Gulf Journal of Mathematics
Vol 21, Issue 1 (2025) 512-521

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v21i1.3500

PARAMETRIC METRIC SPACES AND INTEGRAL-TYPE CONTRACTIONS: NEW FIXED POINT THEOREMS WITH APPLICATIONS

DINESH KANNAN[1*], HAWA IBNOUF OSMAN IBNOUF[2] and SHOBA BALASUBRAMANIYAM[3]

ABSTRACT. In this article, we establish new fixed point theorems for self-mappings defined on complete parametric metric spaces. By introducing a generalized integral-type contractive condition involving an auxiliary control function. The approach relies on the concept of convergence and completeness in parametric settings, which naturally generalizes classical metric frameworks. As an application, we demonstrate the existence of solutions to nonlinear integral equations, highlighting the applicability of the proposed framework. Several examples are presented to illustrate the effectiveness and novelty of the obtained results.

Keywords. Fixed point theorem; Parametric metric space; Integral-type contraction; Self-mapping; Nonlinear integral equation

2020 Mathematics Subject Classification. Primary 47H10; Secondary 54H25, 45G10

1. Introduction

Fixed point (FP) theory has become one of the central tools of modern analysis due to its broad applicability in differential equations, optimization, image processing, computer science, and dynamic systems. The Banach contraction principle is the cornerstone of the theory, but many researchers have extended it in order to treat problems that cannot be addressed in the standard metric framework. Among these, integral-type contractions, first introduced by Branciari and later investigated by several authors, play a fundamental role in broadening the scope of classical contraction principles.

Over the last two decades, various generalized distance structures have been studied in FP theory, such as b-metric spaces, cone metric spaces, fuzzy metric spaces, and modular metric spaces. These generalizations have been motivated by the need to model more complex systems, often involving uncertainty, nonlinearity, or additional structural parameters. In particular, the notion of Parametric metric space (PMSs), introduced and developed in [9, 11, 5, 6, 10], has attracted considerable attention. This setting incorporates a parameter into the distance function, thereby generalizing

Date: Received: Sep 5, 2025; Accepted: Sep 27, 2025.
* Corresponding author

the classical and b-metric cases, and provides a flexible framework for convergence, continuity, and stability analysis.

FP results in *PMS*s have been investigated in various contexts. For instance, Hussain et al. [9] obtained fundamental results in fuzzy b-metric and parametric metric settings, while Krishnakumar and Sanatammappa [11], Jain et al. [10] studied coincidence and common FP theorems under nonlinear conditions. Ege and coauthors [5, 6] provided new perspectives and applications in this area. More recently, FP results in related generalized spaces have been explored in [2, 4, 3, 12, 13, 7, 8], highlighting the ongoing vitality of the field.

Despite these advances, the study of *integral-type contractions* in *PMS*s remains relatively unexplored. Since integral-type conditions generalize classical contractive mappings and are naturally suited for integral and functional equations, combining them with the parametric metric setting provides a promising direction for extending the applicability of FP theory.

The main objective of this paper is to establish several new FP theorems for mappings satisfying integral-type contractive conditions in *Com-PMS*s. Our results unify and extend earlier contributions[11], yielding both existence and uniqueness of FPs under a variety of contractive assumptions. Furthermore, to illustrate the utility of our theoretical findings, we apply them to prove the solvability of a nonlinear Fredholm integral equation, which demonstrates the effectiveness of the proposed framework in handling problems arising in applied analysis.

2. Preliminaries

In this section, we recall some basic notions that will be useful throughout the paper. The terminology is adapted to the framework of *PMS*.

Definition 2.1. [8] Let $\mathcal{Y}$ be a nonempty set and let $\rho : \mathcal{Y} \times \mathcal{Y} \times (0, \infty) \to [0, \infty)$ be a function such that for every $\mathfrak{l}, v, w \in \mathcal{Y}$ and for all $\mathfrak{z} > 0$, the following hold:

(1) $\rho(\mathfrak{l}, v, \mathfrak{z}) = 0$ if and only if $\mathfrak{l} = v$,
(2) $\rho(\mathfrak{l}, v, \mathfrak{z}) = \rho(v, \mathfrak{l}, \mathfrak{z})$,
(3) $\rho(\mathfrak{l}, w, \mathfrak{z}) \leq \rho(\mathfrak{l}, v, \mathfrak{z}) + \rho(v, w, \mathfrak{z})$.

Then the pair $(\mathcal{Y}, \rho)$ is called a *PMS*.

Definition 2.2. [8] Let $(\mathcal{Y}, \rho)$ be a *PMS*. A sequence $\{u_n\}$ in $\mathcal{Y}$ is named to converge to $\mathfrak{l} \in \mathcal{Y}$ if for every $\mathfrak{z} > 0$,

$$\lim_{n\to\infty} \rho(u_n, \mathfrak{l}, \mathfrak{z}) = 0.$$

A sequence $\{u_n\}$ is called a Cauchy sequence if for every $\mathfrak{z} > 0$,

$$\lim_{m,n\to\infty} \rho(u_m, u_n, \mathfrak{z}) = 0.$$

The space $(\mathcal{Y}, \rho)$ is named to be complete if every Cauchy sequence converges to some element of $\mathcal{Y}$.

Definition 2.3. Let $(\mathcal{Y}, \rho)$ be a *PMS* and let $\Lambda : \mathcal{Y} \to \mathcal{Y}$ be a self-mapping. An element $\mathfrak{l}^* \in \mathcal{Y}$ is called a FP of Λ if

$$\Lambda(\mathfrak{l}^*) = \mathfrak{l}^*.$$

Definition 2.4. [1] Let $(\mathcal{Y}, \rho)$ be a *PMS*. A self-mapping $\Lambda : \mathcal{Y} \to \mathcal{Y}$ is named to satisfy an integral-type contraction condition if there exists $\beta \in [0, 1)$ and a function

$\psi : [0,\infty) \to [0,\infty)$, non-decreasing with $\psi(\mathfrak{z}) = 0$ if and only if $\mathfrak{z} = 0$, such that

$$\int_0^{\rho(\Lambda\mathfrak{l},\Lambda v,\mathfrak{z})} \psi(\mathfrak{f})\,\rho\mathfrak{f} \;\le\; \beta \int_0^{M(\mathfrak{l},v,\mathfrak{z})} \psi(\mathfrak{f})\,\rho\mathfrak{f},$$

for all $\mathfrak{l}, v \in \mathcal{Y}$, where

$$M(\mathfrak{l},v,\mathfrak{z}) = \max\Big\{\rho(\mathfrak{l},v,\mathfrak{z}), \rho(\mathfrak{l},\Lambda\mathfrak{l},\mathfrak{z}), \rho(v,\Lambda v,\mathfrak{z}), \rho(\mathfrak{l},\Lambda v,\mathfrak{z}), \rho(\Lambda\mathfrak{l},v,\mathfrak{z})\Big\}.$$

3. Main results

Theorem 3.1. *Let $(\mathcal{Y},\rho)$ be a* Com-PMS *and let $\Lambda : \mathcal{Y} \to \mathcal{Y}$ be a continuous self-mapping. Suppose there exists a constant $\beta \in [0,1)$ and a nonnegative Lebesgue–integrable function $\psi : [0,\infty) \to [0,\infty)$ satisfying*

$$\int_0^{\infty} \psi(\mathfrak{z})\,\rho\mathfrak{z} < \infty,$$

such that for all $\mathfrak{l}, v \in \mathcal{Y}$ with $\mathfrak{l} \ne v$ and for every $\mathfrak{z} > 0$, we have

$$\int_0^{\rho(\Lambda\mathfrak{l},\Lambda v,\mathfrak{z})} \psi(\mathfrak{z})\,\rho\mathfrak{z} \;\le\; \beta \max\left\{ \int_0^{\rho(\mathfrak{l},v,\mathfrak{z})} \psi(\mathfrak{z})\,\rho\mathfrak{z},\ \int_0^{\rho(\mathfrak{l},\Lambda\mathfrak{l},\mathfrak{z})} \psi(\mathfrak{z})\,\rho\mathfrak{z},\ \int_0^{\rho(v,\Lambda v,\mathfrak{z})} \psi(\mathfrak{z})\,\rho\mathfrak{z} \right\}.$$

Then Λ has a unique FP in $\mathcal{Y}$.

Proof. Write $\Phi : [0,\infty) \to [0,\infty)$ for

$$\Phi(r) := \int_0^{r} \psi(\mathfrak{z})\,\rho\mathfrak{z}.$$

Then Φ is continuous, nondecreasing, $\Phi(0) = 0$, and $\Phi(r) \le \int_0^\infty \psi(\mathfrak{z})\,\rho\mathfrak{z} < \infty$ for all $r \ge 0$. Fix $\mathfrak{z} > 0$ once and for all, and choose $w_0 \in \mathcal{Y}$ arbitrarily; define the Picard sequence $w_{n+1} := \Lambda w_n$ for $n \ge 0$.

For $n \ge 1$, apply the contractive condition to the pair $(\mathfrak{l}, v) = (w_{n-1}, w_n)$ to obtain

$$\begin{aligned}\Phi\big(\rho(w_n, w_{n+1}, \mathfrak{z})\big) &= \Phi\big(\rho(\Lambda w_{n-1}, \Lambda w_n, \mathfrak{z})\big)\\ &\le \beta \max\Big\{\Phi\big(\rho(w_{n-1}, w_n, \mathfrak{z})\big),\ \Phi\big(\rho(w_{n-1}, w_n, \mathfrak{z})\big),\ \Phi\big(\rho(w_n, w_{n+1}, \mathfrak{z})\big)\Big\}.\end{aligned}$$

If $\Phi\big(\rho(w_n, w_{n+1}, \mathfrak{z})\big)$ realizes the maximum on the right, then
$(1-\beta)\,\Phi\big(\rho(w_n, w_{n+1}, \mathfrak{z})\big) \le 0$,
hence $\Phi\big(\rho(w_n, w_{n+1}, \mathfrak{z})\big) = 0$. Otherwise,

$$\Phi\big(\rho(w_n, w_{n+1}, \mathfrak{z})\big) \le \beta\, \Phi\big(\rho(w_{n-1}, w_n, \mathfrak{z})\big).$$

In either case,

$$\Phi\big(\rho(w_n, w_{n+1}, \mathfrak{z})\big) \le \beta\, \Phi\big(\rho(w_{n-1}, w_n, \mathfrak{z})\big) \qquad (n \ge 1). \tag{3.1}$$

By induction, (3.1) yields

$$\Phi\big(\rho(w_n, w_{n+1}, \mathfrak{z})\big) \le \beta^{\,n}\, \Phi\big(\rho(w_0, w_1, \mathfrak{z})\big) \qquad (n \ge 0),$$

hence $\Phi\big(\rho(w_n, w_{n+1}, \mathfrak{z})\big) \to 0$ as $n \to \infty$.

Next, let $m > n \ge 1$ and apply the contractive condition to $(\mathfrak{l}, v) = (w_{m-1}, w_{n-1})$:

$$\Phi\big(\rho(w_m, w_n, \mathfrak{z})\big) \le \beta \max\big\{\Phi\big(\rho(w_{m-1}, w_{n-1}, \mathfrak{z})\big),\ \Phi\big(\rho(w_{m-1}, w_m, \mathfrak{z})\big),\ \Phi\big(\rho(w_{n-1}, w_n, \mathfrak{z})\big)\big\}.$$

Iterating this estimate n times (each iteration lowers both indices by one in the first term on the right) gives

$$\Phi\big(\rho(w_m, w_n, \mathfrak{z})\big) \leq \beta^n \max\Big\{\Phi\big(\rho(w_{m-n}, w_0, \mathfrak{z})\big),\ \max_{k\geq 0} \Phi\big(\rho(w_{k+1}, w_k, \mathfrak{z})\big)\Big\}.$$

The second maximum equals $\Phi\big(\rho(w_1, w_0, \mathfrak{z})\big)$ by (3.1). Moreover

$$\Phi\big(\rho(w_{m-n}, w_0, \mathfrak{z})\big) \leq \int_0^\infty \psi(\mathfrak{z})\, \rho\mathfrak{z}.$$

Consequently, there is a finite constant

$$C_t := \max\Big\{\int_0^\infty \psi(\mathfrak{z})\, \rho\mathfrak{z},\ \Phi\big(\rho(w_1, w_0, \mathfrak{z})\big)\Big\}$$

such that for all $m > n$,

$$\Phi\big(\rho(w_m, w_n, \mathfrak{z})\big) \leq C_t\, \beta^n. \tag{3.2}$$

Since $\beta \in [0, 1)$, the right-hand side tends to 0 as $n \to \infty$, uniformly in $m > n$. Because Φ is nondecreasing with $\Phi(r) > 0$ for every $r > 0$ (this holds whenever ψ is not almost everywhere zero on any neighborhood of 0), (3.2) implies $\rho(w_m, w_n, \mathfrak{z}) \to 0$ as $m, n \to \infty$.

Thus (w_n) is Cauchy in $(\mathcal{Y}, \rho(\cdot, \cdot, \mathfrak{z}))$ for the chosen $\mathfrak{z} > 0$. Since the above argument holds for an arbitrary $\mathfrak{z} > 0$, we conclude that (w_n) is Cauchy in the sense of Definition 2.2, that is,

$$\lim_{m,n\to\infty} \rho(w_m, w_n, z) = 0 \quad \forall\, z > 0.$$

By completeness of $(\mathcal{Y}, \rho)$, there exists $z \in \mathcal{Y}$ such that

$$\lim_{n\to\infty} \rho(w_n, z, \mathfrak{z}) = 0 \quad \forall\, \mathfrak{z} > 0,$$

i.e., $w_n \to z$ in the PMS.

The relation $w_{n+1} = \Lambda w_n$ and the continuity of Λ give

$$\Lambda z = \Lambda\big(\lim_{n\to\infty} w_n\big) = \lim_{n\to\infty} \Lambda w_n = \lim_{n\to\infty} w_{n+1} = z,$$

so z is a FP of Λ.

For uniqueness, let $z_1, z_2 \in \mathcal{Y}$ be FPs. Applying the contractive condition to $(\mathfrak{l}, v) = (z_1, z_2)$ yields

$$\Phi\big(\rho(z_1, z_2, \mathfrak{z})\big) = \Phi\big(\rho(\Lambda z_1, \Lambda z_2, \mathfrak{z})\big) \leq \beta \max\big\{\Phi\big(\rho(z_1, z_2, \mathfrak{z})\big),\ 0,\ 0\big\} = \beta\, \Phi\big(\rho(z_1, z_2, \mathfrak{z})\big).$$

Hence $(1-\beta)\, \Phi\big(\rho(z_1, z_2, \mathfrak{z})\big) \leq 0$, so $\Phi\big(\rho(z_1, z_2, \mathfrak{z})\big) = 0$ and therefore $\rho(z_1, z_2, \mathfrak{z}) = 0$. Thus $z_1 = z_2$, proving uniqueness. □

Corollary 3.2. *Let $(\mathcal{Y}, \rho)$ be a* Com-PMS *and let $\Lambda : \mathcal{Y} \to \mathcal{Y}$ be continuous. Suppose there exists $\beta \in [0, 1)$ and a nonnegative Lebesgue–integrable function $\psi : [0, \infty) \to [0, \infty)$ with $\int_0^\infty \psi(\mathfrak{z})\, \rho\mathfrak{z} < \infty$ such that for all $\mathfrak{l}, v \in \mathcal{Y}$ and every $\mathfrak{z} > 0$,*

$$\int_0^{\rho(\Lambda\mathfrak{l}, \Lambda v, \mathfrak{z})} \psi(\mathfrak{z})\, \rho\mathfrak{z} \leq \beta \int_0^{\rho(\mathfrak{l}, v, \mathfrak{z})} \psi(\mathfrak{z})\, \rho\mathfrak{z}.$$

Then Λ admits a unique FP in $\mathcal{Y}$.

Proof. Let $\Phi(r) := \int_0^r \psi(\mathfrak{z})\,\rho\mathfrak{z}$ for $r \geq 0$. The hypothesis reads $\Phi\big(\rho(\Lambda\mathfrak{l}, \Lambda v, \mathfrak{z})\big) \leq \beta\,\Phi\big(\rho(\mathfrak{l}, v, \mathfrak{z})\big)$ for all $\mathfrak{l}, v \in \mathcal{Y}$ and all $\mathfrak{z} > 0$. This is a special (simpler) instance of the integral-type inequality in the preceding theorem, where the maximum on the right is realized by $\Phi\big(\rho(\mathfrak{l}, v, \mathfrak{z})\big)$. Therefore all conclusions of that theorem apply: the Picard iteration starting from any $w_0 \in \mathcal{Y}$ yields a sequence converging to a point $z \in \mathcal{Y}$ with $\Lambda z = z$. Uniqueness follows by applying the same inequality to two FPs and observing that $(1-\beta)\Phi\big(\rho(z_1, z_2, \mathfrak{z})\big) \leq 0$, whence $\rho(z_1, z_2, \mathfrak{z}) = 0$ and $z_1 = z_2$. □

Theorem 3.3. *Let $(\mathcal{Y}, \rho)$ be a* Com-PMS *and let $\Lambda : \mathcal{Y} \to \mathcal{Y}$ be continuous. Assume there exist constants $\alpha, \beta \in [0, \frac{1}{2})$ with $\alpha + \beta < \frac{1}{2}$ and a nonnegative Lebesgue–integrable function $\psi : [0, \infty) \to [0, \infty)$ satisfying*

$$\int_0^\infty \psi(\mathfrak{z})\,\rho\mathfrak{z} < \infty.$$

Define $\Phi(r) := \int_0^r \psi(\mathfrak{z})\,\rho\mathfrak{z}$ for $r \geq 0$. Suppose that for all $\mathfrak{l}, v \in \mathcal{Y}$ and every $\mathfrak{z} > 0$ one has

$$\Phi\big(\rho(\Lambda\mathfrak{l}, \Lambda v, \mathfrak{z})\big) \leq \alpha\Big\{\Phi\big(\rho(\mathfrak{l}, \Lambda\mathfrak{l}, \mathfrak{z})\big) + \Phi\big(\rho(v, \Lambda v, \mathfrak{z})\big)\Big\} + \beta\Big\{\Phi\big(\rho(\mathfrak{l}, \Lambda v, \mathfrak{z})\big) + \Phi\big(\rho(\Lambda\mathfrak{l}, v, \mathfrak{z})\big)\Big\}.$$

Then Λ has at least one FP in $\mathcal{Y}$.

Proof. Write $\Phi : [0, \infty) \to [0, \infty)$ for

$$\Phi(r) := \int_0^r \psi(\mathfrak{z})\,\rho\mathfrak{z},$$

so Φ is continuous, nondecreasing and $\Phi(0) = 0$. Fix an arbitrary $\mathfrak{z} > 0$ and choose $y_0 \in \mathcal{Y}$. Define the sequence (y_n) by $y_{n+1} := \Lambda y_n$ for $n \geq 0$.

Apply the contractive inequality to the pair $(\mathfrak{l}, v) = (y_{n-1}, y_n)$ (for $n \geq 1$). Using $y_n = \Lambda y_{n-1}$ this gives

$$\Phi\big(\rho(y_n, y_{n+1}, \mathfrak{z})\big) \leq \alpha\{\Phi(\rho(y_{n-1}, y_n, \mathfrak{z})) + \Phi(\rho(y_n, y_{n+1}, \mathfrak{z}))\} + \beta\{\Phi(\rho(y_{n-1}, y_{n+1}, \mathfrak{z})) + \Phi(0)\}.$$

Because $\Phi(0) = 0$ and Φ is nondecreasing, and since the parametric metric satisfies

$$\rho(y_{n-1}, y_{n+1}, \mathfrak{z}) \leq \rho(y_{n-1}, y_n, \mathfrak{z}) + \rho(y_n, y_{n+1}, \mathfrak{z}),$$

the subadditivity of Φ (which follows from $\psi \geq 0$) yields

$$\Phi\big(\rho(y_{n-1}, y_{n+1}, \mathfrak{z})\big) \leq \Phi\big(\rho(y_{n-1}, y_n, \mathfrak{z})\big) + \Phi\big(\rho(y_n, y_{n+1}, \mathfrak{z})\big).$$

Combining the previous two displays and rearranging terms produces

$$(1-\alpha)\,\Phi\big(\rho(y_n, y_{n+1}, \mathfrak{z})\big) \leq \alpha\,\Phi\big(\rho(y_{n-1}, y_n, \mathfrak{z})\big) + \beta\Big\{\Phi\big(\rho(y_{n-1}, y_n, \mathfrak{z})\big) + \Phi\big(\rho(y_n, y_{n+1}, \mathfrak{z})\big)\Big\}.$$

Collecting the terms involving $\Phi(\rho(y_n, y_{n+1}, \mathfrak{z}))$ on the left gives

$$\frac{1-\alpha-\beta}{1-\alpha}\,\Phi\big(\rho(y_n, y_{n+1}, \mathfrak{z})\big) \leq \frac{\alpha+\beta}{1-\alpha}\,\Phi\big(\rho(y_{n-1}, y_n, \mathfrak{z})\big).$$

Set

$$r := \frac{\alpha+\beta}{1-\alpha-\beta}.$$

The hypothesis $\alpha + \beta < \frac{1}{2}$ implies $0 \leq r < 1$. The last inequality therefore simplifies to the geometric estimate

$$\Phi\big(\rho(y_n, y_{n+1}, \mathfrak{z})\big) \leq r\,\Phi\big(\rho(y_{n-1}, y_n, \mathfrak{z})\big) \qquad (n \geq 1),$$

and by induction

$$\Phi\big(\rho(y_n, y_{n+1}, \mathfrak{z})\big) \le r^n\, \Phi\big(\rho(y_0, y_1, \mathfrak{z})\big) \qquad (n \ge 0).$$

Hence $\Phi\big(\rho(y_n, y_{n+1}, \mathfrak{z})\big) \to 0$ as $n \to \infty$, which implies $\rho(y_n, y_{n+1}, \mathfrak{z}) \to 0$.

Let $m > n \ge 1$. Applying the contractive condition to $(\mathfrak{l}, v) = (y_{m-1}, y_{n-1})$ and using the same subadditivity argument as above yields

$$\Phi\big(\rho(y_m, y_n, \mathfrak{z})\big) \le \alpha\Big\{\Phi\big(\rho(y_{m-1}, y_{n-1}, \mathfrak{z})\big) + \Phi\big(\rho(y_{m-1}, y_m, \mathfrak{z})\big)\Big\}$$
$$+ \beta\Big\{\Phi\big(\rho(y_{m-1}, y_n, \mathfrak{z})\big) + \Phi\big(\rho(y_m, y_{n-1}, \mathfrak{z})\big)\Big\}.$$

Iterating this inequality n times (each iteration lowers both indices in the first term on the right) and using the uniform bound

$$\Phi\big(\rho(y_k, y_{k+1}, \mathfrak{z})\big) \le \Phi\big(\rho(y_0, y_1, \mathfrak{z})\big) \quad \text{for all } k \ge 0$$

(which follows from the geometric estimate), one obtains constants $C_t > 0$ and $0 \le r < 1$ such that

$$\Phi\big(\rho(y_m, y_n, \mathfrak{z})\big) \le C_t\, r^n \qquad (m > n).$$

Because $r^n \to 0$ as $n \to \infty$, the right-hand side tends to 0 uniformly in $m > n$. Since Φ is nondecreasing and $\Phi(r) > 0$ for every $r > 0$ (provided ψ is not a.e. zero on neighborhoods of 0), it follows that $\rho(y_m, y_n, \mathfrak{z}) \to 0$ as $m, n \to \infty$. Thus (y_n) is Cauchy in the complete space $(\mathcal{Y}, \rho(\cdot,\cdot,\mathfrak{z}))$, hence there exists $y^* \in \mathcal{Y}$ with $y_n \to y^*$.

The relation $y_{n+1} = \Lambda y_n$ and the continuity of Λ imply

$$\Lambda y^* = \Lambda\big(\lim_{n\to\infty} y_n\big) = \lim_{n\to\infty} \Lambda y_n = \lim_{n\to\infty} y_{n+1} = y^*,$$

so y^* is a FP of Λ. This proves existence of at least one FP.

Uniqueness is not guaranteed under the present hypotheses; stronger bounds (for instance, a strict contraction of the form $\Phi(\rho(\Lambda\mathfrak{l}, \Lambda v, \mathfrak{z})) \le \kappa\, \Phi(\rho(\mathfrak{l}, v, \mathfrak{z}))$ with $0 \le \kappa < 1$) would yield uniqueness. □

Theorem 3.4. *Let $(\mathcal{Y}, \rho)$ be a* Com-PMS *and let $\Lambda : \mathcal{Y} \to \mathcal{Y}$ be continuous. Assume there exist nonnegative real constants α, β, γ with*

$$\beta + \gamma - 2\alpha > 1, \qquad \gamma - \alpha > 1,$$

and a nonnegative Lebesgue–integrable function $\psi : [0,\infty) \to [0,\infty)$ satisfying

$$\int_0^\infty \psi(\mathfrak{z})\, d\mathfrak{z} < \infty,$$

and such that

$$\Phi(r) := \int_0^r \psi(\mathfrak{z})\, d\mathfrak{z} > 0 \quad \textit{for all } r > 0.$$

Assume further that for every pair $\mathfrak{l}, v \in \mathcal{Y}$ with $\mathfrak{l} \ne v$ and every $\mathfrak{z} > 0$ the following inequality holds:

$$\Phi\big(\rho(\Lambda\mathfrak{l}, \Lambda v, \mathfrak{z})\big) + \alpha\, \Phi\big(\rho(v, \Lambda\mathfrak{l}, \mathfrak{z})\big) \ge \beta\, \frac{\Phi\big(\rho(\mathfrak{l}, \Lambda\mathfrak{l}, \mathfrak{z})\big)\, \Phi\big(\rho(v, \Lambda v, \mathfrak{z})\big)}{\Phi\big(\rho(\mathfrak{l}, v, \mathfrak{z})\big)} + \gamma\, \Phi\big(\rho(\mathfrak{l}, v, \mathfrak{z})\big).$$

Then Λ has a unique fixed point in $\mathcal{Y}$.

Proof. Define $\Phi : [0,\infty) \to [0,\infty)$ by

$$\Phi(r) := \int_0^r \psi(\mathfrak{z})\, d\mathfrak{z},$$

so that Φ is continuous, nondecreasing, $\Phi(0) = 0$, and

$$M := \int_0^\infty \psi(\mathfrak{z})\, d\mathfrak{z} < \infty,$$

hence $\Phi(r) \le M$ for every $r \ge 0$.

Fix an arbitrary $\mathfrak{z} > 0$ and pick $y_0 \in \mathcal{Y}$. Define the iterative sequence by

$$y_{n+1} := \Lambda y_n \qquad (n \ge 0).$$

Apply the inequality from the statement with $\mathfrak{l} = y_{n-1}$ and $v = y_n$ (valid for $n \ge 1$). Since $\Lambda\mathfrak{l} = y_n$ and $\Lambda v = y_{n+1}$, and using $\Phi(0) = 0$, we obtain

$$\Phi\big(\rho(y_n, y_{n+1}, \mathfrak{z})\big) \ge \beta\, \Phi\big(\rho(y_n, y_{n+1}, \mathfrak{z})\big) + \gamma\, \Phi\big(\rho(y_{n-1}, y_n, \mathfrak{z})\big).$$

Rearranging gives

$$(1-\beta)\, \Phi\big(\rho(y_n, y_{n+1}, \mathfrak{z})\big) \ge \gamma\, \Phi\big(\rho(y_{n-1}, y_n, \mathfrak{z})\big).$$

Setting $\kappa := \frac{\gamma}{1-\beta}$, and noting from the numeric conditions that $\kappa > 1$, iterating yields

$$\Phi\big(\rho(y_n, y_{n+1}, \mathfrak{z})\big) \ge \kappa^{n-1}\, \Phi\big(\rho(y_1, y_2, \mathfrak{z})\big).$$

As $n \to \infty$, the right-hand side diverges unless $\Phi(\rho(y_1, y_2, \mathfrak{z})) = 0$. This forces $\rho(y_N, y_{N+1}, \mathfrak{z}) = 0$ for some N, hence y_N is a fixed point of Λ.

For uniqueness, let $z_1, z_2 \in \mathcal{Y}$ be fixed points. Applying the inequality with $(\mathfrak{l}, v) = (z_1, z_2)$ yields

$$(1+\alpha)\, \Phi\big(\rho(z_1, z_2, \mathfrak{z})\big) \ge \gamma\, \Phi\big(\rho(z_1, z_2, \mathfrak{z})\big).$$

But since $\gamma - \alpha > 1$, the only possibility is $\Phi(\rho(z_1, z_2, \mathfrak{z})) = 0$, which forces $\rho(z_1, z_2, \mathfrak{z}) = 0$ and hence $z_1 = z_2$. Thus the fixed point is unique. $\square$

Example and proof. Let $\mathcal{Y} = \mathbb{R}$ and for $\mathfrak{l}, v \in \mathbb{R}$ and $\mathfrak{z} > 0$ define the parametric metric

$$\rho(\mathfrak{l}, v, \mathfrak{z}) := |\mathfrak{l} - v|.$$

Choose the integrable control function $\psi : [0, \infty) \to (0, \infty)$ by

$$\psi(\mathfrak{z}) := e^{-\mathfrak{z}},$$

so that

$$\Phi(r) := \int_0^r \psi(\mathfrak{z})\, \rho\mathfrak{z} = 1 - e^{-r} \qquad (r \ge 0),$$

and $\displaystyle\int_0^\infty \psi(\mathfrak{z})\, \rho\mathfrak{z} = 1 < \infty$.

Define the mapping $\Lambda : \mathbb{R} \to \mathbb{R}$ by

$$\Lambda(x) := \tfrac{1}{2}x \qquad (x \in \mathbb{R}).$$

Proposition 3.5. *For every $\mathfrak{l}, v \in \mathbb{R}$ and every $\mathfrak{z} > 0$ we have*

$$\Phi\big(\rho(\Lambda\mathfrak{l}, \Lambda v, \mathfrak{z})\big) \le \tfrac{1}{2}\, \Phi\big(\rho(\mathfrak{l}, v, \mathfrak{z})\big).$$

Consequently, Λ has a unique FP in $\mathbb{R}$.

Proof. Fix $\mathfrak{l}, v \in \mathbb{R}$ and $\mathfrak{z} > 0$. Since $\rho(\Lambda\mathfrak{l}, \Lambda v, \mathfrak{z}) = |\frac{1}{2}\mathfrak{l} - \frac{1}{2}v| = \frac{1}{2}|\mathfrak{l} - v|$, we compute

$$\Phi\big(\rho(\Lambda\mathfrak{l}, \Lambda v, \mathfrak{z})\big) = 1 - e^{-\frac{1}{2}|\mathfrak{l}-v|} \quad \text{and} \quad \Phi\big(\rho(\mathfrak{l}, v, \mathfrak{z})\big) = 1 - e^{-|\mathfrak{l}-v|}.$$

The elementary inequality $e^{-a/2} \ge e^{-a/1/} = \sqrt{e^{-a}}$ does not directly compare the two Φ values; instead observe the algebraic identity

$$1 - e^{-r/2} = \frac{1 - e^{-r}}{1 + e^{-r/2}}$$

(valid for $r \geq 0$ after multiplying numerator and denominator by $1 - e^{-r/2}$ and simplifying). Putting $r := |\mathfrak{l} - v|$ yields

$$\Phi\big(\rho(\Lambda\mathfrak{l}, \Lambda v, \mathfrak{z})\big) = \frac{\Phi\big(\rho(\mathfrak{l}, v, \mathfrak{z})\big)}{1 + e^{-|\mathfrak{l}-v|/2}}.$$

Since $1 + e^{-|\mathfrak{l}-v|/2} \geq 2$, we obtain

$$\Phi\big(\rho(\Lambda\mathfrak{l}, \Lambda v, \mathfrak{z})\big) \leq \frac{1}{2}\,\Phi\big(\rho(\mathfrak{l}, v, \mathfrak{z})\big).$$

Thus the integral-type contraction inequality holds with constant $\beta = \frac{1}{2}$. (This is precisely the hypothesis of the Corollary of Theorem 3.1 in integral form.)

Now apply the standard Picard iteration argument: start from any $y_0 \in \mathbb{R}$ and define $y_{n+1} = \Lambda y_n$. From the contraction above we get

$$\Phi\big(\rho(y_{n+1}, y_n, \mathfrak{z})\big) \leq \tfrac{1}{2}\,\Phi\big(\rho(y_n, y_{n-1}, \mathfrak{z})\big) \leq \tfrac{1}{2}^{\,n}\,\Phi\big(\rho(y_1, y_0, \mathfrak{z})\big) \xrightarrow[n\to\infty]{} 0.$$

Using the monotonicity and positivity of Φ we deduce $\rho(y_{n+1}, y_n, \mathfrak{z}) \to 0$, and a standard telescoping / Cauchy estimate (or the argument in the corollary proof) shows (y_n) is Cauchy in $\mathbb{R}$ and therefore converges to some $y^* \in \mathbb{R}$. Continuity of Λ yields

$$\Lambda y^* = \lim_{n\to\infty} \Lambda y_n = \lim_{n\to\infty} y_{n+1} = y^*,$$

so y^* is a FP.

For uniqueness, suppose $z_1, z_2 \in \mathbb{R}$ are FPs. Then

$$\Phi\big(\rho(z_1, z_2, \mathfrak{z})\big) = \Phi\big(\rho(\Lambda z_1, \Lambda z_2, \mathfrak{z})\big) \leq \tfrac{1}{2}\,\Phi\big(\rho(z_1, z_2, \mathfrak{z})\big).$$

This implies $(1 - \frac{1}{2})\Phi(\rho(z_1, z_2, \mathfrak{z})) \leq 0$, so $\Phi(\rho(z_1, z_2, \mathfrak{z})) = 0$, hence $\rho(z_1, z_2, \mathfrak{z}) = 0$ and $z_1 = z_2$. Thus the FP is unique. □

Application: a nonlinear Fredholm integral equation. Let $C[0,1]$ denote the Banach space of real-valued continuous functions on $[0,1]$ with the supremum norm

$$\|\mathfrak{l}\| := \sup_{\mathfrak{z}\in[0,1]} |\mathfrak{l}(\mathfrak{z})|.$$

Consider the nonlinear Fredholm integral equation

$$\mathfrak{l}(\mathfrak{z}) = g(\mathfrak{z}) + \int_0^1 K(\mathfrak{z}, \mathfrak{f})\, F\big(\mathfrak{f}, \mathfrak{l}(\mathfrak{f})\big)\, \rho\mathfrak{f}, \qquad \mathfrak{z} \in [0,1], \tag{3.3}$$

where $g \in C[0,1]$, $K : [0,1] \times [0,1] \to \mathbb{R}$ is continuous, and $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous in both variables and Lipschitz in the second variable.

Assume there exist constants $L \geq 0$ and $K_0 \geq 0$ such that for all $\mathfrak{f} \in [0,1]$ and all $\mathfrak{l}, v \in \mathbb{R}$,

$$|F(\mathfrak{f}, \mathfrak{l}) - F(\mathfrak{f}, v)| \leq L\,|\mathfrak{l} - v|, \qquad \sup_{\mathfrak{z},\mathfrak{f}\in[0,1]} |K(\mathfrak{z}, \mathfrak{f})| \leq K_0,$$

and that the product satisfies

$$K_0\, L < 1.$$

Define the operator $\mathfrak{z} : C[0,1] \to C[0,1]$ by

$$(\mathcal{T}\mathfrak{l})(\mathfrak{z}) := g(\mathfrak{z}) + \int_0^1 K(\mathfrak{z}, \mathfrak{f})\, F\big(\mathfrak{f}, \mathfrak{l}(\mathfrak{f})\big)\, \rho\mathfrak{f}, \qquad \mathfrak{z} \in [0,1].$$

Proposition 3.6. *Under the above assumptions the operator $\mathcal{T}$ has a unique FP $\mathfrak{l}^* \in C[0,1]$, and hence (3.3) admits a unique continuous solution on $[0,1]$.*

Proof. Equip $C[0,1]$ with the parametric metric $\rho(\mathfrak{l}, v, \mathfrak{z}) := \|\mathfrak{l} - v\|$ for every $\mathfrak{z} > 0$. (This choice is independent of $\mathfrak{z}$ and is allowed in the parametric metric framework.) Take $\psi \equiv 1$, so that

$$\Phi(r) = \int_0^r \psi(\mathfrak{z})\, \rho\mathfrak{z} = r \qquad (r \geq 0).$$

For arbitrary $\mathfrak{l}, v \in C[0,1]$ and every $\mathfrak{z} \in [0,1]$ we estimate

$$\begin{aligned}\big|(\mathcal{T}\mathfrak{l})(\mathfrak{z}) - (\mathcal{T}v)(\mathfrak{z})\big| &= \left|\int_0^1 K(\mathfrak{z},\mathfrak{f})\big[F(\mathfrak{f},\mathfrak{l}(\mathfrak{f})) - F(\mathfrak{f}, v(\mathfrak{f}))\big]\, \rho\mathfrak{f}\right| \\ &\leq \int_0^1 |K(\mathfrak{z},\mathfrak{f})|\, \big|F(\mathfrak{f},\mathfrak{l}(\mathfrak{f})) - F(\mathfrak{f}, v(\mathfrak{f}))\big|\, \rho\mathfrak{f} \\ &\leq K_0\, L \int_0^1 \|\mathfrak{l} - v\|\, \rho\mathfrak{f} = K_0\, L\, \|\mathfrak{l} - v\|.\end{aligned}$$

Taking the supremum over $\mathfrak{z} \in [0,1]$ yields

$$\|\mathcal{T}\mathfrak{l} - \mathcal{T}v\| \leq K_0\, L\, \|\mathfrak{l} - v\|.$$

With $\psi \equiv 1$ we have $\Phi(\|\mathcal{T}\mathfrak{l} - \mathcal{T}v\|) = \|\mathcal{T}\mathfrak{l} - \mathcal{T}v\|$ and $\Phi(\|\mathfrak{l} - v\|) = \|\mathfrak{l} - v\|$, so the inequality above becomes the integral-type contraction

$$\Phi\big(\rho(\mathcal{T}\mathfrak{l}, \mathcal{T}v, \mathfrak{z})\big) \leq \beta\, \Phi\big(\rho(\mathfrak{l}, v, \mathfrak{z})\big) \qquad \text{with } \beta := K_0 L < 1.$$

Thus the hypotheses of the integral-type corollary (the special case with $\psi \equiv 1$ for all $s0$.) are satisfied. Consequently, $\mathcal{T}$ admits a unique FP $\mathfrak{l}^* \in C[0,1]$. The FP equation $\mathcal{T}\mathfrak{l}^* = \mathfrak{l}^*$ is precisely the integral equation (3.3), so a unique continuous solution exists on $[0,1]$. □

Conclusion

In this paper, we established new fixed point theorems for continuous self-mappings in complete parametric metric spaces under generalized integral-type contractive conditions. These results extend classical fixed point principles, guarantee existence and uniqueness of fixed points in several cases, and are supported by illustrative examples. Finally, their applicability was demonstrated through the solvability of a nonlinear Fredholm integral equation, confirming the practical relevance of the proposed framework.

References

[1] Branciari, A. (2002). A fixed point theorem for mappings satisfying a general contractive condition of integral type. *International Journal of Mathematics and Mathematical Sciences, 29*(9), 531–536. https://doi.org/10.1155/S0161171202007524

[2] Dinesh, K., Abdalrhim, E., Jazmati, M. S., Mohamed, M. A., & Rizk, D. (2025). Fixed point theorems of multivalued mappings of integral type contraction in cone metric space and its applications. *Results in Nonlinear Analysis, 8*(1), 124–130. https://www.nonlinear-analysis.com/index.php/pub/article/view/552

[3] Dinesh, K., Gerbeti, D., Zoto, K., & Ibnouf, H. I. O. (2025). Fixed point theorems in fuzzy metric spaces via generalized rational-type contractions. *International Journal of Mathematics and Computer Science, 20*(4), 1003–1011. https://doi.org/10.69793/ijmcs/04.2025/zoto

[4] Dinesh, K., Suganya, R., Priya, V. B., Suresh, M., & Atkinswestley, A. (2025). Fixed point theorems for multivalued mappings in neutrosophic fuzzy metric spaces. *Neutrosophic Sets and Systems, 90*, 1–6. https://fs.unm.edu/NSS/1FixedPoint.pdf

[5] Ege, O., & Karaca, I. (2017). Fixed point theorems and an application in parametric metric space. *Azerbaijan Journal of Mathematics, 7*, 27–39. https://azjm.org/volumes/0701/0701-3.pdf

[6] Ege, O., & De la Sen, M. (2019). A new perspective on parametric metric spaces. *Mathematics, 7*, Article 1008. https://doi.org/10.3390/math7111008

[7] Gopalakrishnan, T., Almngoshi, H. Z., Koppula, V. K., Dinesh, K., Sonthi, V. K., Sengan, S., & Dadheech, P. (2025). Fractal image compression using digital cone metric space. *Journal of Interdisciplinary Mathematics, 28*(2), 647–658. https://doi.org/10.47974/JIM-2108

[8] Hussain, N., Khaleghizadeh, S., Salimi, P., & Abdou, A. A. N. (2014). A new approach to fixed point results in triangular intuitionistic fuzzy metric spaces. *Abstract and Applied Analysis, 2014*, Article 690139, 16 pages. https://doi.org/10.1155/2014/690139

[9] Hussain, N., Salimi, P., & Parvaneh, V. (2015). Fixed point results for various contractions in parametric and fuzzy b-metric spaces. *Journal of Nonlinear Sciences and Applications, 8*, 719–739. http://dx.doi.org/10.22436/jnsa.008.05.24

[10] Jain, R., Daheriya, R. D., & Ughade, M. (2016). Fixed point, coincidence point and common fixed point theorems under various expansive conditions in parametric metric spaces and parametric b-metric spaces. *Gazi University Journal of Science, 29*, 95–107. https://dergipark.org.tr/tr/download/article-file/230912

[11] Krishnakumar, R., & Sanatammappa, N. P. (2016). Some fixed point theorems in parametric b-metric space. *International Journal of Mathematical Sciences and Engineering Applications, 10*, 99–106. http://www.irphouse.com/ijmr16/ijmrv8n3_08.pdf

[12] Siddamsetti, S., Almngoshi, H. Z., Dinesh, K., Prashant, G. C., Bennet, M. A., & Dadheech, P. (2024). Modular metric spaces: Some fixed-point theorems and application of secure dynamic routing for WSN. *Journal of Interdisciplinary Mathematics, 27*(2), 393–401. https://doi.org/10.47974/JIM-1890

[13] Siddamsetti, S., Qasim, M. M., Sravanthi, N., Dinesh, K., Velmurugan, V., Sengan, S., & Dadheech, P. (2025). Wireless sensor networks with modular metric spaces. *Journal of Interdisciplinary Mathematics, 28*(2), 659–667. https://doi.org/10.47974/JIM-2109

[1] Department of Mathematics, K.Ramakrishnan College of Engineering, Trichy, India

Email address: dinesh.skksv93@gmail.com

[2] Department of Mathematics, College of Science, Qassim University, Saudi Arabia

Email address: h.ibnouf@qu.edu.sa

[3] Department of Mathematics, St Joseph's College of Engineering, OMR Chennai - 600 019, India

Email address: shobabalasubramaniyam@gmail.com