Gulf Journal of Mathematics
Vol 21, Issue 2 (2025) 250-258

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v21i2.3572

CERTAIN PROPERTIES OF THE UNIT GRAPH OVER THE RING Z_{p^k}

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Abstract. The unit graph $G(R)$ is the graph in which two distinct vertices (distinct elements of R) are connected if the sum of the elements is a unit in R. In this paper, we study the unit graph $G(Z_{p^k})$, where $k \geq 1$ and p is a prime number. We provide some numerical invariants of $G(Z_{p^k})$, such as the clique number and the independence number. Furthermore, we investigate the eigensharp condition on the graph $G(Z_{p^k})$.

Keywords. Commutative ring, unit graph, eigensharp property, biclique partition number
2020 Mathematics Subject Classification. Primary 13M05; Secondary 05C25, 05C50

1. Introduction and preliminaries

Assume $k \geq 1$ and p is a prime number. $G(Z_{p^k})$ is the unit graph on the ring Z_{p^k}, where the set of all elements of the ring Z_{p^k} presents the set of the vertex V, and E is the set of edges obtained by connecting any two distinct vertices whose sum is a unit in Z_{p^k}. An element x in the ring Z_{p^k} is said to be a unit if $\gcd(x, p^k) = 1$, implying that $x \neq 0$ $(mod\ p)$. Grimaldi [10] introduced the unit graph $G(Z_n)$ and analyzed its fundamental characteristics, such as the covering numbers and the independence numbers. Further studies on $G(Z_n)$ can be obtained in [17], [6], and [16]. Ashrafi et al. [4] generalized this notion to the unit graph $G(R)$ associated with an arbitrary ring R. Their study explored the diameter, planarity, chromatic index, and girth of $G(R)$. Additional research on unit graphs on rings can be found in [2], [14], and [18].

The graph invariants are numerical values used to define graphical attributes that remain unchanged under isomorphism. For example, for any graph G, we have the radius number $rad(G)$ and the domination number $\gamma(G)$. Since Z_{p^k} is a local non-division ring, from Li and Suwe [13], we conclude that $rad(G(Z_{p^k})) = 2$. Also, as a direct application of [11], the domination number $\gamma(G(Z_{p^k}))$ is fully determined. In a graph G, *clique* refers to a complete subgraph S of G. A subset I of $V(G)$ in which no two different vertices are connected is called an *independent* subset. In this paper, we evaluate the *clique number*; the size of the largest clique in the graph $G(Z_{p^k})$, and the

Date: Received: Sep 18, 2025; Revised: Oct 16, 2026; Accepted: Oct 29, 2025.
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independence number $\beta(G(Z_{p^k}))$; the maximum cardinality of the independent sets in the graph $G(Z_{p^k})$.

In another respect, numerous studies have been conducted on covering a graph with subgraphs from a certain graph family. Some examples include a tree covering, a cycle covering, and a star covering, see [3] and [7]. A complete bipartite subgraph of G is known as *biclique.* A collection $\mathcal{L}_G = \{H_1, H_2, \cdots, H_m\}$ represents a biclique partition of G if for every $e \in E(G)$ there is only one $i = 1, \cdots, m$ such that $H_i \in \mathcal{L}_G$ and $e \in E(H_i)$. The minimum cardinality relative to all biclique partitions of a graph G, denoted by $bp(G)$, is known as the biclique partition number of G. Several studies have studied the biclique partition number for certain graphs, for example, [5] and [15].

Throughout the paper, we write K_n to indicate a complete graph of order n and $K_{n,m}$ to indicate a complete bipartite graph with specified finite partite sets S, T of orders n and m, respectively.

The path graph with $n+1$ vertices is denoted by P_n, and the graph with m vertices and with no edges is denoted by N_m. The *adjacency matrix* of a graph G is a symmetric square matrix Ω of order $|V(G)|$, where the ij-th entry equals 1 whenever $x_i x_j$ is an edge of G; otherwise, it equals zero. For example, the graph P_n has the following adjacency matrix.

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{n\times n}$$

Consequently, Ω's eigenvalues are always real. The abbreviations $\epsilon_+(\Omega)$, $\epsilon_-(\Omega)$, and $\epsilon_0(\Omega)$ represent the number of positive, negative, and zero eigenvalues of Ω. As indicated by [12],

$$bp(G) \geq \max\{\epsilon_+(\Omega), \epsilon_-(\Omega)\}.$$

The number of linearly independent eigenvectors related to the eigenvalue λ_s is the multiplicity of the eigenvalue. If $\lambda_s, 1 \leq s \leq t$, are the different eigenvalues related to Ω having multiplicity q_s, then the *spectrum* of G is given as $\sigma(\Omega) = \{\lambda_1^{q_1}, \lambda_2^{q_2}, \cdots, \lambda_t^{q_t}\}$.

A graph G is known as an eigensharp graph [8], if $bp(G) = \max\{\epsilon_+(\Omega), \epsilon_-(\Omega)\}$. The eigensharp graphs include several groups of graphs. For instance, Tverberg [19], proved that $bp(K_n) = n-1$, whereas the graph K_n has a spectrum $\{-1^{n-1}, (n-1)^1\}$, i.e. K_n is an eigensharp graph. The graph $K_{n,m}$ is also an eigensharp, where its spectrum $\{-\sqrt{nm}, 0^{n+m-2}, \sqrt{mn}\}$ and $bp(K_{n,m}) = 1$.

Suppose F_1 and F_2 are two graphs. $F_1 \vee F_2$ signifies the join of the two graphs such that $V(F_1 \vee F_2) = V(F_1) \cup V(F_2)$ and $E(F_1 \vee F_2)$ include all edges e such that $e \in E(F_1) \cup E(F_2)$ and $e = uv$ for some $u \in V(F_1)$ and $v \in V(F_2)$. The following Proposition was proved in [9].

Proposition 1.1. [9] *If F_1 and F_2 are two regular graphs, then $\epsilon_-(\Omega_{F_1 \vee F_2}) = \epsilon_-(\Omega_{F_1}) + \epsilon_-(\Omega_{F_2}) + 1$, and $\epsilon_+(\Omega_{F_1 \vee F_2}) = \epsilon_+(\Omega_{F_1}) + \epsilon_+(\Omega_{F_2}) - 1$. If F_1 and F_2 are eigensharp graphs such that $bp(F_1) = \epsilon_-(\Omega_{F_1})$ and $bp(F_2) = \epsilon_-(\Omega_{F_2})$, then $F_1 \vee F_2$ is an eigensharp graph.*

Abdelkarim et al. in [1] showed that the unit graphs $G(Z_p)$ and $G(Z_{2p})$ are eigensharp. In addition, they showed that $G({}^{Z_p[x]}/_{\langle x^2 \rangle})$ is eigensharp. In this paper, we investigate the unit graph $G(Z_{p^k})$ for every prime p and every positive integer k, and we identify some numerical invariants of the graph, such as clique number and independence number. Furthermore, we show that $G(Z_{p^k})$ is an eigensharp graph.

2. Main results

2.1. Some Numerical Invariants of the Graph $G(Z_{p^k})$. We study and determine certain numerical invariants of the graph $G(Z_{p^k})$, including the independence number and the clique number. These constants will help to clarify the unit graph's attributes and complexity.

Remark 2.1. If $p = 2$, the vertex set of the graph $G(Z_{2^k})$ is partitioned into the union of the two sets $V_0 = \{2t : 0 \leq t \leq 2^{k-1}-1\}$ and $V_1 = \{1+2t : 0 \leq t \leq 2^{k-1}-1\}$. Therefore, $G(Z_{2^k}) \simeq K_{2^{k-1},2^{k-1}}$ and so, its clique number is equal to 2 and its independence number $\beta(G(Z_{2^k})) = 2^{k-1}$.

Theorem 2.2. *For $p \geq 3$ and $k \in Z^+$, the graph $G(Z_{p^k})$ has the following properties:*

(1) The clique number $w(G(Z_{p^k})) = \lfloor \frac{p}{2} \rfloor p^{k-1} + 1$
(2) The independence number $\beta(G(Z_{p^k})) = p^{k-1}$.

Proof. 1) Every element in the ring Z_{p^k} can be expressed as $i + tp$ for $0 \leq i \leq p-1, 0 \leq t \leq p^{k-1} - 1$. Thus $V(Z_{p^k})$ is divided into the union of sets $V_i = \{i + tp : 0 \leq t \leq p^{k-1} - 1\}, 0 \leq i \leq p-1$. Assume H_{V_i} to be the induced subgraph of the graph $G(Z_{p^k})$ with respect to the vertex set V_i. For $i \geq 1$, if $x, y \in V_i$, then $x = \alpha p + i$ and $y = \beta p + i$ with $0 \leq \alpha, \beta \leq p^{(k-1)} - 1$, and so $x + y = 2i \ mod\ p$. Thus, $x + y$ is a unit in Z_{p^k} and therefore, H_{V_i} is isomorphic to $K_{p^{k-1}}$. Furthermore, if x and y belong to two distinct sets V_i and V_j, respectively, then x and y are adjacent unless $j = p - i$, where the sum is equal to 0 modulo p.

Now, consider the induced subgraph H_M, where

$$M = \{V_1, V_2, \ldots, V_{\frac{p-1}{2}}\} \cup \{x\}$$

and x is an arbitrary vertex from V_0. We show that H_M constitutes the largest clique within the graph $G(Z_{p^k})$. Actually, every two distinct vertices in M are adjacent. Therefore, $H_M \simeq K_n$ where, $n = \lfloor \frac{p}{2} \rfloor p^{k-1} + 1$. In fact, if $x \in V_j$ with $j > \frac{p-1}{2}$, then $0 < p - j < \frac{1-p}{2} + p = \frac{p+1}{2}$. So $V_{p-j} \in M$, and hence any additional vertex inserted outside of M will not be adjacent to a vertex in M. Hence, the result will follow.

2) We show that the largest independent subset in $V(G(Z_{p^k}))$ is the set $V_0 = \{tp : 0 \leq t \leq p^{k-1} - 1\}$. For, assume A to be an independent subset of $V(G(Z_{p^k}))$ such that $A \cap (\bigcup_{i=1}^{p-1} V_i) \neq \phi$. Since $V_i \cong K_{p^{k-1}}$ $\forall i \geq 1$, then $A \nsubseteq V_i$. Also, since V_i and V_j, when $j \neq p - i$, form the partite sets for a complete bipartite graph $K_{p^{k-1},p^{k-1}}$, thus $A = \{x, y\}$ for some $x \in V_i$ and $y \in V_{p-i}$ for some $i \geq 1$. Since the cardinality of V_0 is $p^{k-1} > 2$, therefore $\beta(G(Z_{p^k})) = p^{k-1}$. $\square$

2.2. The Eigensharp Property for the Unit Graph $G(Z_{p^k})$. According to [12], for each simple graph G, the biclique partition number $bq(G)$ satisfies that:

$$bp(G) \geq \max\{\epsilon_-(\Omega), \epsilon_+(\Omega)\}.$$

We intend to show that $bp(G(Z_{p^k})) = \max\{\epsilon_-(\Omega_{G(Z_{p^k})}, \epsilon_+(\Omega_{G(Z_{p^k}))}\}$, and hence $G(Z_{p^k})$ is an eigensharp graph by [8]. Actually, if $k = 1$ and $p = 2$ or 3, the unit graphs $G(Z_2)$ and $G(Z_3)$ are obviously isomorphic to the graphs P_2 and P_3. Hence, the spectrum of $G(Z_2)$ is $\{1^1, -1^1\}$, whereas the spectrum for $G(Z_3)$ is $\{\sqrt{2}^1, 0^1, -\sqrt{2}^1\}$. Therefore, $G(Z_2)$ and $G(Z_3)$ are eigensharp graphs, as $bp(G(Z_2)) = bp(G(Z_3)) = 1 = \max\{\epsilon_+(\Omega_{P_n}), \epsilon_-(\Omega_{P_n})\}$ for $n = 2$ and 3.

Now, suppose that $k \geq 2$. Then $V(Z_{p^k})$ is partitioned into the union of the sets $V_i = \{i + tp : 0 \leq t \leq p^{k-1} - 1\}, 0 \leq i \leq p - 1$. Assume $B(V_i, V_j)$ to be the subgraph associated with the set $V_i \cup V_j$ such that e, for any edge, connects a vertex in V_i to a vertex in V_j. Since x is a unit in Z_{p^k} if and only if $x \neq 0(mod\ p)$, hence for $i \geq 0$ and $j \neq p-i$, $B(V_i, V_j) \simeq K_{p^{k-1},p^{k-1}}$. For $i \geq 1, B(V_i, V_i) \simeq K_{p^{k-1}}$ and $B(V_i, V_{p-i}) \simeq N_{p^{k-1}}$.

Remark 2.3. Consider the graph $G(Z_{2^k})$. According to Remark 2.1, the vertex set $V(G(Z_{2^k}))$ is a union of the two sets $V_0 = \{2t : 0 \leq t \leq 2^{k-1} - 1\}$ and $V_1 = \{1 + 2t : 0 \leq t \leq p^{k-1} - 1\}$. Clearly, $B(V_i, V_i) \simeq N_{p^{k-1}}$ for $i = 0, 1$, and hence, $G(Z_{2^k}) = B(V_0, V_1)$. Therefore, $G(Z_{2^k}) \simeq K_{2^{k-1},2^{k-1}}$ and so its spectrum is $\{-\sqrt{2^{2k-2}}, 0^{2^k}, \sqrt{2^{2k-2}}\}$ with $bp(G(Z_{2^k})) = 1$. Thus, the graph $G(Z_{2^k})$ is an eigensharp graph.

Let A be the adjacency matrix for the graph $G(Z_{p^k})$, with $p \geq 3$ and $k \geq 2$. Let $\mathbb{L}(i, j)$ denote the adjacency matrix for the induced subgraph corresponding to the set $V_i \cup V_j$, $0 \leq i < j \leq p - 1$. Note that $\mathbb{L}(i, j)$ is a square matrix of size p^{k-1} and $\mathbb{L}(i, j) = \mathbb{L}(j, i)$ $\forall i, j$. Therefore,

$$\Omega_{p^k} = \begin{bmatrix} \mathbb{L}(0,0) & \mathbb{L}(0,1) & \cdots & \mathbb{L}(0,p-1) \\ \mathbb{L}(1,0) & \mathbb{L}(1,1) & \cdots & \mathbb{L}(1,p-1) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{L}(p-1,0) & \mathbb{L}(p-1,1) & \cdots & \mathbb{L}(p-1,p-1) \end{bmatrix}_{p^k \times p^k}.$$

Let $\mathbb{JA}_m$ denote the square matrix $m \times m$ where $a_{ij} = 1$ $\forall i, j$. Let $\Omega_{N_{p^{k-1}}}$ and $\Omega_{K_{p^{k-1}}}$ be the adjacency matrices for the graphs $N_{p^{k-1}}$ and $K_{p^{k-1}}$, respectively. Thus, the structure of the matrix A becomes as follows:

$$A = \begin{bmatrix}
\Omega_{N_{p^{k-1}}} & \mathbb{JA}_{p^{k-1}} & \mathbb{JA}_{p^{k-1}} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbb{JA}_{p^{k-1}} & \mathbb{JA}_{p^{k-1}} \\
\mathbb{JA}_{p^{k-1}} & \Omega_{K_{p^{k-1}}} & \mathbb{JA}_{p^{k-1}} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbb{JA}_{p^{k-1}} & \Omega_{N_{p^{k-1}}} \\
\mathbb{JA}_{p^{k-1}} & \mathbb{JA}_{p^{k-1}} & \Omega_{K_{p^{k-1}}} & \mathbb{JA}_{p^{k-1}} & \cdots & \cdots & \cdots & \mathbb{JA}_{p^{k-1}} & \Omega_{N_{p^{k-1}}} & \mathbb{JA}_{p^{k-1}} \\
\vdots & \vdots & \mathbb{JA}_{p^{k-1}} & \ddots & \ddots & . & \iddots & \iddots & \mathbb{JA}_{p^{k-1}} & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & . & \iddots & \iddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \iddots & . & \iddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \iddots & \iddots & . & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \mathbb{JA}_{p^{k-1}} & \iddots & \iddots & . & \ddots & \ddots & \mathbb{JA}_{p^{k-1}} & \vdots \\
\mathbb{JA}_{p^{k-1}} & \mathbb{JA}_{p^{k-1}} & \Omega_{N_{p^{k-1}}} & \mathbb{JA}_{p^{k-1}} & \cdots & \cdots & \cdots & \mathbb{JA}_{p^{k-1}} & \Omega_{K_{p^{k-1}}} & \mathbb{JA}_{p^{k-1}} \\
\mathbb{JA}_{p^{k-1}} & \Omega_{N_{p^{k-1}}} & \mathbb{JA}_{p^{k-1}} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbb{JA}_{p^{k-1}} & \Omega_{K_{p^{k-1}}}
\end{bmatrix}_{p^k \times p^k}$$

Let $M = \bigcup_{i=1}^{p-1} V_i$, let $G_0 = B(V_0, M)$, and let $G_1 = B(M, M)$. The graph $G(Z_{p^k})$ can thus be defined as $G_0 \vee G_1$.

Lemma 2.4. *For the graph* $G_1, bp(G_1) \leq \lfloor \frac{p}{2} \rfloor (2p^{k-1} - 1) - 1$.

Proof. Let $C_i = V_i \cup V_{p-i}$ and let $D_i = \left(\bigcup_{j=i+1}^{(p-i)-1} V_j \right)$. Since each vertex in C_i connects all the vertex in D_i, the graph $B(C_i, D_i) \simeq K_{|C_i|,|D_i|}$ where, $|C_i| = 2p^{k-1}$ and $|D_i| = (p-2i-1)p^{k-1}$. Consider $H = \{B(C_i, D_i) : 1 \le i \le \lfloor \frac{p}{2} \rfloor - 1\}$. H is an essential component of a biclique cover, as the subgraphs $B(C_i, D_i)$ contain different edges for different values of i. Note that the edges related to the complete graphs $\{B(V_j, V_j) : 1 \le j \le p-1\}$ are the only edges not covered by H. Hence, for each $1 \le j \le p-1$, we define $\{S(j, x) : x \in V_j\}$ as the set of disjoint stars in the graph $B(V_j, V_j)$ generated by the vertex x. Then, for each j, the graph $B(V_j, V_j)$ is partitioned into the union of $p^{k-1} - 1$ stars, where one of its vertices is eliminated. Thus, $F = \{B(C_i, D_i), S(j, x) : 1 \le i \le \lfloor \frac{p}{2} \rfloor - 1, 1 \le j \le p-1, x \in V_j\}$ forms a biclique partition of the graph G_1 of cardinality $(\lfloor \frac{p}{2} \rfloor - 1) + (p-1)(p^{k-1} - 1)$. Therefore, $bp(G(Z_{p^k})) \le \lfloor \frac{p}{2} \rfloor (2p^{k-1} - 1) - 1$. □

Next, we show that $\max\{\epsilon_-(\Omega_1), \epsilon_+(\Omega_1)\} = \lfloor \frac{p}{2} \rfloor (2p^{k-1} - 1) - 1$ and, hence, G_1 is an eigensharp graph.

Lemma 2.5. *In the graph* $G_1, \max\{\epsilon_-(\Omega_1), \epsilon_+(\Omega_1)\} = \lfloor \frac{p}{2} \rfloor (2p^{k-1} - 1) - 1.$

Proof. Let $\alpha = p^k - p^{k-1}$. Observe that the adjacency matrix of the graph G_1 is:

$$\Omega_1 = \begin{bmatrix} \Omega_{K_{p^{k-1}}} & \mathbb{JA}_{p^{k-1}} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbb{JA}_{p^{k-1}} & \Omega_{N_{p^{k-1}}} \\ \mathbb{JA}_{p^{k-1}} & \Omega_{K_{p^{k-1}}} & \mathbb{JA}_{p^{k-1}} & \cdots & \cdots & \cdots & \mathbb{JA}_{p^{k-1}} & \Omega_{N_{p^{k-1}}} & \mathbb{JA}_{p^{k-1}} \\ \vdots & \mathbb{JA}_{p^{k-1}} & \ddots & \ddots & . & \iddots & \iddots & \mathbb{JA}_{p^{k-1}} & \vdots \\ \vdots & \vdots & \ddots & \ddots & . & \iddots & \iddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \iddots & . & \iddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \iddots & \iddots & . & \ddots & \ddots & \vdots & \vdots \\ \vdots & \mathbb{JA}_{p^{k-1}} & \iddots & \iddots & . & \ddots & \ddots & \mathbb{JA}_{p^{k-1}} & \vdots \\ \mathbb{JA}_{p^{k-1}} & \Omega_{N_{p^{k-1}}} & \mathbb{JA}_{p^{k-1}} & \cdots & \cdots & \cdots & \mathbb{JA}_{p^{k-1}} & \Omega_{K_{p^{k-1}}} & \mathbb{JA}_{p^{k-1}} \\ \Omega_{N_{p^{k-1}}} & \mathbb{JA}_{p^{k-1}} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbb{JA}_{p^{k-1}} & \Omega_{K_{p^{k-1}}} \end{bmatrix}_{\alpha \times \alpha}.$$

Let 1_α be an $\alpha \times 1$ matrix with all entries are 1, and let $\mathbf{0}$ be the $1 \times p^{k-1}$ matrix with all entries are zeros. Since $\Omega_1 1_\alpha = (p^k - 2p^{k-1} - 1)1_\alpha$, therefore $(p^k - 2p^{k-1} - 1)$ is an eigenvalue of Ω_1 with multiplicity 1. Furthermore, we note that -1 is an eigenvalue for the matrix $\Omega_{K_{p^{k-1}}}$ having a multiplicity $p^{k-1} - 1$. Let $\{X_r, 1 \le r \le p^{k-1} - 1\}$ be the eigenspace corresponding to -1 with respect to $\Omega_{K_{p^{k-1}}}$, where $X_r = \begin{bmatrix} d_1^r & d_2^r & \cdots & d_{p^{k-1}}^r \end{bmatrix}^t$. Since $\Omega_{K_{p^{k-1}}} X_r = -X_r$, thus, $\forall r$, $\sum_{j \ne s} d_j^r = -d_s^r$, $s = 1, \cdots, p^{k-1}$. Therfore, $\sum_{j=1}^{p^{k-1}} d_j^r = 0$ and hence, $\mathbb{JA}_{p^{k-1}} X_r = \mathbf{0}\ \forall r$.

Now, let $Y_1^r = \begin{bmatrix} X_r^t & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}_{1\times\alpha}^t$, $Y_2^r = \begin{bmatrix} \mathbf{0} & X_r^t & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}_{1\times\alpha}^t$ up to $Y_{p-1}^r = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & X_r^t \end{bmatrix}_{1\times\alpha}^t$. Then $\Omega_1 Y_l^r = -Y_l^r$, $l = 1, \cdots, p-1$. So -1 is an eigenvalue for Ω_1 with multiplicity $(p^{k-1} - 1)(p-1)$.

For example, let $p = 5$ and $k = 2$. Then:

$$\Omega_1 Y_2^3 = \begin{bmatrix} \Omega_{K_5} & \mathbb{JA}_5 & \mathbb{JA}_5 & \Omega_{N_5} \\ \mathbb{JA}_5 & \Omega_{K_5} & \Omega_{N_5} & \mathbb{JA}_5 \\ \mathbb{JA}_5 & \Omega_{N_5} & \Omega_{K_5} & \mathbb{JA}_5 \\ \Omega_{N_5} & \mathbb{JA}_5 & \mathbb{JA}_5 & \Omega_{K_5} \end{bmatrix}_{20\times 20} \begin{bmatrix} \mathbf{0}^t \\ X_3 \\ \mathbf{0}^t \\ \mathbf{0}^t \end{bmatrix}_{20\times 1}$$

$$= \begin{bmatrix} \mathbf{0}^t \\ -X_3 \\ \mathbf{0}^t \\ \mathbf{0}^t \end{bmatrix}_{20\times 1}$$

Next, we show that $(p^{k-1}-1)$ and $-(p^{k-1}+1)$ are eigenvalues for Ω_1 with multiplicities $\frac{p-1}{2}$ and $\frac{p-3}{2}$, respectively.

Let $C_i, 1 \le i \le \lfloor \frac{p}{2} \rfloor$, be an $\alpha \times 1$ matrix such that:

$$C_1 = \begin{bmatrix} \mathbf{1}_{p^{k-1}} \\ \mathbf{0}^t \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \mathbf{0}^t \\ -\mathbf{1}_{p^{k-1}} \end{bmatrix}, C_2 = \begin{bmatrix} \mathbf{0}^t \\ \mathbf{1}_{p^{k-1}} \\ \mathbf{0}^t \\ \vdots \\ \vdots \\ \mathbf{0}^t \\ -\mathbf{1}_{p^{k-1}} \\ \mathbf{0}^t \end{bmatrix}, \cdots, C_{\lfloor \frac{p}{2} \rfloor} = \begin{bmatrix} \mathbf{0}^t \\ \vdots \\ \mathbf{0}^t \\ \mathbf{1}_{p^{k-1}} \\ -\mathbf{1}_{p^{k-1}} \\ \mathbf{0}^t \\ \vdots \\ \mathbf{0}^t \end{bmatrix}$$

.

Since $\{C_j : 1 \le j \le \lfloor \frac{p}{2} \rfloor\}$ are linearly independent vectors such that $\Omega_1 C_j = (p^{k-1} - 1)C_j \ \forall j$, thus $p^{k-1}-1$ is an eigenvalue of the matrix Ω_1 having a multiplicity $\frac{p-1}{2}$ as desired.

On the other hand, let $Q_j, 1 \le j \le \lfloor \frac{p}{2} \rfloor$, be an $\alpha \times 1$ matrix such that:

$$Q_1 = \begin{bmatrix} \mathbf{1}_{p^{k-1}} \\ -\mathbf{1}_{p^{k-1}} \\ \mathbf{0}^t \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \mathbf{0}^t \\ -\mathbf{1}_{p^{k-1}} \\ \mathbf{1}_{p^{k-1}} \end{bmatrix}, Q_2 = \begin{bmatrix} \mathbf{1}_{p^{k-1}} \\ \mathbf{0}^t \\ -\mathbf{1}_{p^{k-1}} \\ \mathbf{0}^t \\ \vdots \\ \vdots \\ \mathbf{0}^t \\ -\mathbf{1}_{p^{k-1}} \\ \mathbf{0}^t \\ \mathbf{1}_{p^{k-1}} \end{bmatrix}, \cdots, Q_{\lfloor \frac{p}{2} \rfloor - 1} = \begin{bmatrix} \mathbf{1}_{p^{k-1}} \\ \mathbf{0}^t \\ \vdots \\ \mathbf{0}^t \\ -\mathbf{1}_{p^{k-1}} \\ -\mathbf{1}_{p^{k-1}} \\ \mathbf{0}^t \\ \vdots \\ \mathbf{0}^t \\ \mathbf{1}_{p^{k-1}} \end{bmatrix}.$$

Then $\{Q_j : 1 \le j \le \lfloor \frac{p}{2} \rfloor - 1\}$ are linearly independent vectors in which $\Omega_1 Q_j = -(p^{k-1}+1)Q_j \ \forall j$, and thus $-(p^{k-1}+1)$ is an eigenvalue of the matrix Ω_1 having a multiplicity $\frac{p-3}{2}$.

Therefore, the spectrum of the graph G_1 is:

$$\sigma(\Omega_1) = \left\{ (p^k - 2p^{k-1} - 1)^1, (-1)^{(p^{k-1}-1)(p-1)}, (p^{k-1}-1)^{\frac{p-1}{2}}, (-p^{k-1}-1)^{\frac{p-3}{2}} \right\},$$

where $p^k - p^{k-1}$ is the degree of the characteristic polynomial related to Ω_1. Now, since $p \ge 3$ and $k \ge 2$, it is obvious that $p^k - 2p^{k-1} - 1$ and $p^{k-1} - 1$ are positive values. Thus $\max\{\epsilon_-(\Omega_1), \epsilon_+(\Omega_1)\} = (p^{k-1}-1)(p-1) + \frac{p-3}{2} = \lfloor \frac{p}{2} \rfloor (2p^{k-1} - 1) - 1$. $\square$

Theorem 2.6. *The graph $G(Z_{p^k})$ is an eigensharp graph.*

Proof. By Lemmas 2.4 and 2.5, we have $\max\{\epsilon_-(\Omega_1), \epsilon_+(\Omega_1)\} = bp(G_1)$ and, hence, G_1 is an eigensharp graph. Since $G_0 \simeq K_{p^{k-1},p^{k-1}}$ is also eigensharp, and since $G(Z_{p^k}) = G_0 \vee G_1$, thus by Proposition 1.1, $G(Z_{p^k})$ is an eigensharp graph. □

3. Conclusion

In this study, we determined the clique number and the independence number for the unit graph $G(Z_{p^k})$ for all prime p and $k \geq 1$. Furthermore, we proved that the graph $G(Z_{p^k})$ is eigensharp. Although eigensharp is essentially theoretical, it has applications in different disciplines. Its applications include network designs and communications, chemistry, molecular graphs, information and quantum physics, coding and design theory, computer science, and machine learning.

We demonstrated that $G(Z_{p^k})$ is isomorphic to the graph $G_0 \vee G_1$, where G_0 and G_1 are special subgraphs of $G(Z_{p^k})$. Studying the adjacency matrices of G_0 and G_1 reveals that $\alpha-(G_0) = 1 = bp(G_0)$, and $\alpha-(G_1) = \lfloor \frac{p}{2} \rfloor (2p^{k-1} - 1) - 1$. Furthermore, we described a biclique partition for G_1 with cardinality $\lfloor \frac{p}{2} \rfloor (2p^{k-1} - 1) - 1$. Therefore, $\alpha-(G_1) = bp(G_1)$, and Proposition 1.1, implying that the graph of $G(Z_{p^k})$ is eigensharp..

Finally, we ask whether the eigensharp property holds for the unit graphs over the rings $Z_{p^r q^s}$, where p, q are distinct primes and $r, s \in Z^+$. We have explored various examples to answer this question, but our research is still ongoing.

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