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AMERICAN OPTION PRICING UNDER AN EXTENDED SLOW-GROWTH VOLATILITY MODEL

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Abstract. This paper extends the Slow Growth Volatility (SGV) model to American option pricing. The SGV framework introduces a nonlinear, state-dependent volatility to reflect observed skew beyond classical models. We establish existence and uniqueness for the penalized PDE and show convergence, as the penalty parameter vanishes, towards the solution of the associated linear complementarity formulation. Numerical implementation employs a rational penalty method with finite difference discretization, and computational results confirm that the SGV framework remains consistent and competitive with Black–Scholes and CEV.

Keywords. Partial differential equation, finite difference method, American options, local volatility

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1. Introduction

The relationship between asset price dynamics and volatility, first explored by Bachelier [3], has remained central in financial research. Empirical evidence of volatility skews and smiles demonstrates the limitations of the classical Black–Scholes framework [7], highlighting the need for more sophisticated models. Several approaches have been introduced to address these limitations, notably stochastic volatility models [16, 15, 8] and jump–diffusion models [21, 4], which enrich the dynamics beyond the constant-volatility assumption. Within this literature, The Slow Growth Volatility (SGV) model [5] is given by

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma(S_t)\, dW_t,$$

where the local volatility function is defined as

$$\sigma(S_t) = \sigma\, S_t^{\alpha-1} \log^{\beta}(1 + S_t),$$

with $\sigma > 0$ a scale parameter, and $\alpha, \beta \in \mathbb{R}$ governing the nonlinear dependence of volatility on the asset level. This structure captures market skew more effectively than constant-volatility models. The SGV framework has also been extended to incorporate jumps for European options [1], thereby accounting for sudden and discontinuous changes in asset prices. This study investigates the pricing of American options within

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the Slow Growth Volatility (SGV) framework. The early exercise constraint induces a free boundary, and the problem may be cast as a linear complementarity problem (LCP) [6, 19, 25]. Analytical solutions are rarely available, and a wide range of numerical approaches have been developed, including finite difference schemes, penalty methods, and operator splitting techniques [26, 11, 27, 17, 2]. However, many of these works rely on classical frameworks, notably the Black–Scholes and CEV models, which do not fully reflect the complexity of observed volatility surfaces. The focus of this study is the adaptation of the SGV model to the valuation of American options. By incorporating a nonlinear volatility structure, we establish a framework that simultaneously addresses the challenges of early exercise and realistic volatility modeling. While penalty methods have previously been applied to American option pricing under Black–Scholes dynamics (see Nielsen et al. [22]), our work adapts this approach to the SGV setting, where—to the best of our knowledge—it has not yet been investigated. We provide theoretical guarantees by proving existence and uniqueness for the penalized problem and demonstrating convergence, as the penalty parameter tends to zero, towards the unique solution of the resulting LCP under degenerate parabolic conditions. This dual perspective offers both analytical justification and practical methodology for American option valuation under SGV. Recent research continues to refine methodologies for option pricing under increasingly complex dynamics. He and Lin [14] investigate exchange options under random liquidity conditions combined with regime-switching; Farkas, Ferrari, and Ulrych [12] analyze autocallable products within local–stochastic volatility frameworks; and Itkin [18] develops semi-analytic methods for time-dependent jump–diffusion models. In addition, approaches based on machine learning, and especially deep neural networks, have recently become increasingly important for enhancing the speed and accuracy of option pricing. These advances underline the continuing importance of innovative approaches to American option pricing and provide context for our proposed extension of the SGV model. The structure of the paper is as follows. Section 2 introduces the SGV model and formulates the American option pricing task as a linear complementarity framework. Section 3 develops the penalty approximation and establishes existence, uniqueness, and convergence. Section 4 discusses boundary conditions and the discretization schemes. Section 5 reports numerical experiments comparing SGV with classical models. Section 6 concludes.

2. Diffusion Model and Obstacle Problem

We consider a financial market with a deterministic risk-free asset and one risky asset whose price, under the risk–neutral measure $\mathbb{Q}$, satisfies

$$dS_t = rS_t\,dt + a(S_t)\,dW_t^{\mathbb{Q}}, \qquad S_0 > 0, \tag{2.1}$$

where $r > 0$ is the constant interest rate and $(W_t^{\mathbb{Q}})_{t\geq 0}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ satisfying the usual assumptions.

Volatility structure. In the slow–growth volatility (SGV) model, the diffusion coefficient is specified by

$$a(S) = \sigma S^{\alpha} \log^{\beta}(1 + S),$$

with $\sigma > 0$ a volatility scale and $\alpha, \beta \in \mathbb{R}$ the SGV parameters. For large S, the power term S^{α} dominates, while the logarithmic factor $\log^{\beta}(1+S)$ modulates the growth. This structure encompasses well-known models as special cases: the Black–Scholes volatility ($\alpha = 1, \beta = 0$) and the CEV volatility ($\alpha \neq 1, \beta = 0$). The logarithmic modulation is mild but provides additional flexibility for capturing implied volatility skew.

Structural assumptions. Throughout we impose

$$\alpha > 0, \qquad \alpha + \beta \geq 0. \tag{2.2}$$

The first condition ensures that volatility does not vanish for large S, so that the asset retains randomness in the tails. The second condition prevents blow–up at the origin: if $\alpha + \beta < 0$, then $a(S) \sim \sigma S^{\alpha+\beta}$ would diverge as $S \downarrow 0$. When $\alpha + \beta = 0$, the volatility remains bounded away from zero near the origin; when $\alpha+\beta > 0$, the process is degenerately parabolic at $S = 0$ but remains well–posed in the sense of Fichera's boundary theory (see, e.g., [13]).
Generator. By Itô's formula, the infinitesimal generator acting on smooth functions $v = v(S)$ is

$$\mathcal{L}v(S) = rS\, v'(S) + \tfrac{1}{2}\sigma^2 S^{2\alpha} \log^{2\beta}(1 + S)\, v''(S). \tag{2.3}$$

This operator will play the central role in the formulation of the PDE associated with contingent claims.
American put payoff. Let $K > 0$ be the strike price, and define the payoff

$$\Phi(S) = (K - S)_+ = \max\{K - S, 0\}.$$

The option value $u(t, S)$ can be represented probabilistically through the Snell envelope

$$u(t, S) = \sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}^{\mathbb{Q}} \left[e^{-r(\tau - t)} \Phi(S_\tau) \,\middle|\, S_t = S \right],$$

where $\mathcal{T}_{[t,T]}$ is the family of $(\mathcal{F}_t)$-adapted stopping times taking values in $[t, T]$ (see, e.g., [19, 6, 23]).
Obstacle problem. Classical arguments from dynamic programming and variational inequality theory (see, e.g., [6, 13]) imply that u satisfies the variational inequality

$$\min\{-\partial_t u - \mathcal{L}u + ru,\ u - \Phi\} = 0, \tag{2.4}$$

posed on $(0, T) \times (0, \infty)$ with terminal condition $u(T, \cdot) = \Phi$. At $S = 0$, Fichera's criterion prescribes the boundary condition $u(t, 0) = K$. As $S \to \infty$, one enforces the decay condition $u(t, S) \to 0$, consistent with the vanishing value of an out-of-the-money put.
Summary. Under assumptions (2.2), the SGV model leads to a degenerate parabolic obstacle problem of the form (2.4), which will be approximated by a penalization method in Section 3.

3. Penalty Approximation: Existence, Uniqueness, and Convergence

In order to address the obstacle condition inherent in the American option framework and to facilitate the subsequent numerical analysis, we adopt a rational penalty scheme as an approximation of the associated linear complementarity problem. This penalized formulation leads to a parabolic PDE that can be studied with classical analytical methods. Within this setting, we establish appropriate sub- and super-solution bounds, obtain polynomial–exponential growth estimates, demonstrate existence and uniqueness of the penalized solution, and finally show that, as $\varepsilon \downarrow 0$, the solution converges to that of the original variational inequality.

Standing assumptions. In what follows we work under the structural conditions

$$\alpha > 0, \qquad \alpha + \beta \geq 0,$$

which ensure non-degeneracy for large S and prevent blow-up of the diffusion coefficient near the origin. In addition, we fix a penalty parameter C satisfying

$$C \;\geq\; rK, \tag{3.1}$$

a lower bound that will be required for the barrier arguments below.
Notation. Let $K > 0$ denote the strike price. We set

$$\Phi(S) := (K - S)_+, \qquad q(S) := K - S,$$

where Φ represents the payoff and q is the linear auxiliary function introduced in the penalty term. Clearly, $\Phi \geq q$, with equality $\Phi = q$ whenever $S \leq K$, while for $S > K$ we have $\Phi = 0$ and $q < 0$.

3.1. **Time reversal and obstacle formulation.**

The valuation problem of Section 2 is backward in time with terminal condition at $t = T$. We introduce the forward time variable

$$\tau := T - t, \qquad U(\tau, S) := u(T - \tau, S),$$

so that $\tau = 0$ corresponds to maturity. This converts the backward parabolic problem with terminal condition at $t = T$ into a forward Cauchy problem in τ, reversing only the time derivative while keeping the spatial operator unchanged. From now on, we denote by

$$\mathcal{L}v := rS\,\partial_S v + \tfrac{1}{2}\sigma^2 S^{2\alpha} \log^{2\beta}(1 + S)\,\partial_{SS} v \tag{3.2}$$

the *spatial* part of the generator. The forward obstacle problem (variational inequality) reads

$$\partial_\tau U - \mathcal{L}U + rU \;\geq 0, \qquad U \geq \Phi, \qquad \big(\partial_\tau U - \mathcal{L}U + rU\big)\,(U - \Phi) = 0,$$

with $U(0, \cdot) = \Phi$, $U(\tau, 0) = K$ and $\lim_{S\to\infty} U(\tau, S) = 0$.

3.2. **Penalized formulation and well-posedness.**

In order to approximate the variational inequality associated with the American put, we employ a rational penalty scheme. For $\varepsilon > 0$ we consider the Cauchy–Dirichlet problem

$$\partial_\tau U_\varepsilon - \mathcal{L}U_\varepsilon + rU_\varepsilon = \frac{\varepsilon C}{U_\varepsilon + \varepsilon - q(S)}, \qquad U_\varepsilon(0, S) = \Phi(S), \tag{3.3}$$

on $(\tau, S) \in (0, T] \times (0, \infty)$, supplemented with the boundary conditions

$$U_\varepsilon(\tau, 0) = K, \qquad \lim_{S\to\infty} U_\varepsilon(\tau, S) = 0, \qquad 0 \leq \tau \leq T.$$

Here $q(S) := K - S$ is the linear function used in the penalty. Since $q \leq \Phi$, the penalization enforces $U_\varepsilon \gtrsim q$ near the contact region, while in the continuation region $U_\varepsilon \gg q$ the right-hand side of (3.3) is of order $O(\varepsilon)$ and therefore negligible. This ensures that the penalized problem consistently approximates the obstacle problem as $\varepsilon \to 0$.

Standing assumptions revisited. We recall the structural conditions introduced in Section 2, namely

$$\alpha > 0, \qquad \alpha + \beta \geq 0.$$

The first requirement guarantees non-degeneracy of the volatility for large S. The second excludes the case $\alpha + \beta < 0$, which would correspond to an explosive diffusion coefficient $\sigma S^{\alpha} \log^{\beta}(1 + S) \sim \sigma S^{\alpha+\beta}$ at the origin. Under these assumptions the operator $\mathcal{L}$ is degenerate parabolic at $S = 0$ (if $\alpha + \beta > 0$) or remains strictly parabolic (if $\alpha + \beta = 0$), and Fichera's theory justifies the boundary condition $U_\varepsilon(\tau, 0) = K$. At infinity, the logarithmic modulation ensures that the growth is slower than $S^{\alpha+\varepsilon}$ for any $\varepsilon > 0$, so that solutions remain in the polynomial–exponential growth class considered below.

Well-posedness. The nonlinearity in (3.3) is monotone in U_ε, since

$$\frac{\partial}{\partial U_\varepsilon}\Big(- \frac{\varepsilon C}{U_\varepsilon + \varepsilon - q(S)} + rU_\varepsilon\Big) = r + \frac{\varepsilon C}{(U_\varepsilon + \varepsilon - q(S))^2} > 0.$$

Hence the operator is *proper* in the viscosity sense. Together with the continuity of the coefficients and the structural assumptions, this yields comparison for viscosity sub/supersolutions of (3.3) (see e.g. [10, 13]). In Theorem 3.6 we also provide an L^2_λ energy argument establishing uniqueness within the growth class of Lemma 3.4.

3.3. **Barrier functions: sub- and super-solutions.** As a first step, we construct two explicit functions that play the role of barriers and do not depend on ε.

Lemma 3.1. *Define $v_0(\tau, S) := q(S) := K - S$ and $u_0(\tau, S) := K$. Then v_0 is a viscosity subsolution and u_0 is a viscosity supersolution of the penalized problem* (3.3) *in $(0, T] \times (0, \infty)$. Furthermore,*

$$v_0(0, \cdot) \leq \Phi \leq u_0(0, \cdot), \qquad v_0(\tau, 0) \leq K \leq u_0(\tau, 0), \qquad \lim_{S\to\infty} v_0(\tau, S) \leq 0 \leq \lim_{S\to\infty} u_0(\tau, S),$$

and $v_0 \leq u_0$ pointwise on $[0, T] \times (0, \infty)$.

Proof. Subsolution $v_0 = q$. Since q is affine in S and independent of τ, one has $\partial_\tau q = 0$, $\partial_S q = -1$, and $\partial_{SS} q = 0$. Hence

$$\partial_\tau q - \mathcal{L}q + rq = rK \quad \text{for } S > 0.$$

Moreover $q + \varepsilon - q = \varepsilon$, so the penalty equals C. Thus

$$\partial_\tau q - \mathcal{L}q + rq - \frac{\varepsilon C}{q + \varepsilon - q} = rK - C \leq 0$$

whenever $C \geq rK$ (see (3.1)). Therefore q is a subsolution in the interior. At the singular point $S = K$, q is only Lipschitz continuous; the viscosity subsolution inequality follows by testing with smooth functions touching q from above at (τ, K).

Supersolution $u_0 \equiv K$. Here $\partial_\tau u_0 = 0$, $\mathcal{L}u_0 = 0$, and

$$\partial_\tau u_0 - \mathcal{L}u_0 + ru_0 = rK.$$

Since $u_0 + \varepsilon - q = S + \varepsilon$,

$$\partial_\tau u_0 - \mathcal{L}u_0 + ru_0 - \frac{\varepsilon C}{u_0 + \varepsilon - q} = rK - \frac{\varepsilon C}{S + \varepsilon} \geq rK - C \geq 0.$$

Hence u_0 is a supersolution in the interior.

Boundary ordering. At $\tau = 0$, one has $q(S) \leq \Phi(S) \leq K$. At $S = 0$, $q(\tau, 0) = K = u_0(\tau, 0)$. As $S \to \infty$, $q(\tau, S) = K - S \to -\infty \leq 0 \leq K = u_0(\tau, S)$. Finally, $q \leq K$ pointwise, so $v_0 \leq u_0$. $\square$

Lemma 3.2 (Lower bound induced by the rational penalty)**.** *For any $\varepsilon \in (0,1]$ and any $C \geq rK$, the viscosity solution U_ε of (3.3) satisfies*

$$U_\varepsilon(\tau, S) \;\geq\; q(S) - \varepsilon, \qquad (\tau, S) \in [0,T] \times (0,\infty).$$

In particular, as $\varepsilon \downarrow 0$, one has $U_\varepsilon \to U \geq q$ locally uniformly.

Proof. Let $\theta \in (0,1)$ and introduce the auxiliary function

$$B_\theta(\tau, S) := q(S) - \theta\varepsilon.$$

Suppose for contradiction that $U_\varepsilon - B_\theta$ attains a strictly negative minimum at some $(\hat{\tau}, \hat{S})$. By the definition of viscosity solutions, there exists a smooth test function touching $U_\varepsilon - B_\theta$ from above at $(\hat{\tau}, \hat{S})$, which yields

$$\partial_\tau U_\varepsilon - \mathcal{L}U_\varepsilon + rU_\varepsilon \;\leq\; \partial_\tau B_\theta - \mathcal{L}B_\theta + rB_\theta = -\mathcal{L}q + rq - r\theta\varepsilon.$$

On the other hand, using the penalized equation (3.3) and the monotonicity of the map $x \mapsto \frac{\varepsilon C}{x}$, one obtains

$$\partial_\tau U_\varepsilon - \mathcal{L}U_\varepsilon + rU_\varepsilon = \frac{\varepsilon C}{U_\varepsilon + \varepsilon - q} \;\geq\; \frac{\varepsilon C}{B_\theta + \varepsilon - q} = \frac{\varepsilon C}{(q - \theta\varepsilon) + \varepsilon - q} = \frac{C}{1-\theta}.$$

Combining the two inequalities gives

$$\frac{C}{1-\theta} \;\leq\; -\mathcal{L}q + rq - r\theta\varepsilon.$$

Since $q(S) = K - S$, one has $-\mathcal{L}q + rq \equiv rK$. Hence it suffices to impose $C \geq rK$. With this choice, for $\varepsilon \leq 1$ and θ close to 1, the previous inequality cannot hold, yielding a contradiction. Therefore $U_\varepsilon \geq q - \theta\varepsilon$ for all $\theta \in (0,1)$, and letting $\theta \uparrow 1$ gives the desired estimate $U_\varepsilon \geq q - \varepsilon$. □

Remark 3.3 (Choice of the penalty constant). Since $-\mathcal{L}q + rq \equiv rK$ for $q(S) = K - S$, the same threshold $C \geq rK$ ensures both the barrier property of Lemma 3.1 and the lower bound in Lemma 3.2. In particular,

$$\boxed{C \geq rK}$$

is sufficient throughout Section 3.

3.4. **Growth estimate.** For large values of S, the diffusion coefficient in $\mathcal{L}$ behaves like $S^{2\alpha}(\log(1+S))^{2\beta}$. Because $\alpha > 0$, this expression diverges, but its growth remains slower than any power $S^{2\alpha+\varepsilon}$ once the logarithmic factor is taken into account. In order to keep the penalized solution under control, we introduce a polynomial–exponential weight that dominates this behavior.

Lemma 3.4 (Growth estimate)**.** *There exist $p \geq 1$, a constant $\lambda > 0$, and a constant $C_1 > 0$ independent of ε such that every solution U_ε of (3.3) satisfies*

$$0 \;\leq\; U_\varepsilon(\tau, S) \;\leq\; C_1(1 + S^p)e^{-\lambda S}, \qquad (\tau, S) \in [0,T] \times (0,\infty).$$

In particular, $U_\varepsilon(\tau, S) \leq \tilde{C}_1(1 + S^p)$ for some $\tilde{C}_1 > 0$ independent of ε.

Proof. Step 1. Choice of a weight. Fix $p \geq 1$ and define

$$V(S) = (1+S)^p e^{-\lambda S}, \qquad S > 0,$$

with $\lambda > 0$ to be determined. Writing $g(S) = \log V(S) = p\log(1+S) - \lambda S$, one finds

$$\frac{V'}{V} = g'(S) = \frac{p}{1+S} - \lambda,$$

$$\frac{V''}{V} = g''(S) + (g'(S))^2 = -\frac{p}{(1+S)^2} + \left(\frac{p}{1+S} - \lambda\right)^2 = \frac{p^2-p}{(1+S)^2} - \frac{2\lambda p}{1+S} + \lambda^2.$$

Step 2. Action of the operator. As V is time-independent,

$$\mathcal{L}V - rV = rS\,V' + \tfrac{1}{2}\sigma^2 S^{2\alpha}\log^{2\beta}(1+S)\,V'' - rV.$$

Dividing by $V > 0$ yields

$$\frac{\mathcal{L}V - rV}{V} = r\Big(\frac{pS}{1+S} - \lambda S - 1\Big) + \tfrac{1}{2}\sigma^2 S^{2\alpha}\log^{2\beta}(1+S)\Big(\tfrac{p^2-p}{(1+S)^2} - \tfrac{2\lambda p}{1+S} + \lambda^2\Big).$$

Denote this expression by $\Psi_{\lambda,p}(S)$.

Step 3. Lower bound on $\Psi_{\lambda,p}$. As $S \to \infty$,

$$\Psi_{\lambda,p}(S) \sim \tfrac{1}{2}\sigma^2\lambda^2 S^{2\alpha}\log^{2\beta}(1+S) \to +\infty,$$

because $\alpha > 0$. As $S \downarrow 0$, all terms remain bounded under the assumption $\alpha + \beta \geq 0$ (see Section 2). Thus $\Psi_{\lambda,p}$ attains a finite minimum on $[0,\infty)$,

$$m_{\lambda,p} := \inf_{S\geq 0} \Psi_{\lambda,p}(S) > -\infty.$$

Hence there exists $C_*(\lambda,p) \geq 0$ with

$$\mathcal{L}V - rV \;\geq\; -C_*(\lambda,p)\,V \quad \text{for all } S > 0.$$

Step 4. Construction of a supersolution. Step 4. Construction of a supersolution. Fix $A > 0$ and set $B(S) = A\,V(S)$, $\tilde{B}(\tau,S) = B(S) + M\tau$, with

$$M := C_*(\lambda,p)A + C.$$

Choose A large enough so that $B(S) \geq q(S)$ for all $S \geq 0$; hence $\tilde{B} \geq q$ and

$$0 \leq \frac{\varepsilon C}{\tilde{B} + \varepsilon - q(S)} \leq C.$$

Then

$$\partial_\tau \tilde{B} - \mathcal{L}\tilde{B} + r\tilde{B} = M - A(\mathcal{L}V - rV) \;\geq\; C + AC_*(1+V) \;\geq\; C.$$

Since $0 \leq \frac{\varepsilon C}{\tilde{B}+\varepsilon-q(S)} \leq C$, it follows that

$$\partial_\tau \tilde{B} - \mathcal{L}\tilde{B} + r\tilde{B} - \frac{\varepsilon C}{\tilde{B} + \varepsilon - q(S)} \;\geq\; 0,$$

so $\tilde{B}$ is a supersolution of (3.3).

Step 5. Comparison argument. On truncated domains $Q_R = (0,T] \times (0,R)$ with $U_\varepsilon(\tau,R) = 0$, the comparison principle yields $U_\varepsilon \leq \tilde{B}$ in Q_R. For any fixed (τ,S) and any $R > S$, this bound holds at (τ,S); hence letting $R \to \infty$ preserves the estimate pointwise. At $\tau = 0$, $\Phi(S) \leq B(S)$ once A is chosen large enough. At $S = 0$, $\tilde{B}(\tau,0) = A + M\tau \geq K$ for sufficiently large A. Therefore $\tilde{B}$ dominates both boundary and initial data. Applying the comparison principle gives

$$U_\varepsilon(\tau,S) \leq \tilde{B}(\tau,S) \leq C_1 V(S),$$

for some $C_1 > 0$ independent of ε. Finally, nonnegativity follows by comparison with the trivial subsolution 0. □

3.5. **Existence for the penalized problem.**

Theorem 3.5 (Existence)**.** *Under the structural assumptions and for every $\varepsilon \in (0,1]$, the penalized equation* (3.3) *admits a nonnegative solution U_ε belonging to the growth class of Lemma 3.4.*

Proof. Step 1. Truncation of the domain. Let $R > 0$ and define the bounded cylinder

$$Q_R := (0,T] \times (0,R).$$

On this truncated domain we solve (3.3) with boundary conditions

$$U_\varepsilon(\tau,0) = K, \qquad U_\varepsilon(\tau,R) = 0, \qquad U_\varepsilon(0,S) = \Phi(S), \quad S \in (0,R).$$

Step 2. Iterative construction. Initialize with $U_\varepsilon^{(0)} := v_0 = q(S) = K - S$. For $m \geq 0$, define $U_\varepsilon^{(m+1)}$ as the unique solution of the linear parabolic equation

$$\partial_\tau U_\varepsilon^{(m+1)} - \mathcal{L} U_\varepsilon^{(m+1)} + r U_\varepsilon^{(m+1)} = \frac{\varepsilon C}{U_\varepsilon^{(m)} + \varepsilon - q(S)} \quad \text{in } Q_R,$$

with the prescribed initial and boundary values. Standard parabolic theory (see [20, 13]) ensures well-posedness at each step.

Step 3. Monotonicity and bounds. By Lemma 3.1 and comparison, the sequence satisfies

$$v_0 \;\leq\; U_\varepsilon^{(m)} \;\leq\; U_\varepsilon^{(m+1)} \;\leq\; u_0 \quad \text{on } Q_R, \qquad u_0 \equiv K.$$

Hence $\{U_\varepsilon^{(m)}\}$ is monotone increasing and bounded. Therefore the pointwise and locally uniform limit

$$U_{\varepsilon,R} := \lim_{m\to\infty} U_\varepsilon^{(m)}$$

exists and solves the nonlinear penalized problem on Q_R.

Step 4. Extension to the whole space. From Lemma 3.4, the family $\{U_{\varepsilon,R}\}_{R>0}$ satisfies uniform growth estimates, independent of R. Thus it is equibounded and equicontinuous on compact subsets of $(0,T] \times (0,\infty)$. By the Arzelà–Ascoli theorem, there exists a subsequence that converges locally uniformly to a function U_ε, which solves (3.3) for all $S > 0$.

Step 5. Positivity. Since $\Phi \geq 0$ and the iteration preserves order, each $U_\varepsilon^{(m)} \geq 0$, whence $U_\varepsilon \geq 0$.

This proves the existence of a nonnegative solution U_ε in the growth class of Lemma 3.4. □

3.6. **Uniqueness for the penalized problem.**

Theorem 3.6 (Uniqueness)**.** *For each $\varepsilon \in (0,1]$, the solution U_ε of* (3.3) *is unique in the growth class of Lemma 3.4.*

Proof. Step 1. Difference of two solutions. Let U_1, U_2 be two solutions in the growth class and set $W := U_1 - U_2$. Subtracting the equations yields

$$\partial_\tau W - \mathcal{L} W + rW = \varepsilon C \left(\frac{1}{U_1 + \varepsilon - q} - \frac{1}{U_2 + \varepsilon - q} \right) =: g.$$

Step 2. Sign of the nonlinearity. The map $v \mapsto (v + \varepsilon - q)^{-1}$ is strictly decreasing, hence

$$g\,W \;\leq\; 0 \qquad \text{pointwise on } (0,T] \times (0,\infty).$$

Step 3. Weighted energy space. Define the weighted norm

$$\|f\|_{L^2_\lambda}^2 := \int_0^\infty |f(S)|^2 e^{-\lambda S}\, dS, \qquad \lambda > 0.$$

Multiplying the PDE for W by $We^{-\lambda S}$ and integrating by parts over $(0,\infty)$ gives

$$\frac{1}{2}\frac{d}{d\tau}\|W(\tau,\cdot)\|_{L^2_\lambda}^2 + \frac{1}{2}\Big\|\sigma S^\alpha(\log(1+S))^\beta \partial_S W\Big\|_{L^2_\lambda}^2 + r\|W(\tau,\cdot)\|_{L^2_\lambda}^2 \;\le\; \int_0^\infty gWe^{-\lambda S}\, dS.$$

Step 4. Boundary terms. The integration by parts is justified since: (i) at $S = 0$, the inflow condition (Fichera's criterion with $\alpha + \beta > 0$) and the boundary value $W(\tau, 0) = 0$ imply vanishing boundary contributions; (ii) as $S \to \infty$, the exponential weight and the polynomial growth bound of Lemma 3.4 ensure $We^{-\lambda S} \to 0$ sufficiently fast.

Step 5. Conclusion. Since $gW \le 0$, the right-hand side is nonpositive, hence

$$\frac{d}{d\tau}\|W(\tau,\cdot)\|_{L^2_\lambda}^2 \;\le\; 0.$$

As $W(0,\cdot) = 0$, we obtain $\|W(\tau,\cdot)\|_{L^2_\lambda} = 0$ for all $\tau \in [0,T]$, and therefore $W \equiv 0$. □

3.7. **Convergence as $\varepsilon \to 0$.**

Theorem 3.7 (Convergence to the obstacle problem)**.** *When $\varepsilon \downarrow 0$, the sequence of penalized solutions U_ε to (3.3) converges locally uniformly on $[0,T] \times (0,\infty)$ towards a function U, which corresponds to the unique viscosity solution associated with*

$$\min\{\partial_\tau U - \mathcal{L}U + rU,\ U - \Phi\} = 0,$$

subject to $U(0,\cdot) = \Phi$, $U(\tau, 0) = K$, and $\lim_{S\to\infty} U(\tau, S) = 0$.

Proof. Step 1. Monotonicity and compactness. For $0 < \varepsilon_1 < \varepsilon_2$ and any $V \ge q$,

$$\frac{\varepsilon_1 C}{V + \varepsilon_1 - q} \;\le\; \frac{\varepsilon_2 C}{V + \varepsilon_2 - q}.$$

Thus, by comparison, $U_{\varepsilon_1} \le U_{\varepsilon_2}$ on $[0,T] \times (0,\infty)$ whenever $0 < \varepsilon_1 < \varepsilon_2$, i.e. as $\varepsilon \downarrow 0$ the family (U_ε) is monotone *nonincreasing*. By Lemmas 3.1, 3.2, and 3.4, the family is locally bounded and equicontinuous; hence, by Arzelà–Ascoli, there is subsequential local uniform convergence.

Step 2. Half-relaxed limits. Define

$$U^*(\tau,S) = \limsup_{\substack{\varepsilon\downarrow 0\\ (\tau',S')\to(\tau,S)}} U_\varepsilon(\tau',S'), \qquad U_*(\tau,S) = \liminf_{\substack{\varepsilon\downarrow 0\\ (\tau',S')\to(\tau,S)}} U_\varepsilon(\tau',S').$$

We show that U^* and U_* are a viscosity subsolution and a viscosity supersolution, respectively.

Step 3. Initial data and lower bound. Lemma 3.2 gives $U_\varepsilon \ge q - \varepsilon$. Since $U_\varepsilon(0,\cdot) = \Phi$ and $\Phi \ge q$, passing to the limit yields

$$U_* \ge q, \qquad U^*(0,\cdot) = U_*(0,\cdot) = \Phi.$$

Step 4. Subsolution property of U^*. Let $\varphi \in C^\infty$ touch U^* from above at $(\bar\tau, \bar S)$. If $U^*(\bar\tau,\bar S) > \Phi(\bar S)$, then for (τ, S) near $(\bar\tau,\bar S)$ the denominator in (3.3) remains bounded away from zero, so the penalty term is $O(\varepsilon)$. By the stability of viscosity inequalities (see [9]),

$$\partial_\tau\varphi(\bar\tau,\bar S) - \mathcal{L}\varphi(\bar\tau,\bar S) + r\varphi(\bar\tau,\bar S) \le 0.$$

If instead $U^*(\bar\tau,\bar S)=\Phi(\bar S)$, the obstacle condition ensures

$$\min\{\partial_\tau\varphi-\mathcal{L}\varphi+r\varphi,\ U^*-\Phi\}\le 0.$$

Step 5. Supersolution property of U_*. Let $\psi\in C^\infty$ touch U_* from below at $(\bar\tau,\bar S)$. If $U_*(\bar\tau,\bar S)>\Phi(\bar S)$, the penalty term vanishes in the limit and

$$\partial_\tau\psi(\bar\tau,\bar S)-\mathcal{L}\psi(\bar\tau,\bar S)+r\psi(\bar\tau,\bar S)\ge 0.$$

If $U_*(\bar\tau,\bar S)=\Phi(\bar S)$, the obstacle condition guarantees

$$\min\{\partial_\tau\psi-\mathcal{L}\psi+r\psi,\ U_*-\Phi\}\ge 0,$$

so in particular $U_*\ge\Phi$. Combined with Step 7 below, this also yields $U^*\ge\Phi$.

Step 6. Boundary and far-field behavior. At $S=0$, the condition $U_\varepsilon(\tau,0)=K$ passes to the limit in the viscosity sense (inflow boundary when $\alpha+\beta>0$, strictly parabolic when $\alpha+\beta=0$). As $S\to\infty$, Lemma 3.4 shows $U_\varepsilon(\tau,S)\to 0$ uniformly on compact time intervals; hence $U^*(\tau,\infty)=U_*(\tau,\infty)=0$.

Step 7. Identification of the limit. By the comparison principle for the obstacle problem, $U^*\le U_*$. Since trivially $U_*\le U^*$, we deduce $U^*=U_*=U$. Monotonicity in ε then ensures convergence of the entire family $U_\varepsilon\to U$, and uniqueness follows from the same comparison principle. □

4. Practical Numerical Schemes

This section is devoted to the numerical resolution of the American put option problem under the SGV framework. The approach relies on the penalized PDE formulation introduced in Section 3. We begin by specifying the boundary conditions on a truncated spatial domain, after which we present two finite difference discretizations: an explicit and an implicit upwind scheme. Special attention is paid to stability, positivity, and the preservation of the early–exercise constraint.

4.1. Boundary Conditions.

We restrict the spatial domain to the finite interval $S\in(0,\overline{S})$, where $\overline{S}$ is chosen large compared to the strike K, and employ the forward time variable $\tau\in(0,T]$ introduced earlier. On this domain, the penalized PDE satisfied by the option value $P(\tau,S)$ takes the form

$$\partial_\tau P-rS\,\partial_S P-\tfrac12\sigma^2S^{2\alpha}\log^{2\beta}(1+S)\,\partial_{SS}P+rP=\frac{\varepsilon C}{P+\varepsilon-q(S)},\qquad(\tau,S)\in(0,T]\times(0,\overline{S}).$$

Here the obstacle function is $q(S)=K-S$, and the penalty parameter is scaled with a constant $C\ge rK$ (see (3.1)).

The mathematical problem is naturally posed on the open interval $(0,\overline{S})$, with boundary conditions prescribed at both extremities. For the discrete approximation, however, it is convenient to work with the closed interval $[0,\overline{S}]$, so that the boundary nodes are incorporated into the computational grid.

The initial and boundary conditions are therefore

$$P(0,S)=(K-S)_+,\qquad S\in[0,\overline{S}],$$

$$P(\tau,0)=K,\qquad P(\tau,\overline{S})=0,\qquad \tau\in(0,T].$$

The choice $P(\tau,0)=K$ corresponds to the inflow boundary in the sense of Fichera's classification, ensuring consistency with the continuous formulation.

4.2. **Finite Difference Method.** For the spatial discretization we introduce grid points $S_j = jh$ for $j = 0, 1, \ldots, M+1$, with mesh size $h = \overline{S}/(M+1)$. The interval $[0,T]$ is divided into $N+1$ supintervals, setting $\tau_n = n\Delta\tau$ for $n = 0, 1, \ldots, N+1$, where $\Delta\tau = T/(N+1)$. We denote by P_j^n the numerical approximation of $P(\tau_n, S_j)$.

To simplify notation, we set

$$a_j := \tfrac{1}{2}\sigma^2 S_j^{2\alpha} \log^{2\beta}(1+S_j), \qquad b_j := rS_j, \qquad q_j := K - S_j.$$

The discrete initial condition is imposed at $n=0$ as

$$P_j^0 = (K - S_j)_+, \qquad j = 0, \ldots, M+1,$$

and the boundary conditions at later time levels are

$$P_0^n = K, \qquad P_{M+1}^n = 0, \qquad n = 1, \ldots, N+1.$$

4.2.1. *Explicit Scheme (upwind on $+\, rS\,\partial_S P$).* For the time discretization we apply a forward Euler method, while the diffusion operator is approximated using centered differences. The convective contribution $+\, b_j \partial_S P$, with $b_j = rS_j$, is handled by an upwind discretization, namely

$$\partial_S P(\tau_n, S_j) \;\approx\; \frac{P_{j+1}^n - P_j^n}{h}.$$

This leads to the following explicit finite difference scheme, valid for $j = 1, \ldots, M$:

$$\frac{P_j^{n+1} - P_j^n}{\Delta\tau} - a_j \frac{P_{j+1}^n - 2P_j^n + P_{j-1}^n}{h^2} + b_j \frac{P_{j+1}^n - P_j^n}{h} + rP_j^n - \frac{\varepsilon C}{P_j^n + \varepsilon - q_j} = 0, \tag{4.1}$$

where $a_j = \frac{1}{2}\sigma^2 S_j^{2\alpha}\log^{2\beta}(1+S_j)$ and $q_j = K - S_j$.

Equivalently, the update can be expressed in the compact form

$$\boxed{\begin{aligned} P_j^{n+1} &= \left(\tfrac{a_j\Delta\tau}{h^2}\right)P_{j-1}^n + \left(1 - \tfrac{2a_j\Delta\tau}{h^2} - \tfrac{b_j\Delta\tau}{h} - r\Delta\tau\right)P_j^n \\ &\quad + \left(\tfrac{a_j\Delta\tau}{h^2} + \tfrac{b_j\Delta\tau}{h}\right)P_{j+1}^n + \frac{\varepsilon C\Delta\tau}{P_j^n + \varepsilon - q_j}. \end{aligned}}$$

For later analysis, it is convenient to define the update map f acting on neighboring nodes as

$$f(P^-, P, P^+, q, S) := \left(\tfrac{a\Delta\tau}{h^2}\right)P^- + \left(1 - \tfrac{2a\Delta\tau}{h^2} - \tfrac{b\Delta\tau}{h} - r\Delta\tau\right)P + \left(\tfrac{a\Delta\tau}{h^2} + \tfrac{b\Delta\tau}{h}\right)P^+ + \frac{\varepsilon C\Delta\tau}{P + \varepsilon - q},$$

with $a = a(S)$ and $b = rS$.

Lemma 4.1 (Monotonicity of the explicit update)**.** *If $r \ge 0$, $P \ge q$, and*

$$\Delta\tau \;\le\; \frac{h^2}{\overline{\sigma}_{\alpha\beta} + b_{\max}h + rh^2 + (C/\varepsilon)h^2}, \qquad \overline{\sigma}_{\alpha\beta} := \max_j 2a_j, \quad b_{\max} := r\overline{S}, \tag{4.2}$$

then f is nondecreasing with respect to each of its first three arguments, that is

$$\frac{\partial f}{\partial P^-} \ge 0, \qquad \frac{\partial f}{\partial P} \ge 0, \qquad \frac{\partial f}{\partial P^+} \ge 0.$$

Proof. We compute directly

$$\frac{\partial f}{\partial P^-} = \frac{a\Delta\tau}{h^2} \ge 0, \qquad \frac{\partial f}{\partial P^+} = \frac{a\Delta\tau}{h^2} + \frac{b\Delta\tau}{h} \ge 0,$$

$$\frac{\partial f}{\partial P} = 1 - \left(\frac{2a}{h^2} + \frac{b}{h} + r\right)\Delta\tau - \frac{\varepsilon C\Delta\tau}{(P+\varepsilon-q)^2} \;\ge\; 1 - \left(\frac{2a}{h^2} + \frac{b}{h} + r + \frac{C}{\varepsilon}\right)\Delta\tau.$$

Condition (4.2) guarantees that the right–hand side is nonnegative. ∎

Theorem 4.2 (Discrete obstacle and positivity: explicit upwind)**.** *If $C \geq rK$ (cf. (3.1)), $\overline{S} \geq K$, and the CFL condition (4.2) is satisfied, then the explicit solution fulfills*

$$P_j^n \ \geq \ (K - S_j)_+, \qquad j = 1, \dots, M, \quad n = 0, \dots, N+1.$$

Proof. We argue by induction over n.

Initialization ($n = 0$). By construction, $P_j^0 = (K - S_j)_+ = (q_j)_+$ for all j. Moreover, $P_0^n = K \geq (K - S_0)_+$ and $P_{M+1}^n = 0 = (K - S_{M+1})_+$ hold at the boundaries.

Induction step. Suppose $P_j^n \geq (q_j)_+$ for all $j = 1, \dots, M$. From the update relation

$$P_j^{n+1} = f\big(P_{j-1}^n, P_j^n, P_{j+1}^n, q_j, S_j\big)\,, \tag{4.3}$$

and the monotonicity of f, it suffices to evaluate the scheme on the lower bounds of the arguments.

Case 1: $q_j \geq 0$ (i.e. $S_j \leq K$). Here $(q_j)_+ = q_j$, and substituting (q_{j-1}, q_j, q_{j+1}) into (4.3) yields

$$f(q_{j-1}, q_j, q_{j+1}, q_j, S_j) = q_j + \Delta\tau \left[-b_j - r q_j + C \right] = q_j + \Delta\tau\,(C - rK).$$

Since $C \geq rK$, we obtain $P_j^{n+1} \geq q_j = (q_j)_+$.

Case 2: $q_j < 0$ (i.e. $S_j > K$). Now $(q_j)_+ = 0$, and by monotonicity of f with nonnegative arguments,

$$P_j^{n+1} \ \geq \ f(0, 0, 0, q_j, S_j) = \frac{\varepsilon C \Delta\tau}{\varepsilon - q_j} > 0.$$

Hence $P_j^{n+1} \geq 0 = (q_j)_+$.

Thus, in all cases, $P_j^{n+1} \geq (K - S_j)_+$, completing the induction. □

4.2.2. *Implicit Scheme (upwind on $+\, rS\,\partial_S P$).* We now consider a fully implicit time discretization. Time stepping is performed via a backward Euler update, while the diffusion operator is approximated by centered second–order differences. The convective contribution, written as $+\, b_j \partial_S P$ with $b_j := rS_j$, is treated by a forward upwind difference consistent with the sign of b_j:

$$\partial_S P(\tau_{n+1}, S_j) \ \approx \ \frac{P_{j+1}^{n+1} - P_j^{n+1}}{h}.$$

With these choices, the penalized implicit upwind scheme for $j = 1, \dots, M$ reads

$$\frac{P_j^{n+1} - P_j^n}{\Delta\tau} - a_j\, \frac{P_{j+1}^{n+1} - 2P_j^{n+1} + P_{j-1}^{n+1}}{h^2} + b_j\, \frac{P_{j+1}^{n+1} - P_j^{n+1}}{h} + rP_j^{n+1} - \frac{\varepsilon C}{P_j^{n+1} + \varepsilon - q_j} = 0, \tag{4.4}$$

where $a_j = \frac{1}{2}\sigma^2 S_j^{2\alpha} \log^{2\beta}(1 + S_j)$ and $q_j := K - S_j$. Boundary and initial data are imposed at every step as $P_0^{n+1} = K$, $P_{M+1}^{n+1} = 0$, and $P_j^0 = (K - S_j)_+$.

It is convenient to reorganize (4.4) so as to display the tridiagonal structure explicitly:

$$\boxed{\begin{aligned} \Big(1 + \tfrac{2a_j\Delta\tau}{h^2} + \tfrac{b_j\Delta\tau}{h} + r\Delta\tau\Big) P_j^{n+1} &= P_j^n + \Big(\tfrac{a_j\Delta\tau}{h^2}\Big) P_{j-1}^{n+1} \\ &\quad + \Big(\tfrac{a_j\Delta\tau}{h^2} + \tfrac{b_j\Delta\tau}{h}\Big) P_{j+1}^{n+1} + \frac{\varepsilon C\,\Delta\tau}{P_j^{n+1} + \varepsilon - q_j}. \end{aligned}} \tag{4.5}$$

Existence and uniqueness at each time step. Fix n and introduce the nonlinear mapping $\mathcal{T}:\mathbb{R}^M\to\mathbb{R}^M$ given by the right–hand side of (4.5), normalized by the diagonal factor

$$d_j := 1+\tfrac{2a_j\Delta\tau}{h^2}+\tfrac{b_j\Delta\tau}{h}+r\Delta\tau \;>\; 0.$$

To construct a solution, we use a standard fixed–point (or nonlinear Gauss–Seidel) approach. Given $U\in\mathbb{R}^M$, define $G(U)=V$ as the unique solution of the linear tridiagonal system obtained by freezing the penalty at U:

$$\mathbf{A}\,V=\mathbf{g}(U),$$

with coefficients

$$\begin{cases} A_{j,j-1}=-\frac{a_j\Delta\tau}{h^2}\le 0, \qquad A_{j,j}=1+\frac{2a_j\Delta\tau}{h^2}+\frac{b_j\Delta\tau}{h}+r\Delta\tau>0,\\ A_{j,j+1}=-\Big(\frac{a_j\Delta\tau}{h^2}+\frac{b_j\Delta\tau}{h}\Big)\le 0, \end{cases} \qquad g_j(U)=P_j^n+\frac{\varepsilon C\,\Delta\tau}{U_j+\varepsilon-q_j}.$$

The matrix $\mathbf{A}$ is an L–matrix (nonpositive off–diagonals) with strict row diagonal dominance

$$A_{j,j}-\big(|A_{j,j-1}|+|A_{j,j+1}|\big)=1+r\Delta\tau>0,$$

hence an M–matrix. In particular, $\mathbf{A}$ is invertible with inverse–positivity, so the linear solve $V=G(U)$ is well posed. Moreover, the right–hand side $g_j(U)$ is order–reversing in U_j and bounded below by $\varepsilon C\Delta\tau/(\varepsilon-q_j)$ (the latter is finite since $q_j\le K$). The monotone iteration

$$P^{(m+1)}:=G\big(P^{(m)}\big), \qquad m=0,1,2,\dots,$$

initialized for instance with $P^{(0)}\equiv 0$ or $P^{(0)}\equiv q_+=((K-S)_+)$, generates a bounded monotone sequence converging to a fixed point of G, which solves (4.5). Uniqueness follows by the strict monotonicity of the left–hand side (M–matrix) and the strict decrease of $v\mapsto \varepsilon C/(v+\varepsilon-q)$: subtracting two solutions and using inverse–positivity yields the claim.

Theorem 4.3 (Discrete obstacle and positivity: implicit upwind)**.** *If $C\ge rK$ and $\overline{S}\ge K$, then any solution of (4.4) satisfies*

$$P_j^{n+1}\;\ge\;(K-S_j)_+, \qquad j=1,\dots,M,\;\; n=0,\dots,N.$$

Proof. We argue by induction on n. At $n=0$, $P_j^0=(K-S_j)_+$, and for all n the boundaries obey $P_0^{n+1}=K\ge(K-S_0)_+$ and $P_{M+1}^{n+1}=0=(K-S_{M+1})_+$ since $\overline{S}\ge K$.

Assume $P_j^n\ge(K-S_j)_+=(q_j)_+$ for $j=1,\dots,M$. Set $V_j^{n+1}:=P_j^{n+1}-q_j$. Using (4.5) and $q_{j\pm1}=q_j\mp h$, subtract the discrete identity satisfied by q_j to obtain

$$\begin{aligned}\big(1+r\Delta\tau\big)V_j^{n+1}\ge V_j^n+\frac{\varepsilon C\,\Delta\tau}{V_j^{n+1}+\varepsilon}-rK\,\Delta\tau\\ +\Big(\tfrac{a_j\Delta\tau}{h^2}\Big)\big(V_{j-1}^{n+1}-V_j^{n+1}\big)+\Big(\tfrac{a_j\Delta\tau}{h^2}+\tfrac{b_j\Delta\tau}{h}\Big)\big(V_{j+1}^{n+1}-V_j^{n+1}\big).\end{aligned} \tag{4.6}$$

Let k minimize V^{n+1} over $j=1,\dots,M$. Then $V_{k\pm1}^{n+1}-V_k^{n+1}\ge 0$, so the last line in (4.6) is nonnegative at $j=k$. Define

$$F(V):=(1+r\Delta\tau)\,V-\frac{\varepsilon C\,\Delta\tau}{V+\varepsilon}+rK\,\Delta\tau.$$

Since $F'(V)=1+r\Delta\tau+\frac{\varepsilon C\,\Delta\tau}{(V+\varepsilon)^2}>0$ and $F(0)=\Delta\tau(rK-C)\le 0$ (because $C\ge rK$), from (4.6) at $j=k$ we get

$$F\big(V_k^{n+1}\big)\;\ge\;V_k^n\;\ge\;(q_k)_+-q_k\;\ge\;0.$$

By strict monotonicity of F, this implies $V_k^{n+1} \geq 0$, hence $P_k^{n+1} \geq q_k$. As k is a minimizer, $P_j^{n+1} \geq q_j$ for all j. Finally, $P_j^{n+1} \geq 0$ is obtained analogously (e.g., by testing (4.5) at a minimal component of P^{n+1} and using $P^n \geq 0$). Therefore $P_j^{n+1} \geq \max\{q_j, 0\} = (K - S_j)_+$. □

Remarks. (i) No CFL restriction is needed for the implicit upwind scheme: the coefficient matrix is an M–matrix for any $\Delta\tau > 0$, ensuring stability and monotonicity of the linearized operator.

(ii) The argument hinges on the linear structure of $q(S) = K - S$ (its discrete diffusion vanishes and the drift–rate combination reduces to $-rK$), together with the choice $C \geq rK$ which guarantees $F(0) \leq 0$.

At each time level, the nonlinear tridiagonal system (4.5) is solved by a Successive Over–Relaxation (SOR) procedure (or equivalently a nonlinear Gauss–Seidel with relaxation). This iterative solver is inexpensive per iteration, exploits the tridiagonal structure, and converges robustly to the fixed point of the rational penalty formulation. All numerical experiments reported below were obtained with this implicit upwind discretization combined with SOR.

5. Numerical Results and Model Comparison

We now present a set of numerical experiments illustrating the performance of the SGV model in pricing American put options. Our analysis combines tabulated results and graphical illustrations, and systematically compares the SGV framework with the classical Black–Scholes (BS) and Constant Elasticity of Variance (CEV) models. Two main aspects are investigated: (i) the relative pricing behavior across different models, and (ii) the effect of the SGV parameters α and β on American option values.

The computations are carried out using the rational penalty formulation of Section 3. The penalized PDE is discretized by an implicit finite difference scheme and solved iteratively via the SOR method, which ensures stability and rapid convergence. Unless otherwise stated, the baseline parameters are $K = 50$, $r = 5\%$, $\sigma = 25\%$, and maturity $T = 7/12$.

Comparison across models. Table 1 reports American put prices under the BS, CEV, and SGV models for underlying asset levels $S \in [10, 100]$. For each model, the relevant parameter pair (α, β) is specified in the header. The table is complemented by Figure 1, which provides a visual comparison of the three models under the implicit scheme with SOR. Notice that differences between BS, CEV, and SGV become increasingly significant as the stock price approaches and exceeds the strike.

Robustness across numerical methods. To evaluate the consistency of the numerical implementation, Table 2 reports option prices obtained using three different discretizations: implicit finite differences, an explicit finite difference scheme, and a binomial tree. The experiment is conducted at fixed asset price $S = 55$ for several configurations of (α, β). The results confirm that the implicit scheme is both stable and accurate, while the explicit scheme remains consistent under suitable CFL restrictions.

Sensitivity to SGV parameters. Figure 2 examines how option values respond to variations in α and β. Panel (a) shows the effect of increasing α with $\beta = 0$: the American put price rises monotonically and converges toward the BS value (≈ 1.5849). This corresponds to the CEV case, where α controls the elasticity of variance. As

TABLE 1. American put option prices under BS, CEV, and SGV models, computed by the rational penalty method (implicit scheme with SOR).

Stock price S	**BS** ($\alpha = 1,\ \beta = 0$)	**CEV** ($\alpha = 0.9,\ \beta = 0$)	**SGV** ($\alpha = 1,\ \beta = 0.1$)
10	40	40	40
20	30	30	30
30	20	20	20
40	10.0360	10	10.1675
50	3.2185	2.1342	3.7150
60	0.7159	0.1800	1.0504
70	0.1235	0.0076	0.2508
80	0.0178	1.976e-04	0.0535
90	0.0023	7.122e-06	0.0107
100	2.923e-04	2.941e-06	0.0021

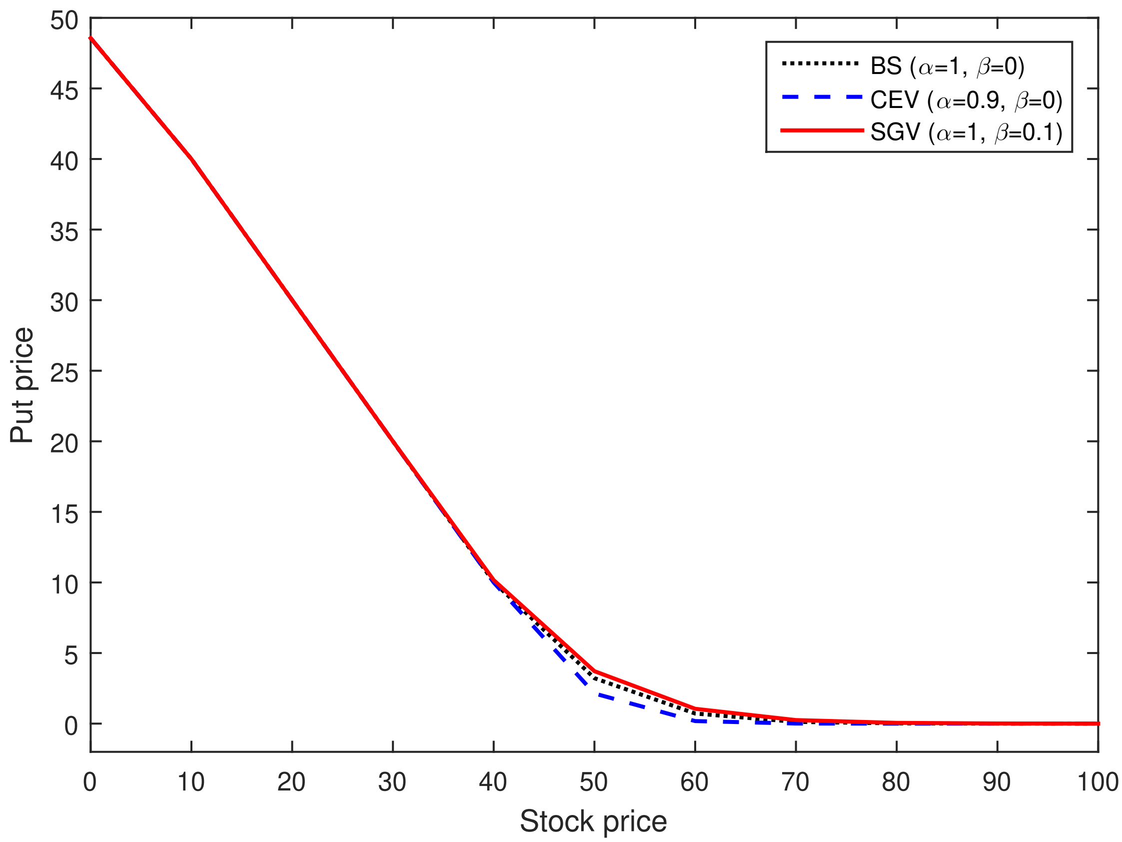

FIGURE 1. Comparison of American put prices across BS, CEV, and SGV models, illustrating the impact of model dynamics on option values.

$\alpha \to 1$, the model recovers the BS benchmark, confirming the interpolation property of SGV.

TABLE 2. American put option values obtained under implicit, explicit, and binomial methods for various parameter settings.

Model	**Parameters** (α, β)	**Implicit**	**Binomial**	**Explicit**
CEV $(\alpha < 1)$	(0.5, 0)	0.0000	2.0126e–07	0.0055
	(0.9, 0)	0.5623	0.6094	0.6184
	(0.99, 0)	1.4465	1.4527	1.4552
SGV $(\alpha < 1,\ \beta < 0)$	(0.9, –0.4)	0.1231	0.0757	0.0822
	(0.9, –0.1)	0.4877	0.4032	0.4119
SGV $(\alpha < 1,\ \beta > 0)$	(0.9, 0.1)	0.9595	0.8779	0.8864
	(0.9, 0.4)	2.1399	2.1399	2.1407
BS $(\alpha = 1,\ \beta = 0)$	(1, 0)	1.5849	1.5781	1.5795
SGV $(\alpha = 1,\ \beta < 0)$	(1, –1)	0.0177	0.0037	0.0090
	(1, –0.4)	0.4492	0.3817	0.3889
	(1, –0.01)	1.5442	1.5331	1.5354

Panel (b) displays the impact of varying β with $\alpha = 1$. When $\beta = 0$, the SGV reduces exactly to BS. Negative β amplifies volatility for small asset levels, which tends to suppress early exercise incentives and thus lowers the option value. Positive β increases volatility for large asset levels, producing higher option prices and a marked skew. This asymmetry illustrates the ability of the SGV model to capture features that lie beyond the reach of classical BS or CEV dynamics.

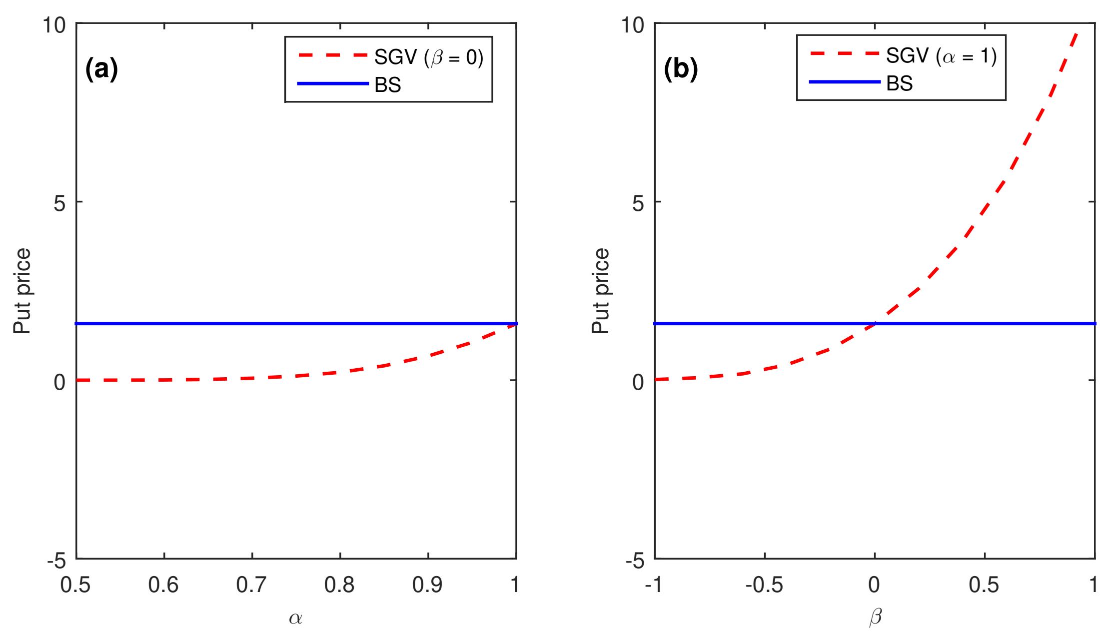

FIGURE 2. Sensitivity of American put prices to SGV parameters. Panel (a): variation with respect to α at fixed $\beta = 0$. Panel (b): variation with respect to β at fixed $\alpha = 1$.

6. Conclusion

We have extended the Slow Growth Volatility (SGV) model to the valuation of American options. The early exercise feature was formulated as a linear complementarity problem and treated numerically via a rational penalty approach combined with finite difference discretizations. Existence, uniqueness, and convergence of the penalized formulation were established, and numerical solutions were computed by an implicit upwind scheme with SOR iterations.

The numerical results demonstrate that the SGV model constitutes a flexible and robust framework. Through the parameters (α, β), the SGV interpolates between the CEV and BS models, while offering the additional capability to reproduce skew effects observed in practice. This positions SGV as a valuable extension of classical local volatility models.

Future research directions include the incorporation of jump–diffusion dynamics, applications to multi–asset and path–dependent derivatives, and the integration of machine learning methods to accelerate calibration and improve numerical efficiency.

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