \cup \{0\}$ is an ideal of R. ☐

Proposition 5.3. *Let R be a $*$-ring. If $\Gamma'_s(R)$ contains a cut-vertex, then $x+y \in V(\Gamma'_s(R)) \cup \{0\})$ for all $x, y \notin V(\Gamma'_s(R)) \cup \{0\})$.*

Proof. Let a be a cut-vertex in $\Gamma'_s(R)$. By Theorem 5.1, $\{0, a^n\}$ is an ideal in R for some positive integer n. This gives $Ra^n = I = \{0, a^n\}$ and for any $x \in R, xRa^n = xI = \{0, xa^n\} \subseteq I$. If $x, y \in R \setminus V(\Gamma'_s(R))$, then $xRa^n \neq \{0\}$ and $xRa^n \subseteq I$, which implies $xRa^n = \{a^n\}$. Similarly, if $-y \in R \setminus V(\Gamma'_s(R))$, then $-yRa^n \neq \{0\}$ and $-yRa^n \subseteq I$, so $-yRa^n = \{a^n\}$. Consequently, $xRa^n = -yRa^n = \{a^n\}$, which implies $(x + y)Ra^n = \{0\}$. Thus, $x + y \in R \setminus V(\Gamma'_s(R))$. ☐

The following example demonstrates that the existence of a cut-vertex is necessary in Proposition 5.3.

Example 5.4. (1) Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$ with the identity involution. Then $V(\Gamma_s'(R)) = \{(0,1),(0,2),(1,0),(2,0)\}$. Note that R does not contain a cut-vertex. For $x = (2,2)$, $y = (2,2)$, both $x, y \notin V(\Gamma_s'(R))$, and $x + y = (1,1) \notin V(\Gamma_s'(R)) \cup \{0\}$.

(2) Let $R = M_2(\mathbb{Z}_3)$ be a $*$-ring with transpose involution. Consider the matrices $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Note that R does not contain a cut-vertex, and $X, Y \notin V(\Gamma_s'(R))$. Also, $X + Y = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \notin V(\Gamma_s'(R)) \cup \{0\}$.

The following example shows that the converse of Proposition 5.3 is not true.

Example 5.5. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ with the identity involution. Then $V(\Gamma_s'(R)) = \{(0,1),(1,0)\}$. For $x = (1,1)$, $y = (1,1)$, we have $x, y \notin V(\Gamma_s'(R))$, but $x + y = (0,0) \in V(\Gamma_s'(R)) \cup \{0\}$. However, $V(\Gamma_s'(R))$ does not contain a cut-vertex.

If $a \neq 0$, we say that $r_R(aR)$ (right annihilator of aR), is *properly maximal* if for every $b \in R \setminus \{0, a\}$, $r_R(aR) \nsubseteq r_R(bR)$.

Proposition 5.6. *Let R be a $*$-ring and a be a cut-vertex in $\Gamma_s'(R)$. Then $r_R(a^nR)$ is properly maximal in R.*

Proof. Assume there exists $b \in R \setminus \{0, a\}$ such that $r_R(a^nR) \subseteq r_R(bR)$. Let x be a vertex adjacent to a. Then $a^nRx^* = \{0\}$ or $x^mRa^* = \{0\}$ for some positive integers m, n. If $a^nRx^* = \{0\}$, then $x^* \in r_R(a^nR) \subseteq r_R(bR)$, which implies $bRx^* = \{0\}$. Thus x is adjacent to b. If $x^mRa^* = \{0\}$, then $aR(x^m)^* = \{0\}$, implying $a^nR(x^m)^* = \{0\}$. Hence $(x^m)^* \in r_R(a^nR) \subseteq r_R(bR)$, which leads to $bR(x^m)^* = \{0\}$. Consequently, $x^mRb^* = \{0\}$, and x is adjacent to b. Thus, every vertex adjacent to a is also adjacent to b, contradicting the assumption that a is not a cut-vertex. Therefore, $r_R(a^nR)$ is properly maximal for some positive integer n. $\square$

The following example demonstrates that the converse of Proposition 5.6 is not true.

Example 5.7. Let $R = \mathbb{Z}_{14}$ be a $*$-ring with the identity involution. Here, $r_R((\overline{2})^nR) = \{\overline{0}, \overline{7}\}$ is properly maximal for any positive integer n, but $\overline{2}$ is not a cut-vertex in $\Gamma_s'(R)$.

Lemma 5.8. *Let R be a generalized p.q.-Baer $*$-ring. If $\Gamma_s'(R)$ contains a cut-vertex, then R contains at most one nonzero nilpotent element. Moreover, if $\Gamma_s'(R)$ contains a cut-vertex and a is a nilpotent element in R, then a is a cut-vertex with $a^2 = 0$.*

Proof. Assume $x, y \in R$ are distinct, nonzero nilpotent elements. Since $y \in V(\Gamma_s'(R)) \setminus \{x\}$, the graph $\Gamma_s'(R) \setminus \{x\}$ is connected, implying x is not a cut-vertex. Similarly, y is not a cut-vertex. If $a \in V(\Gamma_s'(R)) \setminus \{x, y\}$, then $\Gamma_s'(R) \setminus \{a\}$ is connected, so $a \in V(\Gamma_s'(R)) \setminus \{x, y\}$ is also not a cut-vertex. This contradicts the assumption that $\Gamma_s'(R)$ contains a cut-vertex. Hence R contains at most one nonzero nilpotent element. Now, suppose a is the only nonzero nilpotent element

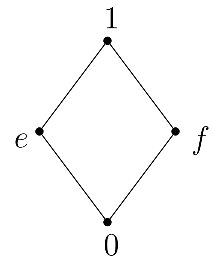

FIGURE 2. Lattice of central projections in $M_2(\mathbb{Z}_6)$

in R, and R contains a cut-vertex. Clearly, a must be a cut-vertex. If $a^2 \neq 0$, then a^2 is also nilpotent, which means R contains more than one nilpotent element, a contradiction. Therefore $a^2 = 0$. □

Theorem 5.9. *Let R be a generalized p.q.-Baer $*$-ring and let a (non-nilpotent element) be a cut-vertex in $\Gamma'_s(R)$. Then a^n is an atom in the lattice of central projections in R, for some positive integer n.*

Proof. We first show that a^n is a central projection. Let $e = GC(a)$. Then $r_R(a^nR) = r_R(eR)$ for some positive integer n. If $a^n \neq e$, then $e \in R \setminus \{0, a^n\}$, and $r_R(a^nR) \subseteq r_R(eR)$, which contradicts the fact that a is a cut-vertex. Thus $a^n = e = GC(a)$, implying a^n is a central projection. Next, let f be a central projection in R such that $f \leq a^n$, i.e., $f = a^nf = fa^n$. Then $r_R(a^nR) \subseteq r_R(fR)$. Since $r_R(a^nR)$ is properly maximal, we conclude that $f = 0$ or $f = a^n$. Hence a^n is an atom in the lattice of central projections in R. □

The converse of Theorem 5.9 is not true.

Example 5.10. Let $R = M_2(\mathbb{Z}_6)$ with the transpose involution. Then R is a p.q-Baer $*$-ring, and the set of all central projections in R is:

$$CP(R) = \left\{0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, e = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, f = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right\}.$$

Here, observe that $e = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is an atom in the lattice of central projections $L(CP(R))$. However, e is not a cut-vertex in $\Gamma'_s(R)$. Since R contains more than one nonzero nilpotent element, R does not have a cut-vertex.

Example 5.11. Let $R = \mathbb{Z}_{12}$ be a $*$-ring with the identity involution. Here $(\overline{2})^2 = \overline{4}$ is an atom in the lattice of projections in R. However, $\overline{2}$ is not a cut-vertex in $\Gamma'_s(R)$.

A vertex of degree 1 is called a *pendant* vertex. In the following result, we prove that the existence of a pendant vertex gives the existence of a cut-vertex.

Lemma 5.12. *Let R be a generalized p.q.-Baer $*$-ring and a is a pendant vertex in $\Gamma'_s(R)$. Then $\Gamma'_s(R)$ contains a cut-vertex.*

Proof. Let $a \in V(\Gamma'_s(R))$ be a pendant vertex. Since a is not a nilpotent element in R, $GC(a)$ is nontrivial. Clearly, a and $1 - GC(a)$ are adjacent vertices in $\Gamma'_s(R)$. Hence $1 - GC(a)$ is a cut-vertex in $\Gamma'_s(R)$. □

6. Uniquely complementedness and Complement

Vishweswaran [23] studied the complement and the connectedness of the complement of the zero-divisor graph of commutative rings. In this section, we investigate when the GSZDG of generalized p.q.-Baer $*$-rings is complemented. Also, we explore the diameter and girth of the complement of the GSZDG for these rings.
We recall the following definitions:
Let G be a graph. The set of vertices adjacent to a vertex a in G is denoted by $\mathcal{N}(a)$. For two non-adjacent vertices a and b of G, $a \approx b$ if and only if a and b are adjacent to the same vertices, i.e., $\mathcal{N}(a) = \mathcal{N}(b)$. This relation " $\approx$ " is an equivalence relation on G. Two distinct vertices a and b of G are said to be *orthogonal*, if a and b are adjacent and they don't have any common neighbor. It is denoted by $a \perp b$. A graph G is said to be *complemented*, if for each vertex a in G, there exist a vertex b of G (called a *complement* of a) such that $a \perp b$. A graph G is said to be *uniquely complemented* if G is complemented, and whenever $a \perp b$ and $a \perp c$, then $b \approx c$.

Lemma 6.1. *Let R be a generalized p.q.-Baer $*$-ring and $a, b \in V(\Gamma'_s(R))$. If $a \approx b$, then $GC(a) = GC(b)$.*

Proof. Suppose $a \approx b$, i.e., $\mathcal{N}(a) = \mathcal{N}(b)$ and a, b are non-adjacent. Let $x \in \mathcal{N}(a) = \mathcal{N}(b)$. Then we have $GC(a)RGC(x) = \{0\}$ if and only if $GC(b)RGC(x) = \{0\}$. This implies $a^nRGC(x) = \{0\}$ if and only if $b^mRGC(x) = \{0\}$ for some positive integers m, n. Thus $GC(x) \in r_R(a^nR)$ if and only if $GC(x) \in r_R(b^mR)$. Since $r_R((a^nR)) = (1 - GC(a))R$ and $r_R((b^mR)) = (1 - GC(b))R$, we conclude that $(1 - GC(a))R = (1 - GC(b))R$. Hence $GC(a) = GC(b)$. $\square$

The following example demonstrates that the converse of the above lemma is not true.

Example 6.2. Let $R = \mathbb{Z}_{72}$ be a $*$-ring with the identity involution. Here, $GC(\overline{2}) = \overline{64}$ and $GC(\overline{8}) = \overline{64}$. Observe that $\overline{3} \in \mathcal{N}(\overline{8})$, but $(\overline{3})^nR(\overline{2}) \neq \{0\}$ and $(\overline{2})^nR(\overline{3}) \neq \{0\}$ for any positive integer n. Therefore $\overline{3} \notin \mathcal{N}(\overline{2})$. Hence $\mathcal{N}(\overline{2}) \neq \mathcal{N}(\overline{8})$.

Corollary 6.3. *Let R be a generalized p.q.-Baer $*$-ring. For two central projections e and f in R, if $e \approx f$ in $\Gamma'_s(R)$, then $e = f$.*

For a p.q.-Baer $*$-ring R, $\Gamma^*_s(R)$ is uniquely complemented (see [13]). However, $\Gamma'_s(R)$ need not be complemented.

Example 6.4. Let $R = M_2(\mathbb{Z}_3)$ be a p.q.-Baer $*$-ring, and $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{Z}_3)$. Since E_{12} is a nilpotent element, it is adjacent to all other elements, and E_{21} is a common neighbor of all vertices. Hence, E_{12} does not have a complement

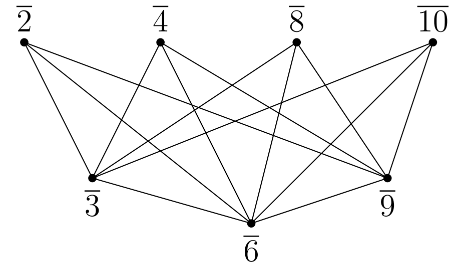

FIGURE 3. The generalized strong zero-divisor graph $\Gamma'_s(\mathbb{Z}_{12})$

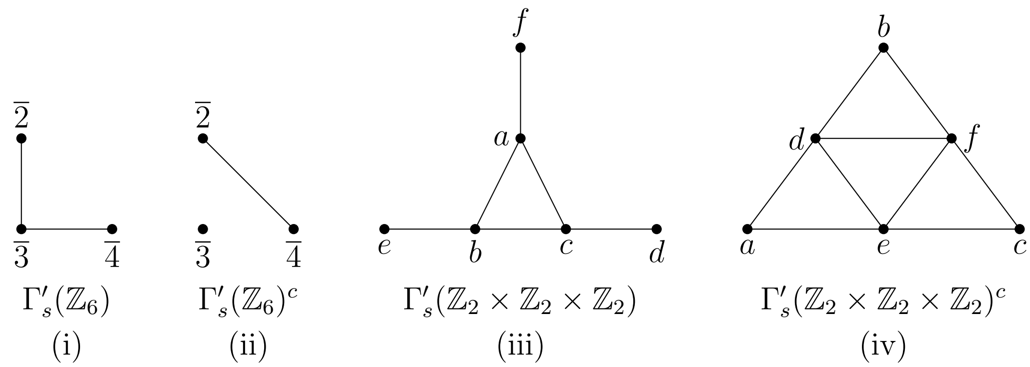

FIGURE 4. GSZDG and its complement

Example 6.5. Let $R = \mathbb{Z}_{12}$ with the identity involution be a generalized p.q.-Baer $*$-ring. Here, $\overline{6}$ is the only nonzero nilpotent element in R. Note that $\overline{6}$ does not have a complement in $\Gamma'_s(R)$ (see Figure 3).

The following are some observations about the complementedness of $\Gamma'_s(R)$.

Remark 6.6.
(1) If R contains a nilpotent element, then $\Gamma'_s(R)$ is not complemented.
(2) If R is a reduced p.q.-Baer $*$-ring, then $\Gamma'_s(R) = \Gamma^*_s(R)$ and hence $\Gamma'_s(R)$ is uniquely complemented. Thus $\Gamma'_s(R)$ is uniquely complemented if and only if R is reduced generalized p.q.-Baer $*$-ring.

Next, we characterize generalized p.q.-Baer $*$-rings for which the complement of $\Gamma'_s(R)$, denoted by $\Gamma'_s(R)^c$ is connected. In the following example, we demonstrate the GSZDG and its complement for the $*$-ring.

Example 6.7. Let $R_1 = \mathbb{Z}_6$ and $R_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ be $*$-rings with the identity involution. Both R_1 and R_2 are p.q.-Baer $*$-rings. For R_1, $V(\Gamma'_s(R_1)) = \{\overline{2}, \overline{3}, \overline{4}\}$. Observe that $\Gamma'_s(R_1)$ is connected, but the complement $\Gamma'_s(R_1)^c$ is disconnected. For R_2, $V(\Gamma'_s(R_2) = \{a = (\overline{1}, \overline{0}, \overline{0}), b = (\overline{0}, \overline{1}, \overline{0}), c = (\overline{0}, \overline{0}, \overline{1}), d = (\overline{1}, \overline{1}, \overline{0}), e = (\overline{1}, \overline{0}, \overline{1}), f = (\overline{0}, \overline{1}, \overline{1})\}$. Here, both $\Gamma'_s(R_2)$ and $\Gamma'_s(R_2)^c$ are connected (see Figure 4).

Thus, the complement of the GSZDG of a generalized p.q.-Baer $*$-ring need not always be connected.
Note that if R contains a nilpotent element, then $\Gamma'_s(R)$ is disconnected.

Theorem 6.8. [14] *For a p.q.-Baer $*$-ring R, the complement $\Gamma^*_s(R)^c$ is connected if and only if $|CP(R)| \geq 6$.*

Corollary 6.9. *Let R be a generalized p.q.-Baer $*$-ring. If $\Gamma'_s(R)^c$ is connected, then $|CP(R)| \geq 6$.*

Proof. Let $\Gamma'_s(R)^c$ be a connected graph. Suppose R contains a nilpotent element, then $\Gamma'_s(R)^c$ is disconnected. Therefore, Rmust be reduced. In this case, R is a p.q.-Baer $*$-ring and $\Gamma^*_s(R) = \Gamma'_s(R)$. By Theorem 6.8, $|CP(R)| \geq 6$. $\square$

From the above corollary, for a generalized p.q.-Baer $*$-ring R, if $|CP(R)| < 6$, then $\Gamma'_s(R)^c$ is disconnected. However, the converse is not true. That is, if $|CP(R)| \geq 6$, $\Gamma'_s(R)^c$ need not be connected.

Example 6.10. Let $R = M_2(\mathbb{Z}_3)$ with the transpose involution. The set of projections in R is

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right\}.$$

The number of projections in R is 6, but $\Gamma'_s(R)^c$ is disconnected. But by Theorem 6.8, $\Gamma^*_s(R)^c$ is connected.

Remark 6.11. Observe that, if R contains a nilpotent element, then $\Gamma'_s(R)^c$ is disconnected. That is, if $\Gamma'_s(R)^c$ is connected, then R does not contain a nilpotent element, and hence R is reduced.

Recall the following result from [14].

Theorem 6.12 ([14])**.** *Let R be a p.q.-Baer $*$-ring. If $\Gamma^*_s(R)^c$ is connected, then $diam(\Gamma^*_s(R)^c) = 2$ and $gr(\Gamma^*_s(R)^c) = 3$.*

Corollary 6.13. *Let R be a generalized p.q.-Baer $*$-ring. If $\Gamma'_s(R)^c$ is connected, then $diam(\Gamma'_s(R)^c) = 2$ and $gr(\Gamma'_s(R)^c) = 3$.*

Proof. Suppose $\Gamma'_s(R)^c$ is connected, then by Remark 6.11, R is a reduced ring. Therefore $\Gamma'_s(R) = \Gamma^*_s(R)$. Consequently, $\Gamma'_s(R)^c = \Gamma^*_s(R)^c$. By Theorem 6.12, $diam(\Gamma'_s(R)^c) = 2$ and $gr(\Gamma'_s(R)^c) = 3$. $\square$

Recall the following result.

Lemma 6.14 ([14])**.** *Let R be a p.q.-Baer $*$-ring such that $\Gamma^*_s(R)^c$ is connected, then it is not complemented.*

Corollary 6.15. *Let R be a generalized p.q.-Baer $*$-ring such that $\Gamma'_s(R)^c$ is connected, then it is not complemented.*

Proof. Suppose $\Gamma'_s(R)^c$ is connected, then R is a reduced ring. Therefore $\Gamma'_s(R)^c = \Gamma^*_s(R)^c$. By Lemma 6.14, $\Gamma'_s(R)^c$ is not complemented. $\square$

Conflict of interest: Authors declare that they have no competing interests.
Acknowledgements: We sincerely appreciate the editor and anonymous referees for their detailed and valuable suggestions, which enhanced the presentation of this paper.

References

1. Ahmadi, M., Golestani, N., & Moussavi, A. (2020). Generalized quasi-Baer $*$-rings and Banach $*$-algebras. Communications in Algebra, 48, 2207-2247. https://doi.org/10.1080/00927872.2019.1710841.
2. Anderson, D., & Livingston, P. (1999). The zero-divisor graph of a commutative ring. Journal of Algebra, 217, 434-447. https://doi.org/10.1016/S0021-8693(03)00435-6.
3. Axtell, M., Stickles, J., & Trampbachls, W. (2009). Zero-divisor ideals and realizable zero-divisor graphs. Involve, a Journal of Mathematics, 2, 17–27. https://doi.org/10.2140/involve.2009.2.17.
4. Axtell, M., Baeth, N., & Stickles, J. (2011). Cut vertices in zero-divisor graphs of finite commutative rings. Communications in Algebra, 39(6), 2179-2188. https://doi.org/10.1080/00927872.2010.488681.
5. Beck, I. (1988), Coloring of commutative rings. Journal of Algebra, 116, 208-226. https://doi.org/10.1016/0021-8693(88)90202-5.
6. Behboodi, M., Beyranvand, R., & Khabazian, H. (2008). Strong zero-divisors of non-commutative rings. Journal of Algebra and its Applications, 2, 25-44. https://doi.org/10.1142/S0219498809003540.
7. Berberian, S. K. (1972). Baer $*$-Rings, Springer-Verlag, Berlin, New York.
8. Birkenmeier, G. F., Kim, J. Y., & Park, J. K. (2001). Polynomial extensions of Baer and quasi-Baer rings. Journal of Pure and Applied Algebra, 159(1), 25-42. https://doi.org/10.1016/S0022-4049(00)00055-4.
9. Birkenmeier, G. F., Park, J. K., & Rizvi, S. T. (2013). Extensions of Rings and Modules (New York, Birkhäuser).
10. Khairnar, A., & Waphare, B. N. (2016). Zero-Divisor Graphs of Laurent Polynomials and Laurent Power Series. Journal of Algebra and its Applications, 345-349. https://doi.org/10.1007/978-981-10-1651-6_21.
11. Khairnar, A., & Waphare, B. N. (2018). Unitification of weakly p.q.-Baer $*$-rings. Southeast Asian Bulletin of Mathematics, 42, 387-400. https://doi.org/10.24330/ieja.1518558.
12. Khairnar, A., & Waphare, B. N. (2018). A Sheaf Representation of Principally Quasi-Baer $*$-Rings. Algebras and Representation Theory, 22, 79-97. https://doi.org/10.1007/s10468-017-9758-0.
13. Kumbhar, N., Khairnar, A., & Waphare, B. N. (2023). Strong zero-divisor graph of rings with involution, Asian-European Journal of Mathematics, 16(10), 2350179. https://doi.org/10.1142/S1793557123501796.
14. Kumbhar, N., Khairnar, A., & Waphare, B. N. (2024). Strong zero-divisor graph of p.q.-Baer $*$-rings, arXiv:2408.05949 [math.CO]. https://doi.org/10.48550/arXiv.2408.05949.
15. Lande, A., & Khairnar, A. (2024). Generalized zero-divisor graph of $*$-rings. arXiv:2403.10161v1 [math.CO]. https://doi.org/10.48550/arXiv.2403.10161.
16. Lande, A., & Khairnar, A. (2024). On the spectrum of generalized zero-divisor graph of the ring $\mathbb{Z}_{p^\alpha q^\beta}$. Communications in Mathematics and Applications, 15(3), 1031-1044. https://doi.org/10.26713/cma.v15i3.2737.
17. Lande, A., & Khairnar A. (2025). Idempotent graph of 2×2 matrix ring with involution. Gulf Journal of Mathematics, 19(2), 168-180. https://doi.org/10.56947/gjom.v19i2.2665.
18. Lande, A., Khairnar, A., & Gutman, I. (2025). The zero-divisor graph of 2×2 matrix ring and its energies. Filomat, 39(22), 7559-7570. https://doi.org/10.2298/FIL2522559L.

19. More, S., & Khairnar, A. (2025). On Generalized p.q.-Baer $*$-rings, arXiv:2509.02584 [math.RA]. https://doi.org/10.48550/arXiv.2509.02584.
20. Patil, A., & Waphare, B. N. (2018). The zero-divisor graph of a ring with involution. Journal of Algebra and its Applications, 17(03), p. 1850050. https://doi.org/10.1142/S0219498818500500.
21. Redmond, S. (2002). The zero-divisor graph of a non-commutative ring. International Journal of Commutative Rings, 1, 203-211.
22. Redmond, S. P. (2012). Cut Vertices and Degree One Vertices of Zero-Divisor Graphs. Communications in Algebra 40(8), 2749-2756. https://doi.org/10.1080/00927872.2011.585192.
23. Visweswaran, S. (2011). Some results on the complement of the zero-divisor graph of a commutative ring. Journal of Algebra and its Applications, 10(3), 573-595. https://doi.org/10.1142/S0219498811004781.
24. Waphare, B. N., & Khairnar, A. (2015). Semi-Baer modules. Journal of Algebra and its Applications, 14(10), 1550145(12 pages). https://doi.org/10.1142/S0219498815501455.

[1] Department of Mathematics, MES Abasaheb Garware College (Autonomous), Pune, India.
Email address: anil_maths2004@yahoo.com

[2] Department of Mathematics, MES Abasaheb Garware College (Autonomous), Pune, India.
Email address: anita7783@gmail.com