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FULLY COUPLED MCKEAN-VLASOV FORWARD-BACKWARD SDES DRIVEN BY MIXED BROWNIAN MOTIONS

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ABSTRACT. In this paper, we prove the existence and uniqueness of a solution for a general McKean-Vlasov fully coupled forward-backward stochastic differential equation driven by mixed Brownian motions, where the coefficients depend, nonlinearly, on both the state process as well as of its probability law. Using the continuation method, we establish our results under suitable conditions.

Keywords. Mixed Brownian motion, fractional Itô formula, Malliavin derivative, McKean-Vlasov FBSDEs
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1. INTRODUCTION

The study of forward-backward stochastic differential equations (FBSDEs) began in the context of stochastic optimal control, where they appeared as adjoint equations. Bismut [7] pioneered this field by introducing the first decoupled linear FBSDEs system. Nonlinear FBSDEs were later developed by Antonelli [2], who proved the existence and uniqueness of solutions to FBSDEs for small time intervals using a fixed-point theorem, while also examining the stability properties of these solutions under perturbations of the coefficients. For arbitrary time horizons, three fundamental approaches have been established to analyze the solvability of fully coupled FBSDEs systems. The first is introduced by Ma et al. [27], who investigated the adapted solutions to a class of non-degenerate FBSDEs. The authors demonstrated that solutions can be obtained over any time interval using a direct four-step scheme. Later, Delarue [17] developed the approach obtained in [27]. Hu and Peng [22] first proposed the continuation method. This methodology was subsequently extended by Yong [39], who studied this kind of system via the continuation method, but retained coefficient monotonicity and further refined in 1999 by Peng and Wu [35]. Pardoux and Tang [34] proposed the third approach, based on contraction mapping principles. For more general

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non-Markovian FBSDEs where these methods fail, Ma et al. [28] introduced a unified framework to establish solvability.

The real motivation for studying the McKean-Vlasov stochastic differential equations (MV-SDEs) comes from their connection to the Vlasov kinetic equation in plasma dynamics (see Kac [26]). McKean [31] later introduced a probabilistic counterpart to this equation. Buckdahn et al. [8, 9] conducted pioneering work on nonlinear McKean-Vlasov backward stochastic differential equations (MV-BSDEs). Since then, MV-BSDEs have been extensively studied by various researchers. For instance, Carmona and Delarue [12] employed the method of continuation, to study optimal control problems involving McKean-Vlasov diffusion processes. Meanwhile, Bensoussan et al. [4] established the existence and uniqueness of solutions to McKean-Vlasov forward-backward stochastic differential equations (MV-FBSDEs) under a general monotonicity conditions. These equations have attracted significant attention in recent literature (see, e.g., Bensoussan et al. [5], Carmona & Delarue [11], Chassagneux et al. [13], Chen et al. [15, 14], Tian & Yu [38] and the references mentioned therein).

However, fractional Brownian motion (FrBm) has been widely employed in various applied fields, including mathematical finance, due to its distinctive properties. Pioneering work by Hurst [24, 25] and Mandelbrot & Van Ness [29]) established the foundation for these applications. In particular, FrBm exhibits several remarkable characteristics: it is neither a semimartingale nor a Markov process when $H \neq \frac{1}{2}$, while maintaining the valuable property of self-similarity. These unique features have made fBm particularly useful in modeling complex systems where such properties are essential. Building on the foundational work on FrBm, Bender [3] pioneered the study of fractional linear BSDEs (Fr-LBSDEs) with a Hurst parameter $H > \frac{1}{2}$. Shortly afterward, Hu & Peng [23] conducted the first investigation of nonlinear fractional BSDEs (or, for short, Fr-BSDEs), proving the existence and uniqueness of solutions for arbitrary Hurst parameter $H > \frac{1}{2}$. Their approach leveraged semi-conditional expectation to circumvent the lack of a martingale representation theory for FrBm. Maticiuc & Nie [30] later developed a rigorous analytical framework, establishing general theoretical results for Fr-BSDEs. Fei [19] extended Hu & Peng [23] findings to BSDEs driven by both fractional and standard Brownian motions (SFr-BSDEs). Meanwhile, Han & Wang [20] explored optimal control systems governed by mixed Brownian motions by combining the Martingale representation theory and the properties of operator fractional Brownian motions. In 2017 Buckdahn & Jing [10] investigated fractional MV-SDE in the setting of a Hurst parameter greater than $\frac{1}{2}$. More recently, Sin et al. [36] used the continuation method to prove the existence and uniqueness of a solution to McKean-Vlasov Fr-FBSDE, where the coefficients involved depend not only on the solutions (X, Y) but also on the distribution $\mathbb{P}_{(X,Y)}$. Subsequently, Sin et al. [37] generalized these results, establishing the existence and uniqueness of solutions to MV-FBSDEs, whose coefficients do not depend on the solution (X, Y, Z) and also on its distribution. In this paper, we extend the study of MV-FBSDEs to the setting of mixed Brownian motion, introducing new analytical challenges and opportunities.

The main motivation for studying McKean-Vlasov SFr-FBSDEs stems from the stochastic maximum principle for the stochastic control problem of systems driven by FrBm. In such problems, the adjoint equation naturally involves by both FrBm and standard Brownian motions. To derive a complete stochastic maximum principle, it becomes essential to establish the existence and uniqueness of solutions for this class of MV-FBSDEs.

Our main contribution in this paper: By the method of continuation we prove the existence and uniqueness of solutions to the following fully-coupled FBSDEs driven by both standard and FrBm with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ (SFr-FBSDEs) of McKean-Vlasov type

$$\begin{cases} dX(t) = \kappa(t, X(t), Y(t), Z_1(t), Z_2(t), \mathbb{P}_{(X(t),Y(t),Z_1(t),Z_2(t))})dt \\ +\sigma_1(t, X(t), Y(t), Z_1(t),\ \mathbb{P}_{(X(t),Y(t),Z_1(t))})dB(t) \\ +\sigma_2(t, X(t), Y(t), Z_2(t), \mathbb{P}_{(X(t),Y(t),Z_2(t))})dB^H(t), \\ -dY(t) = F(t, \eta(t), X(t), Y(t), Z_1(t), Z_2(t), \mathbb{P}_{(X(t),Y(t),Z_1(t),Z_2(t))})dt \\ -Z_1(t)dB(t) - Z_2(t)dB^H(t), \\ X(0) = x_0, \quad Y(T) = H(X(T), \mathbb{P}_{X(T)}), \quad t \in [0, T], \end{cases}$$

where $x_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P})$, X, Y, Z_1 and Z_2 take values in $\mathbb{R}$, which contains a special structure: $\eta(t) = \eta_0 + g(t) + \int_0^t \zeta(s)dB(s) + \int_0^t \beta(s)dB^H(s)$ under conditions inspired by Hu & Peng [22]. Compared to [36, 37], our MV-FBSDEs involves two forward stochastic integrals, one with respect to fractional Brownian motion and one with respect to standard Brownian motion and the functions κ, σ_1, σ_2, F and H include the joint distribution $\mathbb{P}_{(X(t),Y(t),Z_1(t),Z_2(t))}$, $\mathbb{P}_{(X(t),Y(t),Z_1(t))}$, $\mathbb{P}_{(X(t),Y(t),Z_2(t))}$, $\mathbb{P}_{(X(t),Y(t),Z_1(t),Z_2(t))}$, $\mathbb{P}_{X(T)}$, respectively, making our MV-FBSDEs more general.

The stochastic systems influenced by mixed Brownian motions studied in this work are highly interesting. Their ability to model long-range dependence and non-Markovian dynamics in applications like differential games motivates our subsequent application to McKean-Vlasov LQ Stackelberg differential games under an adapted open-loop information structure.

The remainder of the paper is structured as follows: In Section 2, we introduce key notations and preliminary results for FrBM calculus, including the fractional Itô formula for mixed Brownian motion. In Section 3 establishes the existence and uniqueness of solutions to McKean-Vlasov SFr-FBSDEs using the method of continuation.

2. Preliminaries and General Framework

The study of stochastic integrals with respect to FrBm has seen increasingly significant development and progress, finding numerous applications across various fields (see, e.g., Alos [1], Biagini [6], Decreusefond [16], Duncan [18], Hu [21], Mishura & Veretennikov [32] and Nualart & Saussereau [33]).

Consider a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ such that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is the $\sigma-$algebra generated by a fractional Brownian motion $B^H =$

$\{B^H(t), t \geq 0\}$ and a standard Brownian motion $B = \{B(t), t \geq 0\}$. Throughout this paper, we fix the Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ arbitrarily, the notation $|\cdot|$ denotes the absolute value, and $\langle \cdot, \cdot \rangle$ represents the scalar product.

Let ϕ be a function defined on $[0,T] \times [0.T]$ and taking values in $\mathbb{R}_+$, given by

$$\phi(s,t) = H(2H-1)|s-t|^{2H-2}.$$

The fractional Brownian motion B^H satisfies the following properties

$$\mathbb{E}\left(B^H(t)\right) = 0, \qquad \mathbb{E}\left(B^H(t) B^H(s)\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} + |t-s|^{2H}\right)$$

Let θ and η be two continuous functions on $[0,T]$. We define a Hilbert space scalar product $\langle \cdot, \cdot \rangle_{\phi,[0,t]}$ acting on these functions as follows

$$\langle \theta, \eta \rangle_{\phi,[0,t]} = \int_0^t \int_0^t \theta(x) \eta(y) \phi(x,y) \, dx dy, \;\; \|\eta\|_{\phi,[0,t]} = \langle \eta, \eta \rangle_{\phi,[0,t]}.$$

Let $\mathcal{H}$ be the completion of the continuous functions under the norm $\|\cdot\|_{\phi,[0,t]}$ and consider a sequence $(\theta_n)_n$ in $\mathcal{H}$ such that

$$\langle \theta_i, \theta_j \rangle_{\phi,[0,T]} = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol. The space $\mathbb{L}^2([0,T])$ is continuously embedded in $\mathcal{H}$, as shown in (Biagini et al. [6]).

We denote by $\mathcal{P}_T$ the set of all polynomials of an fractional Brownian motion over interval $[0,T]$, i.e., consisting of all elements expressible in the folloowing form

$$F(\omega) = G\left(\int_0^T \theta_1(s) \, dB_s^H, \cdots, \int_0^T \theta_n(s) \, dB_s^H\right),$$

where G is a polynomial function of n variables. The Malliavin derivative operator $\mathcal{D}_t^{\mathcal{H}}$ of a polynomial functional $F = G\left(\int_0^T \theta_1(s) \, dB_s^H, \cdots, \int_0^T \theta_n(s) \, dB_s^H\right)$ is defined as follows

$$\mathcal{D}_t^{\mathcal{H}} F = \sum_{i=1}^n \frac{\partial G}{\partial x_i}\left(\int_0^T \theta_1(s) \, dB_s^H, \cdots, \int_0^T \theta_n(s) \, dB_s^H\right) \theta_i(t), \;\; t \in [0,T].$$

Similarly, the Mallaivin derivative $\mathcal{D}_t V$ of the Brownian functional can be defined as follow

$$V(\omega) = G\left(\int_0^T \theta_1(s) \, dB_s, \cdots, \int_0^T \theta_n(s) \, dB_s\right).$$

Since the divergence operator $\mathcal{D}^{\mathcal{H}} \colon \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathcal{H})$ is closable, we can consider the space $\mathbb{D}_{1,2}$ be the completion of $\mathcal{P}_T$ with the norm

$$\|G\|_{1,2}^2 = \mathbb{E}|G|^2 + \mathbb{E}\left\|\mathcal{D}^{\mathcal{H}} G\right\|_T^2.$$

Now, we introduce the Malliavin $\phi-$derivative $\mathcal{D}^{\mathcal{H}}$ of G by

$$\mathcal{D}_t^{\phi} G = \int_0^T \mathcal{D}_s^{\mathcal{H}} G \cdot \phi(s,t) \, ds.$$

Before stating our principal result, we define the following essential sets:

$$\mathbb{L}_{[0,T]}^{1,2} \colon = \left\{\zeta : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathcal{H} | \;\; \zeta \in \mathbb{L}^2(\Omega, [0,T]), \; \mathbb{E}\left(\|\zeta\|_{\phi,[0,T]}^2 + \int_0^T \int_0^T \left|\mathcal{D}_s^{\phi} \zeta(t)\right| dsdt\right) < +\infty\right\}.$$

$\mathcal{C}_{pol}^{1,3}([0,T]\times\mathbb{R}):=\{\Lambda\in\mathcal{C}^{1,3}([0,T]\times\mathbb{R})|$ all derivatives of Λ are of polynomial growth$\}$.
$\mathcal{V}_{[0,T]}:=\left\{Y(\cdot)=\Lambda(\cdot,\eta)|\ \Lambda\in\mathcal{C}_{pol}^{1,3}([0,T]\times\mathbb{R})\text{ with }\frac{d\Lambda}{dt}\text{ is bounded, }t\in[0,T]\right\}$.

We denote by $\tilde{\mathbb{L}}_{[0,T]}^{1,2}$ and $\tilde{\mathcal{V}}_{[0,T]}$ the completion of $\mathbb{L}_{[0,T]}^{1,2}$ and $\mathcal{V}_{[0,T]}$ respectively under the following norm

$$||Y||_{[0,T]}:=\left(\mathbb{E}\int_0^T|Y(t)|^2\,dt\right)^{\frac{1}{2}}<+\infty.$$

Furthermore, we denote by $\tilde{\mathcal{V}}_{[0,T]}^{1,2}$ the completion of $\mathcal{V}_{[0,T]}$ with respect to the following norm

$$||Z||^2_{\mathbb{L}^{1,2}_{[0,T]}}:=\mathbb{E}\left(\int_0^T|Z(t)|^2\,dB_t^H\right)^2=\mathbb{E}\left(\|Z\|^2_{\phi,[0,T]}+\int_0^T\int_0^T\mathcal{D}_s^{\phi}Z(t)\,\mathcal{D}_t{}^{\phi}Z(s)\,dsdt\right)<+\infty.$$

Now, for any two functions θ, $\zeta\in\mathbb{L}^{1,2}_{[0,T]}$, we define the following inner product

$$\begin{aligned}\langle\theta,\zeta\rangle_{\mathbb{L}^{1,2}_{[0,T]}}:&=\mathbb{E}\left(\int_0^T\theta(t)\,dB_t^H\cdot\int_0^T\zeta(t)\,dB_t^H\right)\\&=\mathbb{E}\left(\int_0^T\int_0^T\theta(s)\cdot\zeta(t)\,\phi(s,t)\,dsdt+\int_0^T\int_0^T\mathcal{D}_t^{\phi}\theta(s)\,\mathcal{D}_s^{\phi}\zeta(t)\,dsdt\right).\end{aligned}$$

We now present the fractional Itô chain rule for stochastic differential equations driven by a combination of standard Brownian motion and fractional Brownian motion (see, Fei [19], [Theorem 3.2]).

Lemma 2.1. *Let $g_i\in\tilde{\mathbb{L}}^{1,2}_{[0,T]}$, $h_i\in\tilde{\mathbb{L}}^{1,2}_{[0,T]}$ and $f_i\in\tilde{\mathcal{V}}^{1,2}_{[0,T]}$, $i=1,2$. And denote*

$$X_i(t)=X_i(0)+\int_0^t g_i(s)ds+\int_0^t h_i(s)dB_s+\int_0^t f_i(s)dB_s^H,\quad t\in[0,T].$$

Then $X_i\in\tilde{\mathbb{L}}^{1,2}_{[0,T]}$ and

$$\mathbb{E}\left(\int_0^T X_2(s)f_1(s)dB_s^H\right)=0,\quad\mathbb{E}\left(\int_0^T X_1(s)f_2(s)dB_s^H\right)=0.$$

Moreover

$$\begin{aligned}\mathbb{E}(X_1(T)X_2(T))&=\mathbb{E}(X_1(0)X_2(0))+\mathbb{E}\left(\int_0^T X_1(s)g_2(s)ds\right)+\mathbb{E}\left(\int_0^T X_2(s)g_1(s)ds\right)\\&\quad+\mathbb{E}\left(\int_0^t\left(h_1(s)\,\mathcal{D}_sX_2(s)\,ds+h_2(s)\,\mathcal{D}_sX_1(s)\right)ds\right)+\langle f_1,f_2\rangle_{\mathbb{L}^{1,2}_{[0,T]}}.\end{aligned}$$

Next, we provide the following lemma which plays an important role to our subsequent analysis.

Lemma 2.2. *Let $\Psi\in\mathbb{L}^{1,2}_{[0,T]}$ and $\Xi(t)=\int_0^t\mathbb{E}(\Psi(s))\,ds$. Then $\mathbb{E}(\Psi)\in\mathbb{L}^{1,2}_{[0,T]}$ and $\Xi\in\mathbb{L}^{1,2}_{[0,T]}$.*

Proof. Using the fact that $\mathcal{D}_t^{\phi}\mathbb{E}\left(\Psi\left(s\right)\right)=0$ and $\Psi\in\mathbb{L}_{[0,T]}^{1,2}$, we obtain

$$\begin{aligned}\left\|\mathbb{E}\left(\Psi\right)\right\|_{\mathbb{L}_{[0,T]}^{1,2}} &= \mathbb{E}\left(\left\|\mathbb{E}\left(\Psi\right)\right\|_{\phi,[0,T]}+\int_0^T\int_0^T\mathcal{D}_t^{\phi}\mathbb{E}\left(\Psi\left(s\right)\right)\mathcal{D}_s^{\phi}\mathbb{E}\left(\Psi\left(t\right)\right)dsdt\right)\\ &= \int_0^T\int_0^T\mathbb{E}\left(\Psi\left(s\right)\right)\mathbb{E}\left(\Psi\left(t\right)\right)\phi\left(s,t\right)dsdt\\ &\leq C\int_0^T\int_0^T\phi\left(s,t\right)dsdt\\ &= H\left(2H-1\right)C\int_0^T\int_0^T\left|s-t\right|^{2H-2}dsdt<+\infty.\end{aligned}$$

Therefore $\mathbb{E}\left(\Psi\right)\in\mathbb{L}_{[0,T]}^{1,2}$. Employing an analogous methodology to the preceding analysis, we can derive $\Xi\in\mathbb{L}_{[0,T]}^{1,2}$. □

Now, we denote by $\mathcal{M}(\mathbb{R}^{2d})$ the space of all probability measures π with finite second-order moment over an Euclidean space $(\mathbb{R}^{2d},\mathcal{B}(\mathbb{R}^{2d}))$ for some $d\in\mathbb{N}$. In particular, we endow the space $\mathcal{M}(\mathbb{R}^{2d})$ with the following $2-$Wasserstein distance defined by

$$\mathcal{W}\left(\pi_1,\pi_2\right):=\inf_{\pi\in\Pi(\pi_1,\pi_2)}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^2\pi(dx,dy)\right)^{\frac{1}{2}},$$

where $\Pi(\pi_1,\pi_2)$ denotes the set of all coupling measures; that is, the probability measures $\pi\in\mathcal{M}(\mathbb{R}^{2d})$ with marginals $\pi_1,\pi_2\in\mathcal{M}(\mathbb{R}^d)$ (i.e., $\pi_1{:}\ =\mathbb{P}\circ X_1^{-1}$ and $\pi_2{:}\ =\mathbb{P}\circ X_2^{-1}$).

Lemma 2.3. *In the case X_1 and X_2 are $d-$dimensional random vectors with distributions $\pi_1=\mathbb{P}_{X_1}$ and $\pi_2=\mathbb{P}_{X_2}$, we have*

$$\left|\mathbb{E}\left(X_1\right)-\mathbb{E}\left(X_2\right)\right|\leq\mathcal{W}(\pi_1,\pi_2)\leq\sqrt{\mathbb{E}\left|X_1-X_2\right|^2},$$

Proof. For the left hand inequality: We have

$$\mathbb{E}\left(X_1\right)-\mathbb{E}\left(X_2\right)=\mathbb{E}\left(X_1-X_2\right),$$

now, take the absolute value and apply the triangle inequality, we obtain

$$\begin{aligned}\left|\mathbb{E}\left(X_1\right)-\mathbb{E}\left(X_2\right)\right| &\leq \mathbb{E}\left(|X_1-X_2|\right)\\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d}|x_1-x_2|d\left(\mathbb{P}_{(X_1,Y_2)}(x_1,x_2)\right)\\ &\leq \inf_{\pi\in\Pi(\pi_1,\pi_2)}\int_{\mathbb{R}^d\times\mathbb{R}^d}|x_1-x_2|\pi(dx_1,dx_2),\end{aligned}$$

Using the following standard result related to the Wasserstein distance

$$\inf_{\pi\in\Pi(\pi_1,\pi_2)}\int_{\mathbb{R}^d\times\mathbb{R}^d}|x_1-x_2|\pi(dx_1,dx_2)\leq\mathcal{W}\left(\pi_1,\pi_2\right),$$

we find the proof of the required inequality.

For the right hand inequality: By a direct consequence of the definition, we get

$$\begin{aligned}\mathcal{W}(\pi_1,\pi_2)^2 &= \inf_{\pi\in\Pi(\pi_1,\pi_2)}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}|x_1-x_2|^2\pi(dx_1,dx_2)\right)\\ &\le \int_{\mathbb{R}^d\times\mathbb{R}^d}|x_1-x_2|^2 d\left(\mathbb{P}_{(X_1,Y_2)}(x_1,x_2)\right)\\ &= \mathbb{E}\left|X_1-X_2\right|^2.\end{aligned}$$

And with this we which achieves the proof of this lemma. □

With these preliminaries established, we can now state our principal result.

3. Existence and uniqueness of solution for McKean-Vlasov SFr-FBSDEs

This paper investigates the solvability of the following fully coupled McKean-Vlasov SFr-FBSDE

$$\begin{cases} dX(t)=\kappa(t,X(t),Y(t),Z_1(t),Z_2(t),\mathbb{P}_{(X(t),Y(t),Z_1(t),Z_2(t))})dt\\ +\sigma_1(t,X(t),Y(t),Z_1(t),\ \mathbb{P}_{(X(t),Y(t),Z_1(t))})dB(t)\\ +\sigma_2(t,X(t),Y(t),Z_2(t),\mathbb{P}_{(X(t),Y(t),Z_2(t))})dB^H(t),\\ -dY(t)=F(t,\eta(t),X(t),Y(t),Z_1(t),Z_2(t),\mathbb{P}_{(X(t),Y(t),Z_1(t),Z_2(t))})dt\\ -Z_1(t)dB(t)-Z_2(t)dB^H(t),\\ X(0)=x_0,\quad Y(T)=H(X(T),\mathbb{P}_{X(T)}),\quad t\in[0,T]\,,\end{cases}\tag{3.1}$$

where $x_0\in\mathbb{L}^2(\Omega,\mathcal{F}_0,\mathbb{P})$ and $\kappa, F, \sigma_1,\sigma_2, H$ are functions such as

$$\begin{aligned}\kappa &: [0,T]\times\mathbb{R}^4\times\mathcal{M}(\mathbb{R}^4)\to\mathbb{R},\\ F &: [0,T]\times\mathbb{R}^5\times\mathcal{M}(\mathbb{R}^4)\to\mathbb{R},\\ \sigma_1 &: [0,T]\times\mathbb{R}^3\times\mathcal{M}(\mathbb{R}^3)\to\mathbb{R},\\ \sigma_2 &: [0,T]\times\mathbb{R}^3\times\mathcal{M}(\mathbb{R}^3)\to\mathbb{R},\\ H &: \mathbb{R}\times\mathcal{M}(\mathbb{R})\to\mathbb{R},\end{aligned}$$

and

$$\eta(t)=\eta_0+g\,(t)+\int_0^t\zeta(s)dB(s)+\int_0^t\beta(s)dB^H(s),\quad t\in[0,T],\tag{3.2}$$

where η_0 is a constant and $g(t)$, $\zeta(s)$, $\beta(s)$ be deterministic constants or functions.

We are now going to present a definition for the fully coupled McKean-Vlasov FBSDEs driven by mixed Brownian motion.

Definition 3.1. The quadruple process $\{(X(t),Y(t),Z_1(t),Z_2(t)),t\in[0,T]\}$ is called a solution for McKean-Vlasov SFr-FBSDEs (3.1) if $(X,Y,Z_1,Z_2)\in\tilde{\mathbb{L}}^{1,2}_{[0,T]}\times\tilde{\mathbb{L}}^{1,2}_{[0,T]}\times\tilde{\mathbb{L}}^{1,2}_{[0,T]}\times\tilde{\mathcal{V}}^{1,2}_{[0,T]}$ and the quadruple also satisfies system (3.1), $\mathbb{P}$-almost surely.

To ensure clarity throughout this work, we adopt the following notational conventions:

$$
\begin{aligned}
&U: \ = (X, Y, Z_1, Z_2)\,,\\
&A(s,\eta(s), X(s), Y(s), Z_1(s), Z_2(s), \mathbb{P}_{X(s),Y(s),Z_1(s),Z_2(s)})\\
&: \ = \begin{pmatrix} -F(s,\eta(s), X(s), Y(s), Z_1(s), Z_2(s), \mathbb{P}_{X(s),Y(s),Z_1(s),Z_2(s)}) \\ \kappa(s, X(s), X(s), Y(s), Z_1(s), Z_2(s), \mathbb{P}_{X(s),Y(s),Z_1(s),Z_2(s)}) \\ \sigma_1(s, X(s), Y(s), Z_1(s), \mathbb{P}_{X(s),Y(s),Z_1(s)}) \\ \sigma_2(s, X(s), Y(s), Z_2(s), \mathbb{P}_{X(s),Y(s),Z_2(s)}) \end{pmatrix} : \ = \begin{pmatrix} -\bar{F}(s) \\ \bar{\kappa}(s) \\ \bar{\sigma}_1(s) \\ \bar{\sigma}_2(s) \end{pmatrix},\\
&\langle A, U\rangle: \ = \mathbb{E}\left(\int_0^T \left(-\bar{F}(s)X(s) + \bar{\kappa}(s)Y(s) + \bar{\sigma}_1 Z_1\right) ds + (\bar{\sigma}_2, Z_2)_{\mathbb{L}^{1,2}_{[0,T]}}\right),\\
&\|A\|^2 := \|F\|^2_{[0,T]} + \|\kappa\|^2_{[0,T]} + \|\sigma_1\|^2_{[0,T]} + \|\sigma_2\|^2_{\mathbb{L}^{1,2}_{[0,T]}},\\
&\|U\|^2_{[0,T]} := \|X\|^2_{[0,T]} + \|Y\|^2_{[0,T]} + \|Z_1\|^2_{[0,T]} + \|Z_2\|^2_{\mathbb{L}^{1,2}_{[0,T]}},\\
&\|\mathbb{E}[U]\|^2_{[0,T]} := \|\mathbb{E}[X]\|^2_{[0,T]} + \|\mathbb{E}[Y]\|^2_{[0,T]} + \|\mathbb{E}[Z_1]\|^2_{[0,T]} + \|\mathbb{E}[Z_2]\|^2_{\phi,[0,T]},\\
&\|\mathcal{W}(\mathbb{P}_U, \mathbb{P}_{\acute{U}})\|^2_{[0,T]} := \|\mathcal{W}(\mathbb{P}_X, \mathbb{P}_{X'})\|^2_{[0,T]} + \|\mathcal{W}(\mathbb{P}_Y, \mathbb{P}_{Y'})\|^2_{[0,T]} + \|\mathcal{W}(\mathbb{P}_{Z_1}, \mathbb{P}_{Z_1'})\|^2_{[0,T]}\\
&+\|\mathcal{W}(\mathbb{P}_{Z_2}, \mathbb{P}_{Z_2'})\|^2_{\phi,[0,T]},\\
&\|U\|^2_{\mathcal{R}} := \|U\|^2_{[0,T]} + \|\mathbb{E}[U]\|^2_{[0,T]} + \mathbb{E}|X(T)|^2,
\end{aligned}
$$

where

$$
\begin{aligned}
&\|\mathcal{W}(\mathbb{P}_\Gamma, \mathbb{P}_{\acute{\Gamma}})\|^2_{[0,T]} := \int_0^T \mathcal{W}(\mathbb{P}_\Gamma(s), \mathbb{P}_{\acute{\Gamma}}(s))^2 ds, \quad \forall \Gamma \in \{X, Y, Z\}, \quad \text{and}\\
&\|\mathcal{W}(\mathbb{P}_{Z_2}, \mathbb{P}_{Z_2'})\|^2_{\phi,[0,T]} := \int_0^T \int_0^T \mathcal{W}(\mathbb{P}_{Z_2}(s), \mathbb{P}_{Z_2'}(s))\mathcal{W}(\mathbb{P}_{Z_2}(t), \mathbb{P}_{Z_2'}(t))\phi(s,t)dsdt.
\end{aligned}
$$

For the coefficients of McKean-Vlasov SFr-FBSDE (3.1), we impose the following conditions

Assumption A1. There exist two constants $\mu > 0$ and $K > 1$, for all $U, \acute{U} \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$

i): $\left\langle A - \acute{A}, U - \acute{U}\right\rangle \leq -\mu\left(\left\|U - \acute{U}\right\|^2_{[0,T]} + \|\mathcal{W}(\mathbb{P}_U, \mathbb{P}_{\acute{U}})\|^2_{[0,T]}\right).$

ii): $\left\|A - \acute{A}\right\|^2_{[0,T]} \leq K\left(\left\|U - \acute{U}\right\|^2_{[0,T]} + \left\|\mathbb{E}\left(U - \acute{U}\right)\right\|^2_{[0,T]}\right).$

Assumption A2. We assume that there exist two constants $\mu_1 > 0$, $K_1 > 1$ such that

$$
\begin{cases}
\mathbb{E}\left(\left(H(\xi, \mathbb{P}_\xi) - H(\bar{\xi}, \mathbb{P}_{\bar{\xi}'})\right)(\xi - \bar{\xi})\right) \geq \mu_1 \mathbb{E}|\xi - \bar{\xi}|^2,\\
\mathbb{E}|H(\xi, \mathbb{P}_\xi) - H(\bar{\xi}, \mathbb{P}_{\bar{\xi}})|^2 \leq K_1\left(\mathbb{E}|\xi - \bar{\xi}|^2 + \mathcal{W}(\mathbb{P}_\xi, \mathbb{P}_{\bar{\xi}})^2\right),
\end{cases}
$$

for all $\xi, \bar{\xi} \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

To prove the existence and uniqueness of solutions to the McKean-Vlasov SFr-FBSDE, consider the solution $(X^\lambda, Y^\lambda, Z_1^\lambda, Z_2^\lambda) \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$ for the following McKean-Vlasov SFr-FBSDE corresponding to the parameter

$\lambda \in [0,1]$

$$\begin{cases} dX(t) = \left[\lambda\kappa(t,U(t),\mathbb{P}_{U(t)}) - \mu(1-\lambda)(Y(t)+\mathbb{E}\left(Y\left(t\right)\right)) + \varphi(t)\right]dt \\ +\left(\lambda\sigma_1(t,X(t),Y(t),Z_1(t),\mathbb{P}_{(X(t),Y(t),Z_1(t))}) - \mu(1-\lambda)(Z_1(t)+\mathbb{E}\left(Z_1\left(t\right)\right)) + \Phi(t)\right)dB(t) \\ +\left(\lambda\sigma_2(t,X(t),Y(t),Z_2(t),\mathbb{P}_{(X(t),Y(t),Z_2(t))}) - \mu(1-\lambda)(Z_2(t)+\mathbb{E}\left(Z_2\left(t\right)\right)) + \psi(t)\right)dB^H(t), \\ -dY(t) = \left(\lambda F\left(t,\eta(t),U\left(t\right),\mathbb{P}_{U(t)}\right) + \mu(1-\lambda)(X(t)+\mathbb{E}\left(X_t\right)+\eta(t)) + \varsigma(t)\right)dt \\ -Z_1(t)dB(t) - Z_2(t)dB^H(t), \\ X(0) = x_0, \quad Y(T) = \lambda H(X(T),\mathbb{P}_{X(T)}) + (1-\lambda)X(T) + \xi, \quad t\in[0,T], \end{cases} \tag{3.3}$$

where $(\varphi,\varsigma,\Phi,\psi) \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$ and $\xi \in \mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{P})$. When $\varphi = \Phi = \psi = \varsigma = 0$ and $\lambda = 1$, it is obvious that the existence of solution of system (3.3) implies that of system (3.1). When $\lambda = 0$, McKean-Vlasov SFr-FBSDE (3.3) becomes the following linear McKean-Vlasov FBSDE driven by mixed Brownian motion

$$\begin{cases} dX(t) = \left[-\mu(Y(t)+\mathbb{E}\left(Y_t\right)) + \varphi(t)\right]dt + \left[-\mu\left(Z_1(t)+\mathbb{E}\left(Z_1\left(t\right)\right)\right) + \Phi\left(t\right)\right]dB(t) \\ +\left[-\mu\left(Z_2(t)+\mathbb{E}\left(Z_2\left(t\right)\right)\right) + \psi(t)\right]dB^H(t), \\ -dY(t) = \left[\mu(X(t)+\mathbb{E}\left(X_t\right)+\eta(t)) + \varsigma(t)\right]dt - Z_1(t)dB(t) - Z_2(t)dB^H(t), \\ X(0) = x_0, \quad Y(T) = X(T) + \xi, \quad t\in[0,T]. \end{cases} \tag{3.4}$$

Lemma 3.2. *Let $(\varphi,\phi,\psi) \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \mathbb{L}^{2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$. Then the following McKean-Vlasov stochastic differential equation driven by mixed Brownian motion*

$$\begin{cases} dX(t) = \left(-\mu(X(t)+\mathbb{E}\left(X_t\right)) + \varphi(t)\right)dt + \Phi\left(t\right)dB(t) + \psi(t)dB^H(t), \\ X(0) = x_0, \quad t\in[0,T], \end{cases}$$

admits a solution $X \in \tilde{\mathbb{L}}^{1,2}_{[0,T]}$.

Proof. As the triplet $(\varphi,\Phi,\psi) \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$, there exist a sequence $(\varphi^n,\Phi^n,\psi^n) \in \mathbb{L}^{1,2}_{[0,T]} \times \mathbb{L}^{2}_{[0,T]} \times \mathcal{V}^{1,2}_{[0,T]}$, $n\in\mathbb{N}$, $(\varphi^n,\Phi^n,\psi^n)$ that converges to (φ,Φ,ψ) in $\tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$. Then φ^n,Φ^n,ψ^n has the form $\varphi^n(\cdot) = \varsigma^n(\cdot,\eta(\cdot))$, $\Phi^n(\cdot) = \varkappa^n(\cdot,\eta(\cdot))$ and $\psi^n(\cdot) = \rho^n(\cdot,\eta(\cdot))$, where η is the process specified in equation (3.2).
Now, with $x \in \mathbb{L}^{1,2}_{[0,T]}$, we examine the following McKean-Vlasov stochastic integral equation with mixed Brownian motion

$$X(t) = x_0 + \int_0^t \left(-\mu(x(s)+\mathbb{E}\left(x\left(s\right)\right)) + \varphi^n(s)\right)ds + \int_0^t \Phi^n\left(s\right)dB(s) + \int_0^t \psi^n(s)dB^H(s),$$

the results of Lemma 2.1 and Lemma 2.2 guarantee that $X \in \mathbb{L}^{1,2}_{[0,T]}$, we shall demonstrate that the mapping defined by

$$\begin{array}{lll} \Theta: & \mathbb{L}^{1,2}_{[0,T]} & \to \mathbb{L}^{1,2}_{[0,T]} \\ & x & \to X, \end{array}$$

constitutes a strict contraction on the space $X \in \mathbb{L}^{1,2}_{[0,T]}$. Setting $\Theta x' := X'$, $\hat{x} := x - x'$ and $\hat{X} := X - X'$, we get

$$\hat{X}(t) = \int_0^t \left(-\mu(\hat{x}(s) + \mathbb{E}\left(\hat{x}\left(s\right)\right)\right) ds, \quad 0 \leq t \leq T.$$

By applying Itô's formula to the process $e^{-\varrho t}\hat{X}(t)^2$, yields

$$\mathbb{E}\left(e^{-\varrho T}\hat{X}(T)^2\right) \leq -\frac{\varrho}{2}\int_0^T e^{-\varrho s}\mathbb{E}\left(\hat{X}(s)\right)^2 ds + \frac{4\mu^2}{\varrho}\int_0^T e^{-\varrho s}\mathbb{E}\left(\hat{x}(s)\right)^2 ds.$$

Therefore

$$\int_0^T e^{-\varrho s}\mathbb{E}\left(\hat{X}(s)\right)^2 ds \leq \frac{8\mu^2}{\varrho^2}\int_0^T e^{-\varrho s}\mathbb{E}\left(\hat{x}(s)\right)^2 ds,$$

by selecting the constant $\varrho > 0$ sufficiently large, we establish that the mapping Θ constitutes a strict contraction on the space $\mathbb{L}^{1,2}_{[0,T]}$.

Now, we construct a sequence of processes $X^n \in \mathbb{L}^{1,2}_{[0,T]}$, $n \in \mathbb{N}$ recursively by choosing $X^1 \in \mathbb{L}^{1,2}_{[0,T]}$, and by defining $X^{n+1} \in \mathbb{L}^{1,2}_{[0,T]}$ through the McKean-Vlasov stochastic integral equation driven by mixed Brownian motion

$$X^{n+1}(t) = x_0 + \int_0^t [-\mu(X^n(s) + \mathbb{E}\left(X^n(s)\right)) + \varphi^n(s)]ds + \int_0^t \Phi^n\left(s\right)dB(s) + \int_0^t \psi^n(s)dB^H(s).$$

Since the operator Θ is contractive, we may conclude that $X^n \in \mathbb{L}^{1,2}_{[0,T]}$, $n \in \mathbb{N}$ is a Cauchy sequence in $\mathbb{L}^{1,2}_{[0,T]}$ and there exists $X \in \mathbb{L}^{1,2}_{[0,T]}$. Therefore the sequence X^n converges to some X in $\mathbb{L}^{1,2}_{[0,T]}$ which satisfies

$$X(t) = x_0 + \int_0^t [-\mu(X(s) + \mathbb{E}\left(X(s)\right)) + \varphi(s)]ds + \int_0^t \Phi\left(s\right)dB(s) + \int_0^t \psi(s)dB^H(s).$$

The proof is completed. □

By leveraging the results of the previous lemmas, we can now demonstrate the following lemma.

Lemma 3.3. *Let* $(\varphi, \varsigma, \Phi, \psi) \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$ *and* $\xi \in \mathbb{L}^2\left(\Omega, \mathcal{F}_T, \mathbb{P}\right)$. *Then, there exists a unique solution* $(X, Y, Z_1, Z_2) \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$ *to the McKean-Vlasov SFr-FBSDE* (3.3).

Proof. To establish our main results, we first address the following linear McKean-Vlasov SFr-BSDE

$$\begin{cases} -d\varkappa(t) = [-\mu(\varkappa(t) + \mathbb{E}\left(\varkappa\left(t\right)\right) - \eta(t)) + \varphi(t) + \varsigma(t)]dt + \acute{q}_1(t)dB(t) + \acute{q}_2(t)dB^H(t), \\ \varkappa(T) = \xi, \quad t \in [0, T]\,. \end{cases} \tag{3.5}$$

Consequently, the system (3.5) admits a unique solution $(\varkappa, \acute{q}_1, \acute{q}_2) \in \tilde{\mathcal{V}}_{[0,T]} \times \tilde{\mathcal{V}}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$ (see Fei et al. [19]). Furthermore, we obtain $(q_1, q_2) \in \tilde{\mathcal{V}}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$ satisfying

$$q_1'(t) = \Phi(t) - (1+\mu)q_1(t) - \mu\left(\mathbb{E}\left(q_1\left(t\right)\right)\right), \tag{3.6}$$

$$q_2'(t) = \psi(t) - (1+\mu)q_2(t) - \mu\left(\mathbb{E}\left(q_2\left(t\right)\right)\right) \tag{3.7}$$

Indeed, from (3.7), we get $(1+\mu)q_2(t) = -q_2'(t) - \mu\mathbb{E}\left(q_2\left(t\right)\right) + \psi(t)$. We now take the expectation on both sides, which gives

$$\mathbb{E}\left(q_2\left(t\right)\right) = \frac{1}{(1+2\mu)}(\mathbb{E}\left(\psi(t)\right) - \mathbb{E}\left(\acute{q}_2\left(t\right)\right)).$$

Substituting the expression for $\mathbb{E}\left(q_2\left(t\right)\right)$ into (3.7), we derive

$$(1+\mu)q_2(t) = \psi(t) - q_2'(t) - \frac{\mu}{(1+2\mu)}\left(\mathbb{E}\left(\psi(t)\right) - \mathbb{E}\left(\acute{q}_2\left(t\right)\right)\right).$$

Following similar arguments, we derive equation (3.6). Consequently, we establish that $\left(q_1\left(t\right), q_2\left(t\right)\right) \in \tilde{\mathcal{V}}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$ the triple (p, q_1, q_2) constitutes a solution to the following McKean-Vlasov SFr-BSDE

$$\begin{cases} -d\varkappa(t) = [-\mu(\varkappa(t) + \mathbb{E}\left(\varkappa\left(t\right)\right) - \eta(t)) + \varphi(t) + \varsigma(t)]dt \\ +[\phi(t) - (1+\mu)q_1(t) - \mu\mathbb{E}\left(q_1\left(t\right)\right)]dB(t) + [\psi(t) - (1+\mu)q_2(t) - \mu\mathbb{E}\left(q_2\left(t\right)\right)]dB^H(t), \\ \varkappa(T) = \xi, \quad t \in [0,T]. \end{cases} \tag{3.8}$$

Now, proceeding with the solution $(\varkappa, q_1, q_2) \in \tilde{\mathcal{V}}_{[0,T]} \times \tilde{\mathcal{V}}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$ to equation (3.8), we now consider the following McKean-Vlasov SFr-SDE

$$\begin{cases} dX(t) = [-\mu\left[X(t) + \mathbb{E}\left(X\left(t\right)\right) + \varkappa(t) + \mathbb{E}\left(\varkappa\left(t\right)\right)\right] + \varphi(t)]dt \\ +[-\mu(q_1(t) + \mathbb{E}\left(q_1\left(t\right)\right)) + \Phi(t)]dB(t) + [-\mu(q_2(t) + \mathbb{E}\left(q_2\left(t\right)\right)) + \psi(t)]dB^H(t), \\ X(0) = x_0, \quad t \in [0,T]. \end{cases} \tag{3.9}$$

By Lemma 3.2, equation (3.9) admits a solution $X \in \tilde{\mathbb{L}}^{1,2}_{[0,T]}$. Putting

$$(X(\cdot), Y(\cdot), Z_1(\cdot), Z_2\left(\cdot\right)) := (X(\cdot), X(\cdot) + \varkappa(\cdot), q_1(\cdot), q_2\left(\cdot\right)),$$

we consequently obtain that the solution $(X(\cdot), Y(\cdot), Z_1(\cdot), Z_2\left(\cdot\right))$ belong in the space $\tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$ and

$$\begin{cases} dX(t) = \left(-\mu\left[X(t) + \mathbb{E}\left(X\left(t\right)\right) + \varkappa(t) + \mathbb{E}\left(\varkappa\left(t\right)\right)\right] + \varphi(t)\right)dt \\ +[-\mu(q_1(t) + \mathbb{E}\left(q_1\left(t\right)\right)) + \Phi(t)]dB(t) + [-\mu(q_2(t) + \mathbb{E}\left(q_2\left(t\right)\right)) + \psi(t)]dB^H(t) \\ = [-\mu(Y(t) + \mathbb{E}\left(Y\left(t\right)\right)) + \varphi(t)]dt + [-\mu(Z_1(t) + \mathbb{E}\left(Z_1\left(t\right)\right)) + \Phi(t)]dB(t) \\ +[-\mu(Z_2(t) + \mathbb{E}\left(Z_2\left(t\right)\right)) + \psi(t)]dB^H(t), \\ X(0) = x_0, \quad t \in [0,T], \end{cases}$$

and

$$\begin{cases} dY(t) = \left(-\mu\left[X(t) + \mathbb{E}\left(X\left(t\right)\right) + \varkappa(t) + \mathbb{E}\left(\varkappa\left(t\right)\right)\right] + \varphi(t)\right)dt \\ +[-\mu(q_1(t) + \mathbb{E}\left(q_1\left(t\right)\right)) + \Phi(t)]dB(t) + [-\mu(q_2(t) + \mathbb{E}\left(q_2\left(t\right)\right)) + \psi(t)]dB^H(t) \\ +[\mu\left(\varkappa(t) + \mathbb{E}\left(\varkappa\left(t\right)\right)\right) - \eta(t)) - \varphi(t) - \varsigma(t)]dt \\ -[\Phi(t) - (1+\mu)q_1(t) - \mu\mathbb{E}\left(q_1\left(t\right)\right)]dB(t) - [\psi(t) - (1+\mu)q_2(t) - \mu\mathbb{E}\left(q_2\left(t\right)\right)]dB^H(t) \\ = [-\mu(X(t) + \mathbb{E}\left(X\left(t\right)\right) + \eta(t)) - \varsigma(t)]dt + q_1(t)dB(t) + q_2(t)dB^H(t), \\ Y(T) = X(T) + \varkappa(T) = X(T) + \xi, \quad t \in [0,T]. \end{cases}$$

Hence $(X(\cdot), Y(\cdot), Z_1(\cdot), Z_2\left(\cdot\right)) \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$ constitutes a solution to the equation (3.4).

Now, it remains to prove the uniqueness of the solution to equation (3.4). Let $(X^k(t), Y^k(t), Z_1^k(t), Z_2^k(\cdot))$, $k = 1, 2$ be two solutions of (3.4). We denote

$$\hat{U} := (\hat{X}(t), \hat{Y}(t), \hat{Z}_1(t), \hat{Z}_2(\cdot)) := (X^1(t)-X^2(t), Y^1(t)-Y^2(t), Z_1^1(t)-Z_1^2(t), Z_2^1(t)-Z_2^2(t)).$$

Then it satisfies the following equation

$$\begin{cases} d\hat{X}(t) = -\mu(\hat{Y}(t) + \mathbb{E}\left(\hat{Y}(t)\right))dt - \mu(\hat{Z}_1(t) + \mathbb{E}\left(\hat{Z}_1(t)\right))dB(t) - \mu(\hat{Z}_2(t) + \mathbb{E}\left(\hat{Z}_2(t)\right))dB^H(t), \\ -d\hat{Y}(t) = \mu(\hat{X}(t) + \mathbb{E}\left(\hat{X}(t)\right))dt - \hat{Z}_1(t)dB(t) + \hat{Z}_2(t)dB^H(t), \\ \hat{X}(0) = 0, \quad \hat{Y}(T) = \hat{X}(T), \quad t \in [0, T]. \end{cases}$$

By using Lemma 2.1 and Lemma 2.3, we get

$$\begin{aligned} & \mathbb{E}\left(\hat{X}(T)\hat{Y}(T)\right) \\ = & -\mu\mathbb{E}\left[\int_0^T \hat{X}(s)(\hat{X}(s) + \mathbb{E}\left(\hat{X}(s)\right))ds\right] - \mu\mathbb{E}\left[\int_0^T \hat{Y}(s)(\hat{Y}(s) + \mathbb{E}\left(\hat{Y}(s)\right))ds\right] \\ & -\mu\mathbb{E}\left[\int_0^T \hat{Z}_1(s)(\hat{Z}_1(s) + \mathbb{E}\left(\hat{Z}_1(s)\right))ds\right] - \mu\left(\hat{Z}_2, \hat{Z}_2 + \mathbb{E}\left(\hat{Z}_2\right)\right)_{\mathbb{L}^{1,2}_{[0,T]}} \\ = & -\mu\left(\left\|\hat{U}\right\|^2_{[0,T]} + \|\mathbb{E}\left(\hat{U}\right)\|^2_{[0,T]}\right) \le -\mu\left\|\hat{U}\right\|^2_{[0,T]}. \end{aligned}$$

The equality $\mathbb{E}\left(\hat{X}(T)\hat{Y}(T)\right) = \mathbb{E}\left(\hat{X}(T)\right)^2$ implies $\mu\|\hat{U}\|^2_{[0,T]} \le 0$, from which we conclude $\hat{X}(t) \equiv 0$, $\hat{Y}(t) \equiv 0$, $\hat{Z}_1(t) \equiv 0$ and $\hat{Z}_2(t) \equiv 0$ almost everywhere in $[0, T] \times \Omega$, this completing the proof. $\square$

Now with the result of Lemma 3.2 and Lemma 3.3, we can derive the following continuation theorem.

Theorem 3.4. *Let Assumptions **A1** and **A2** hold, and there exists a constant $\lambda_0 \in [0, 1)$ such that for any $x_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ and any tuple $(\varphi, \varsigma, \Phi, \psi) \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$, the system (3.3) admits a unique solution $(X^{\lambda_0}, Y^{\lambda_0}, Z_1^{\lambda_0}, Z_2^{\lambda_0})$ in the space $\tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$. Then, there exists $\theta_0 \in (0, 1)$ depending solely on the parameters μ, μ_1, K, K_1 and T, such that for any $\theta \in (0, \theta_0]$, the McKean-Vlasov SFr-FBSDE (3.3) admits a unique solution $(X^{\lambda_0+\theta}, Y^{\lambda_0+\theta}, Z_1^{\lambda_0+\theta}, Z_2^{\lambda_0+\theta}) \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$.*

Proof. For each tuple $u = (x, y, z_1, z_2) \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$, consider the following fractional McKean-Vlasov SFr-FBSDE

$$\begin{cases} dX(t) = \left[\lambda_0\kappa(t, U(t), \mathbb{P}_{U(t)}) - \mu(1-\lambda_0)(Y(t) + \mathbb{E}(Y(t))) + \bar{\varphi}(t)\right]dt \\ +\left[\lambda_0\sigma_1(t, X(t), Y(t), Z_1(t), \mathbb{P}_{(X(t),Y(t),Z_1(t))}) - \mu(1-\lambda_0)(Z_1(t) + \mathbb{E}(Z_1(t))) + \bar{\Phi}(t)\right]dB(t) \\ +\left[\lambda_0\sigma_2(t, X(t), Y(t), Z_2(t), \mathbb{P}_{(X(t),Y(t),Z_2(t))}) - \mu(1-\lambda_0)(Z_2(t) + \mathbb{E}(Z_2(t))) + \bar{\psi}(t)\right]dB^H(t), \\ -dY(t) = \left[\lambda_0 F(t, \eta(t), U(t), \mathbb{P}_{U(t)}) + \mu(1-\lambda_0)(X(t) + \mathbb{E}(X(t)) + \eta(t)) + \bar{\varsigma}(t)\right]dt \\ -Z_1(t)dB(t) - Z_2(t)dB^H(t), \\ X(0) = x_0, \quad t \in [0, T], \; Y(T) = \lambda_0 H(X(T), \mathbb{P}_{X(T)}) + (1-\lambda_0)X(T) + \bar{\xi}, \end{cases} \tag{3.10}$$

where

$$
\begin{array}{rcl}
\bar{\varphi}(t) & = & \theta\left(\kappa(t,u(t),\mathbb{P}_{u(t)}) + \mu(y(t)+\mathbb{E}\left(y\left(t\right)\right))\right) + \varphi(t), \\
\bar{\Phi}\left(t\right) & = & \theta\left(\sigma_1(t,x(t),y(t),z_1(t),\mathbb{P}_{(x(t),y(t),z_1(t))}) + \mu(z_1(t)+\mathbb{E}\left(z_1\left(t\right)\right))\right) + \Phi(t), \\
\bar{\psi}(t) & = & \theta\left(\sigma_2(t,x(t),y(t),z_2(t),\mathbb{P}_{(x(t),y(t),z_2(t))}) + \mu(z_2(t)+\mathbb{E}\left(z_2\left(t\right)\right))\right) + \psi(t), \\
\bar{\varsigma}(t) & = & \theta\left(F(t,\eta(t),u(t),\mathbb{P}_{u(t)}) - \mu(x(t)+\mathbb{E}\left(x\left(t\right)\right)+\eta(t))\right) + \varsigma(t), \\
\bar{\xi} & = & \theta\left(H(x(T),\mathbb{P}_{x(T)}) - x(T)\right) + \xi.
\end{array}
$$

Under the assumptions **A1** and **A2**, the McKean-Vlasov equation (3.10) admits a unique solution $U := (X, Y, Z_1, Z_2)$ in the space $\tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$ for each $u := (x, y, z_1, z_2) \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$. We now prove that the mapping defined by

$$
\begin{array}{rlcl}
\Theta_\theta : & \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]} & \rightarrow & \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]} \\
 & (x, y, z_1, z_2) & \rightarrow & (X, Y, Z_1, Z_2),
\end{array}
$$

is contractive.

We define the mapping Θ_θ such that $\Theta_\theta \acute{u} := \acute{U}$ where $\acute{u} := (\acute{x}, \acute{y}, \acute{z}_1, \acute{z}_2)$ and $\acute{U} := \left(\acute{X}, \acute{Y}, \acute{Z}_1, \acute{Z}_2\right)$,

$$
\begin{aligned}
&\hat{\kappa}(t) := \kappa(t, u\left(t\right), \mathbb{P}_{u(t)}) - \kappa(t, \acute{u}\left(t\right), \mathbb{P}_{\acute{u}(t)}),\ \ \tilde{\kappa}(t) := \kappa(t, U\left(t\right), \mathbb{P}_{U(t)}) - \kappa(t, \acute{U}\left(t\right), \mathbb{P}_{\acute{U}(t)}), \\
&\hat{\sigma}_1(t) := \sigma_1(t, x(t), y(t), z_1(t), \mathbb{P}_{(x(t),y(t),z_1(t))}) - \sigma_1(t, \acute{x}(t), \acute{y}(t), \acute{z}_1(t), \mathbb{P}_{(\acute{x}(t),\acute{y}(t),\acute{z}_1(t))}), \\
&\tilde{\sigma}_1(t) := \sigma_1(t, X(t), Y(t), Z_1(t), \mathbb{P}_{(X(t),Y(t),Z_1(t))}) - \sigma_1(t, \acute{X}(t), \acute{Y}(t), \acute{Z}_1(t), \mathbb{P}_{\left(\acute{X}(t),\acute{Y}(t),\acute{Z}_1(t)\right)}), \\
&\hat{\sigma}_2(t) := \sigma_2(t, x(t), y(t), z_2(t), \mathbb{P}_{(x(t),y(t),z_2(t))}) - \sigma_2(t, \acute{x}(t), \acute{y}(t), \acute{z}_2(t), \mathbb{P}_{(\acute{x}(t),\acute{y}(t),\acute{z}_2(t))}), \\
&\tilde{\sigma}_2(t) := \sigma_2(t, X(t), Y(t), Z_2(t), \mathbb{P}_{(X(t),Y(t),Z_2(t))}) - \sigma_2(t, \acute{X}(t), \acute{Y}(t), \acute{Z}_2(t), \mathbb{P}_{\left(\acute{X}(t),\acute{Y}(t),\acute{Z}_2(t)\right)}), \\
&\hat{F}(t) := F(t, \eta(t), u\left(t\right), \mathbb{P}_{u(t)}) - F(t, \eta(t), \acute{u}\left(t\right), \mathbb{P}_{\acute{u}(t)}), \\
&\tilde{F}(t) := F(t, \eta(t), U\left(t\right), \mathbb{P}_{U(t)}) - F(t, \eta(t), \acute{U}\left(t\right), \mathbb{P}_{\acute{U}(t)}), \\
&\hat{H}(T) := H(x(T), \mathbb{P}_{x(T)}) - H(\acute{x}(T), \mathbb{P}_{\acute{x}(T)}),\ \ \tilde{H}(T) := H(X(T), \mathbb{P}_{X(T)}) - H(\acute{X}(T), \mathbb{P}_{\acute{X}(T)}),
\end{aligned}
$$

we set also

$$
\begin{aligned}
\hat{u} &:= (x - \acute{x}, y - \acute{y}, z_1 = \acute{z}_1, z_2 = \acute{z}_2) := (\hat{x}, \hat{y}, \hat{z}_1, \hat{z}_2), \\
\hat{U} &:= \left(X - \acute{X}, Y - \acute{Y}, Z_1 = \acute{Z}_1, Z_2 = \acute{Z}_2\right) := \left(\hat{X}, \hat{Y}, \hat{Z}_1, \hat{Z}_2\right),
\end{aligned}
$$

Consequently, we observe that the quadruple process $\left(\hat{X}(t), \hat{Y}(t), \hat{Z}_1(t), \hat{Z}_2(t)\right)$ belongs to the product space $\tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$ satisfies $\forall t \in [0,T]$

$$\begin{cases} d\hat{X}(t) = \big[\lambda_0\tilde{\kappa}(t) - \mu(1-\lambda_0)(\hat{Y}(t) + \mathbb{E}\left(\hat{Y}(t)\right)) + \theta(\hat{\kappa}(t) + \mu(\hat{y}(t) + \mathbb{E}(\hat{y}(t))))\big]dt \\ +\big[\lambda_0\tilde{\sigma}_1(t) - \mu(1-\lambda_0)(\hat{Z}_1(t) + \mathbb{E}\left(\hat{Z}_1(t)\right)) + \theta(\hat{\sigma}(t) + \mu(\hat{z}(t) + \mathbb{E}(\hat{z}_1(t))))\big]dB(t) \\ +\big[\lambda_0\tilde{\sigma}_2(t) - \mu(1-\lambda_0)(\hat{Z}_2(t) + \mathbb{E}\left(\hat{Z}_2(t)\right)) + \theta(\hat{\sigma}(t) + \mu(\hat{z}(t) + \mathbb{E}(\hat{z}_2(t))))\big]dB^H(t), \\ -d\hat{Y}(t) = \big[\lambda_0\tilde{F}(t) + \mu(1-\lambda_0)(\hat{X}(t) + \mathbb{E}\left(\hat{X}(t)\right)) + \theta(\hat{F}(t) - \mu(\hat{x}(t) + \mathbb{E}(\hat{x}(t))))\big]dt \\ -\hat{Z}_1(t)dB(t) - \hat{Z}_2(t)dB^H(t), \\ \hat{X}(0) = 0, \quad \hat{Y}(T) = \lambda_0\tilde{H}(T) + (1-\lambda_0)\hat{X}(T) + \theta(\hat{H}(T) - \hat{x}(T)). \end{cases}$$

By applying Lemma 2.1, Lemma 2.3, and hypothesis **A1,** we derive the following estimate

$$\begin{aligned}
&\mathbb{E}\left(\hat{X}(T)\hat{Y}(T)\right) \\
&= \int_0^T \mathbb{E}\big[\lambda_0(-\tilde{F}(s))\hat{X}(s) - \mu(1-\lambda_0)(\hat{X}(t) + \mathbb{E}\left(\hat{X}(t)\right))\hat{X}(s) \\
&+ \theta(-\hat{F}(s))\hat{X}(s) + \theta\mu(\hat{x}(t) + \mathbb{E}(\hat{x}(t)))\hat{X}(s)\big]ds \\
&+ \int_0^T \mathbb{E}\big[\lambda_0\tilde{\kappa}(s)\hat{Y}(s) - \mu(1-\lambda_0)(\hat{Y}(t) + \mathbb{E}\left(\hat{Y}(t)\right))\hat{Y}(s) + \theta\hat{\kappa}(s)\hat{Y}(s) + \theta\mu(\hat{y}(t) + \mathbb{E}(\hat{y}(t)))\hat{Y}(s)\big]ds \\
&+ \int_0^t \mathbb{E}\big[\lambda_0\tilde{\sigma}_1(t) - \mu(1-\lambda_0)(\hat{Z}_1(t) + \mathbb{E}\left(\hat{Z}_1(t)\right)) - \theta(\hat{\sigma}_1(t) + \mu(\hat{z}_1(t) + \mathbb{E}(\hat{z}_1(t))))\mathcal{D}_s\hat{Y}(s)\big]ds \\
&- \int_0^t \mathbb{E}\left[\hat{Z}_1(s)\mathcal{D}_s\hat{X}(s)\right]ds \\
&+ \left\langle \lambda_0\tilde{\sigma}_2 - \mu(1-\lambda_0)(\hat{Z}_2 + \mathbb{E}\left(\hat{Z}_2(t)\right)) + \theta\hat{\sigma}_2 + \theta\mu(\hat{z}_2 + \mathbb{E}(\hat{z}_2(t))), \hat{Z}_2\right\rangle_{\mathbb{L}^{1,2}_{[0,T]}} \\
&\le -\mu\lambda_0(\|\hat{U}\|^2_{[0,T]} + \left\|\mathbb{E}\left(\hat{U}\right)\right\|^2_{[0,T]}) - \mu(1-\lambda_0)(\|\hat{U}\|^2_{[0,T]} + \left\|\mathbb{E}\left(\hat{U}\right)\right\|^2_{[0,T]}) \\
&+ \theta\left\langle \hat{A}, \hat{U}\right\rangle + \theta\mu\left\langle \hat{u}, \hat{U}\right\rangle + \theta\mu\left\langle \mathbb{E}(\hat{u}), \mathbb{E}\left(\hat{U}\right)\right\rangle,
\end{aligned}$$

with

$$\begin{aligned}
\left\langle \hat{A}, \hat{U}\right\rangle :&= \mathbb{E}\left[\int_0^T \left((-\hat{F}(s))\hat{X}(s) + \hat{\kappa}(s)\hat{Y}(s) + \hat{\sigma}_1(s)\hat{Z}_1(s)\right)ds + \left\langle \hat{\sigma}_2, \hat{Z}_2\right\rangle_{\mathbb{L}^{1,2}_{[0,T]}}\right], \\
\left\langle \hat{u}, \hat{U}\right\rangle :&= \mathbb{E}\left[\int_0^T \left(\hat{x}(s)\hat{X}(s) + \hat{y}(s)\hat{Y}(s) + \hat{z}_1(s)\hat{Z}_1(s)\right)ds + \left\langle \hat{z}_2, \hat{Z}_2\right\rangle_{\mathbb{L}^{1,2}_{[0,T]}}\right], \\
\left\langle \mathbb{E}(\hat{u}), \mathbb{E}\left(\hat{U}\right)\right\rangle :&= \int_0^T \left(\mathbb{E}(\hat{x}(t))\mathbb{E}\left(\hat{X}(t)\right) + \mathbb{E}(\hat{y}(t))\mathbb{E}\left(\hat{Y}(t)\right) + \mathbb{E}(\hat{z}_1(s))\mathbb{E}\left(\hat{Z}_1(s)\right)\right)ds \\
&+ \left\langle \mathbb{E}(\hat{z}_2), \mathbb{E}\left(\hat{Z}_2\right)\right\rangle_{\phi,[0,T]}.
\end{aligned}$$

By Assumption **A1** and the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we derive the following inequality

$$\begin{aligned}\left\langle \hat{A}, \hat{U} \right\rangle &\leq \mathbb{E}\left[\int_0^T |\hat{F}(s)||\hat{X}(s)|ds + \int_0^T |\hat{\kappa}(s)||\hat{Y}(s)|ds + \int_0^T |\hat{\sigma}_1(s)| \left|\hat{Z}_1(s)\right| ds\right] \\ &\quad + \|\hat{\sigma}_2\|_{\mathbb{L}^2[0,T]} \|\hat{Z}_2\|_{\mathbb{L}^{1,2}_{[0,T]}} \\ &\leq \frac{K}{2}\left(\|\hat{u}\|^2_{[0,T]} + \|\mathbb{E}[\hat{u}]\|^2_{[0,T]} + \|\hat{U}\|^2_{[0,T]}\right).\end{aligned}$$

Following analogous arguments, we establish that

$$\left\langle \hat{u}, \hat{U} \right\rangle \leq \|\hat{u}\|^2_{[0,T]} + \|\hat{U}\|^2_{[0,T]},$$
$$\left\langle \mathbb{E}(\hat{u}), \mathbb{E}\left(\hat{U}\right) \right\rangle \leq \|\mathbb{E}[\hat{u}]\|^2_{[0,T]} + \|\mathbb{E}[\hat{U}]\|^2_{[0,T]}.$$

Therefore

$$\begin{aligned}\mathbb{E}[\hat{X}(T)\hat{Y}(T)] &\leq -\mu\lambda_0(\|\hat{U}\|^2_{[0,T]} + \left\|\mathbb{E}\left(\hat{U}\right)\right\|^2_{[0,T]}) - \mu(1-\lambda_0)(\|\hat{U}\|^2_{[0,T]} + \left\|\mathbb{E}\left(\hat{U}\right)\right\|^2_{[0,T]}) \\ &\quad + \theta K(\|\hat{u}\|^2_{[0,T]} + \|\mathbb{E}(\hat{u})\|^2_{[0,T]}) + \theta K \left\|\hat{U}\right\|^2_{[0,T]} + \theta\mu(\|\hat{u}\|^2_{[0,T]} + \left\|\hat{U}\right\|^2_{[0,T]}) \\ &\quad + \theta\mu(\|\mathbb{E}(\hat{u})\|^2_{[0,T]} + \left\|\mathbb{E}\left(\hat{U}\right)\right\|^2_{[0,T]}) \qquad (3.11) \\ &\leq -\Delta\left(\left\|\hat{U}\right\|^2_{[0,T]} + \left\|\mathbb{E}\left(\hat{U}\right)\right\|^2_{[0,T]}\right) + \tilde{\Delta}\left(\|\hat{u}\|^2_{[0,T]} + \|\mathbb{E}(\hat{u})\|^2_{[0,T]}\right).\end{aligned}$$

where $\Delta = \mu - \theta(K+\mu)$ and $\tilde{\Delta} = \theta(K+\mu)$. Moreover, under Assumption **A2**, we have

$$\begin{aligned}\mathbb{E}[\hat{X}(T)\hat{Y}(T)] &= \mathbb{E}\left(\hat{X}(T)\left(\lambda_0\tilde{H}(T) + (1-\lambda_0)\hat{X}(T) + \theta(\hat{H}(T) - \hat{x}(T))\right)\right) \qquad (3.12) \\ &\geq \Lambda\mathbb{E}|\hat{X}(T)|^2 + \theta\mathbb{E}[\hat{X}(T)\hat{H}(T)] - \theta\mathbb{E}[\hat{X}(T)\hat{x}(T)].\end{aligned}$$

where $\Lambda = \mu_1\lambda_0 + (1-\lambda_0) \geq \min(\mu_1, 1) = \vartheta$ and by using (3.11), (3.12) and Assumption **A2**, we can obtain

$$\begin{aligned}&\vartheta\mathbb{E}|\hat{X}(T)|^2 + \Delta\left(\|\hat{U}\|^2_{[0,T]} + \|\mathbb{E}[\hat{U}]\|^2_{[0,T]}\right) \\ &\leq \tilde{\Delta}(\|\hat{u}\|^2_{[0,T]} + \|\mathbb{E}[\hat{u}]\|^2_{[0,T]}) - \theta\mathbb{E}[\hat{X}(T)\hat{H}(T)] + \theta\mathbb{E}[\hat{X}(T)\hat{x}(T)] \\ &\leq \tilde{\Delta}(\|\hat{u}\|^2_{[0,T]} + \|\mathbb{E}[\hat{u}]\|^2_{[0,T]}) + \frac{\theta}{2}\left(\mathbb{E}(|\hat{X}(T)|^2 + |\hat{H}(T)|^2) + \mathbb{E}(|\hat{X}(T)|^2 + |\hat{x}(T)|^2)\right) \\ &\leq \tilde{\Delta}(\|\hat{u}\|^2_{[0,T]} + \|\mathbb{E}[\hat{u}]\|^2_{[0,T]}) + \theta(K_1+1)(\mathbb{E}|\hat{X}(T)|^2 + \mathbb{E}|\hat{x}(T)|^2).\end{aligned}$$

Therefore, we obtain the following estimate

$$\begin{aligned}&\Delta(\|\hat{U}\|^2_{[0,T]} + \|\mathbb{E}[\hat{U}]\|^2_{[0,T]}) + (\vartheta - \theta(K_1+1))\mathbb{E}|\hat{X}(T)|^2 \\ &\leq \tilde{\Delta}(\|\hat{u}\|^2_{[0,T]} + \|\mathbb{E}[\hat{u}]\|^2_{[0,T]}) + \theta(K_1+1)\mathbb{E}|\hat{x}(T)|^2.\end{aligned}$$

To ensure the conditions $\Delta > \tilde{\Delta}$ and $\vartheta - \theta(K_1+1) > \theta(K_1+1)$ are satisfied, we set $\theta_0 := \min\left(\frac{\mu}{2(K+\mu)}, \frac{\vartheta}{2(K_1+1)}\right)$. It follows immediately that $0 < \theta_0 < 1$. And for any $\theta \in (0, \theta_0]$, there exists a constant $C \in (0,1)$ such that $\|\hat{U}\|_{\mathcal{R}} \leq C\|\hat{u}\|_{\mathcal{R}}$. Note that θ_0 is a constant depending only on μ, μ_1, K, K_1 and T. Consequently, the operator Θ_θ is a contraction under the norm $\|\cdot\|_{\mathcal{R}}$ and therefore possesses a unique fixed point. Hence, the system (3.10) can be written as

$$\begin{cases} dX(t) = \big[(\lambda_0+\delta)\kappa\left(t, U(t), \mathbb{P}_{U(t)}\right) - \mu(1-(\lambda_0+\theta))(Y(t)+\mathbb{E}[Y(t)]) + \varphi(t)\big]dt \\ +\big[\,(\lambda_0+\theta)\,\sigma_1(t, X(t), Y(t), Z_1(t), \mathbb{P}_{(X(t),Y(t),Z_1(t))} \\ -\mu(1-(\lambda_0+\theta))(Z_1(t)+\mathbb{E}\left(Z_1(t)\right)) + \Phi(t)\,\big]dB(t) \\ +\big[(\lambda_0+\theta)\sigma_2\left(t, X(t), Y(t), Z_2(t), \mathbb{P}_{(X(t),Y(t),Z_2(t))}\right) \\ -\mu(1-(\lambda_0+\theta))(Z_2(t)+\mathbb{E}[Z_2(t)]) + \psi(t)\big]dB^H(t), \\ -dY(t) = \big[(\lambda_0+\theta)F\left(t, \eta(t), U(t), \mathbb{P}_{U(t)}\right) \\ +\mu(1-(\lambda_0+\theta))(X(t)+\mathbb{E}[X(t)]+\eta(t)) + \varsigma(t)\big]dt - Z_1(t)dB(t) - Z_2(t)dB^H(t), \\ X(0) = x_0, \\ Y(T) = (\lambda_0+\theta)H(X(T), \mathbb{P}_{X(T)}) + (1-(\lambda_0+\theta))X(T) + \xi, \quad 0 \leq t \leq T. \end{cases}$$

As a result, there exists a unique solution $\left(X^{\lambda_0+\theta}, Y^{\lambda_0+\theta}, Z_1^{\lambda_0+\theta}, Z_2^{\lambda_0+\theta}\right)$ to system (3.3) in the space $\tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$. This concludes the proof. □

Theorem 3.5. *Under Assumption* ***A1*** *and* ***A2****, there exists a unique solution* $(X, Y, Z_1, Z_2) \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$ *to the McKean-Vlasov forward-backward equation* (3.3).

Proof. This proof is subdivided in two parts.
Step 1: Existence. By Lemma 3.3, for every initial condition $x_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P})$ and for any processes $(\varphi, \Phi, \varsigma, \zeta) \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$, the system (3.3) admits a unique solution when $\lambda = 0$. Then, by Theorem 3.4, there exists a threshold $\theta_0 \in (0,1)$ depending only on μ, μ_1, K, K_1 and T, such that for any $\theta \in (0, \theta_0]$, the McKean-Vlasov SFr-FBSDE (3.3) admits a unique solution $\left(X^\theta, Y^\theta, Z_1^\theta, Z_2^\theta\right) \in \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{1,2}_{[0,T]} \times \tilde{\mathbb{L}}^{2}_{[0,T]} \times \tilde{\mathcal{V}}^{1,2}_{[0,T]}$. Select $\theta > 0$ such that $0 < \theta \leq \theta_0$ and $N\theta = 1$ for some integer N. By repeating the above procedure, we establish that the system (3.3) admits a solution xhene $\lambda = 1$. This completes the proof of existence part for the system (3.3).
Step 2: Uniqueness. Let $U^j(t) := \left(X^j(t), Y^j(t), Z_1^j(t), Z_2^j(t)\right),\ j = 1,2$ denote two solutions of the Mckean-Vlasov SFr-FBSDE (3.3) corresponding to $X(0) = x_0$. We define the following notations

$$\begin{aligned}
&\hat{U}(t) := (\hat{X}(t), \hat{Y}(t), \hat{Z}_1(t), \hat{Z}_2(t)) := (X^1(t)-X^2(t), Y^1(t)-Y^2(t), Z_1^1(t)-Z_1^2(t), Z_2^1(t)-Z_2^2(t)),\\
&\tilde{\kappa}(t) := \kappa(t, U^1(t), \mathbb{P}_{U^1(t)}) - \kappa(t, U^2(t), \mathbb{P}_{U^2(t)}),\\
&\tilde{F}(t) := F\left(t, \eta(t), U^1(t), \mathbb{P}_{U^1(t)}\right) - F\left(t, \eta(t), U^2(t), \mathbb{P}_{U^2(t)}\right),\\
&\tilde{\sigma}_1(t) := \sigma_1\left(t, X^1(t), Y^1(t), Z_1^1(t), \mathbb{P}_{\left(X^1(t),Y^1(t),Z_1^1(t)\right)}\right) - \sigma_1\left(t, X^2(t), Y^2(t), Z_1^2(t), \mathbb{P}_{\left(X^2(t),Y^2(t),Z_1^2(t)\right)}\right),\\
&\tilde{\sigma}_2(t) := \sigma_2\left(t, X^1(t), Y^1(t), Z_2^1(t), \mathbb{P}_{\left(X^1(t),Y^1(t),Z_2^1(t)\right)}\right) - \sigma_2\left(t, X^2(t), Y^2(t), Z_2^2(t), \mathbb{P}_{\left(X^2(t),Y^2(t),Z_2^2(t)\right)}\right),\\
&\tilde{H}(T) := H(X_1(T), \mathbb{P}_{X_1(T)}) - H(X_2(T), \mathbb{P}_{X_2(T)}).
\end{aligned}$$

Consequently, we derive the following system

$$\begin{cases} d\hat{X}(t) = \tilde{\kappa}(t)dt + \tilde{\sigma}_1(t)dB(t) + \tilde{\sigma}_2(t)dB^H(t), \\ -d\hat{Y}(t) = \tilde{F}(t)dt - \hat{Z}_1(t)dB(t) - \hat{Z}_2(t)dB^H(t), \\ \hat{X}(0) = 0, \quad \hat{Y}(T) = \tilde{H}(T). \end{cases}$$

Applying Lemma 2.3 in conjunction with Assumption **A2**, we can derive the following estimate

$$\begin{aligned} \mathbb{E}[\hat{X}(T)\hat{Y}(T)] &= \int_0^T \mathbb{E}\left[(-\tilde{F}(s))\hat{X}(s) + \tilde{\kappa}(s)\hat{Y}(s) + \tilde{\sigma}_1(s)\hat{Z}_1(s)\right] ds + \langle \tilde{\sigma}_2, \hat{Z}_2 \rangle_{\mathbb{L}^{1,2}[0,T]} \\ &\leq -\mu(\|\hat{U}\|^2_{[0,T]} + \|\mathcal{W}(\mathbb{P}_U, \mathbb{P}_{\acute{U}})\|^2_{[0,T]}) \leq -\mu\|\hat{U}\|^2_{[0,T]}. \end{aligned}$$

From the inequality $\mathbb{E}[\hat{X}(T)\hat{Y}(T)] \geq \mu_1 \mathbb{E}|\hat{X}(T)|^2 \geq 0$, we have $\mu\|\hat{U}\|^2_{[0,T]} \leq 0$, we conclude that $\hat{X}(t) \equiv 0$, $\hat{Y}(t) \equiv 0$, $\hat{Z}_1(t) \equiv 0$ and $\hat{Z}_2(t) \equiv 0$. This establishes the desired uniqueness result. ☐

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