

POSITIVE m -HOMOGENEOUS POLYNOMIAL IDEALS

ADEL BOUNABAB¹ and KHALIL SAADI^{2*}

ABSTRACT. We introduce and study the concept of positive polynomial ideals between Banach lattices. The paper develops the basic principles of these classes and presents methods for constructing them from given positive operator ideals. In addition, we present concrete examples of positive polynomial ideals that demonstrate the relevance of these classes. In particular, we introduce a new class of positive polynomials, referred to as positive (q, r) -dominated polynomials, which constitutes an important example of a positive polynomial ideal.

Keywords. Banach lattices, Positive p -summing operators, Positive linear ideals, Positive left polynomial ideals, Positive right polynomial ideals, Positive ideals, Positive (q, r) -dominated polynomials.

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1. INTRODUCTION AND PRELIMINARIES

Several attempts have been made to extend the theory of operator ideals to non-linear contexts. After the development of multi-ideal theory for multilinear mappings and homogeneous polynomials [23], attention naturally turned to the study of positive classes. In [15], the positive ideal of linear and multilinear mappings was investigated. These new ideals encompass several important classes of operators, such as positive p -summing operators [5], positive strongly p -summing operators [3], and positive (p, q) -dominated operators [11] in the linear case, as well as Cohen positive strongly p -summing multilinear operators [7], positive Cohen p -nuclear multilinear operators [8], and factorable positive strongly p -summing multilinear operators [9] in the multilinear case. Motivated by these studies, we now propose to define ideals for positive classes of homogeneous polynomials. Our aim is to examine certain families of positive polynomial ideals and to demonstrate how positivity enhances the theory of homogeneous polynomials. Following the same procedure as in [15], we introduce these polynomial ideals in a systematic way. This approach not only unifies several notions already studied in the literature but also opens new directions for extending classical results to the nonlinear and positive framework. As in the linear and multilinear cases, the definition of positive polynomial ideals encompasses several classes, such as Cohen positive strongly p -summing polynomials [17], positive Cohen p -nuclear polynomials

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* Corresponding author

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[18], and positive p -dominated polynomials [17], as well as the new class we introduce here, namely positive (q, r) -dominated polynomials.

The structure of the paper is as follows.

In Section 1, we review the fundamental concepts and terminology used in this paper, including Banach lattices, linear operators, symmetric multilinear forms, and polynomials. We also recall the definition of positive operator ideals, with particular attention to positive p -summing operators, as well as the definition of polynomial ideals. In Section 2, we develop the foundations of positive left ideals, denoted $\mathcal{P}_L^+$, positive right ideals, denoted $\mathcal{P}_R^+$, and positive ideals, denoted $\mathcal{P}^+$, for m -homogeneous polynomials. This framework naturally extends the positive m -linear setting. We then apply the composition method to construct a positive left polynomial ideal from a given positive left operator ideal. In addition, we present the factorization method to generate a positive right ideal of m -homogeneous polynomials from a given positive right operator ideal. In Section 3, we present concrete examples of positive polynomial ideals, such as Cohen positive strongly p -summing polynomials, positive Cohen p -nuclear polynomials, and positive p -dominated polynomials. In particular, we introduce a new class of positive polynomials, referred to as positive (q, r) -dominated polynomials. This class satisfies both the Pietsch Domination Theorem and the Kwapien Factorization Theorem, and constitute an important example of positive polynomial ideals.

Throughout the paper, E, F and G denote Banach lattices and X, Y denote Banach spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). By B_X we denote the closed unit ball of X and by X^* its topological dual. We use the symbol $\mathcal{L}(X; Y)$ for the space of all bounded linear operators from X into Y . For $1 \leq p \leq \infty$, we denote by p^* its conjugate, i.e., $1/p + 1/p^* = 1$. Let E be a Banach lattice with norm $\|\cdot\|$ and order $\leq$. We denote by E^+ the positive cone of E , i.e., $E^+ = \{x \in E : x \geq 0\}$. Let $x \in E$, its positive part is defined by $x^+ := \sup\{x, 0\} \geq 0$ and its negative part is defined by $x^- := \sup\{-x, 0\} \geq 0$. We have $x = x^+ - x^-$ and $|x| = x^+ + x^-$. The dual E^* of a Banach lattice E is a Banach lattice with the natural order

$$x_1^* \leq x_2^* \Leftrightarrow \langle x, x_1^* \rangle \leq \langle x, x_2^* \rangle, \quad \forall x \in E^+.$$

A bounded linear operator $u : E \rightarrow F$ is called positive if $u(x) \in F^+$, whenever $x \in E^+$. By $\mathcal{L}^+(E; F)$ we denote the set of all positive operators from E to F . A linear operator u is called *regular* if there exist $u_1, u_2 \in \mathcal{L}^+(E; F)$ such that

$$u = u_1 - u_2.$$

We denote by $\mathcal{L}^r(E; F)$ the vector space of regular operators from E to F . The vector space $\mathcal{L}^r(E; F)$ is generated by positive operators which is a Banach space with the norm

$$\|u\|_r = \inf \{ \|v\| : v \in \mathcal{L}^+(E; F), |u(x)| \leq v(x), \quad \forall x \in E^+ \},$$

By [19, Section 1.3], if $F = \mathbb{K}$, we have

$$E^* = \mathcal{L}(E, \mathbb{K}) = \mathcal{L}^r(E, \mathbb{K}).$$

The canonical embedding $i : E \rightarrow E^{**}$ such that $\langle i(x), x^* \rangle = \langle x^*, x \rangle$ of E into its second dual E^{**} is an order isometry from E onto a sublattice of E^{**} . If we consider E as a sublattice of E^{**} we have for $x_1, x_2 \in E$

$$x_1 \leq x_2 \iff \langle x_1, x^* \rangle \leq \langle x_2, x^* \rangle, \quad \forall x^* \in E^{*+}.$$

The spaces $\mathcal{C}(K)$ where K is compact and $L_p(\mu)$, ($1 \leq p \leq \infty$) are Banach lattices. Let X be a Banach space. We denote by $\ell_p^n(X)$ the Banach space of all absolutely

p -summable sequences $(x_i)_{i=1}^n \subset X$ with the norm

$$\|(x_i)_{i=1}^n\|_p = \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}},$$

and by $\ell_{p,w}^n(X)$ the Banach space of all weakly p -summable sequences $(x_i)_{i=1}^n \subset X$ with the norm,

$$\|(x_i)_{i=1}^n\|_{p,w} = \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^p \right)^{\frac{1}{p}}.$$

Consider the case where X is replaced by a Banach lattice E , and define

$$\ell_{p,|w|}^n(E) = \{(x_i)_{i=1}^n \subset E : (|x_i|)_{i=1}^n \in \ell_{p,w}^n(E)\} \text{ and } \|(x_i)_{i=1}^n\|_{p,|w|} = \|(|x_i|)_{i=1}^n \|_{p,w}.$$

Let $B_{E^*}^+ = \{x^* \in B_{E^*} : x^* \geq 0\} = B_{E^*} \cap E^{*+}$. If $(x_i)_{i=1}^n \subset E^+$, we have

$$\|(x_i)_{i=1}^n\|_{p,|w|} = \|(x_i)_{i=1}^n\|_{p,w} = \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^n \langle x^*, x_i \rangle^p \right)^{\frac{1}{p}}.$$

Given $m \in \mathbb{N}^*$, we denote by $\mathcal{L}(X_1, \dots, X_m; Y)$ the Banach space of all bounded multilinear operators from $X_1 \times \dots \times X_m$ into Y endowed with the supremum norm. A map $P : X \rightarrow Y$ is an m -homogeneous polynomial if there exists a unique symmetric m -linear operator $\hat{P} : X \times \binom{m}{\dots} \times X \rightarrow Y$ such that

$$P(x) = \hat{P}\left(x, \binom{m}{\dots}, x\right),$$

for every $x \in X$. We denote by $\mathcal{P}(^m X; Y)$, the Banach space of all continuous m -homogeneous polynomials from X into Y endowed with the norm

$$\|P\| = \sup_{\|x\| \leq 1} \|P(x)\| = \inf \{C : \|P(x)\| \leq C \|x\|^m, \quad x \in X\}.$$

We denote by $\mathcal{P}_f(^m X; Y)$ the space of all m -homogeneous polynomials of finite rank, that is

$$\mathcal{P}_f(^m X; Y) = \left\{ \sum_{i=1}^k \varphi_i^m(x) y_i : k \in \mathbb{N}, \varphi_i \in X^*, y_i \in Y, 1 \leq i \leq k \right\}.$$

If $Y = \mathbb{K}$, we write simply $\mathcal{P}(^m X)$. For the general theory of polynomials on Banach spaces, we refer to [14] and [20]. By $X_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi X_m$ we denote the completed projective tensor product of $X_1, \dots, X_m$. If $X = X_1 = \dots = X_m$, we write $\hat{\otimes}_\pi^m X$. By $\otimes_s^m X := X \otimes \binom{m}{\dots} \otimes X$ we denote the m fold symmetric tensor product of X , that is,

$$\otimes_s^m X = \left\{ \sum_{i=1}^n \lambda_i x_i \otimes \binom{m}{\dots} \otimes x_i : n \in \mathbb{N}^*, \lambda_i \in \mathbb{K}, x_i \in X, (1 \leq i \leq n) \right\}.$$

By $\hat{\otimes}_{\pi,s}^m X$ we denote the closure of $\otimes_s^m X$ in $\hat{\otimes}_\pi^m X$. For symmetric tensor products, we refer to [16]. Let $P \in \mathcal{P}(^m X; Y)$, we define its linearization $P_L : \hat{\otimes}_{\pi,s}^m X \rightarrow Y$ by

$$P_L\left(x \otimes \binom{m}{\dots} \otimes x\right) = P(x),$$

for all $x \in X$. Consider the canonical m -homogeneous polynomial $\delta_m : X \rightarrow \hat{\otimes}_{\pi,s}^m X$ defined by

$$\delta_m(x) = x \otimes \binom{m}{\dots} \otimes x.$$

We have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{P} & Y \\ & \searrow \delta_m & \uparrow P_L \\ & & \widehat{\otimes}_{\pi,s}^m X \end{array} \quad (1.1)$$

In other words, $P = P_L \circ \delta_m$. We have $\|P\| = \|P_L\|$ and the following isometric identification holds

$$\mathcal{P}(^m X; Y) = \mathcal{L}(\widehat{\otimes}_{\pi,s}^m X; Y).$$

Blasco [5] introduced the positive generalization of p -summing operators as follows: An operator $T : E \rightarrow X$ is said to be *positive p -summing* ($1 \leq p \leq \infty$) if there exists a constant $C > 0$ that verifies the inequality

$$\|(T(x_i))_{i=1}^n\|_p \leq C \|(x_i)_{i=1}^n\|_{p,w}, \quad (1.2)$$

with $x_1, \dots, x_n \in E^+$. We denote by $\Pi_p^+(E; X)$, the space of all positive p -summing operators from E into X , which is a Banach space with the norm $\pi_p^+(T)$ given by the infimum of the constant $C > 0$ that verify the inequality (1.2). We have $\Pi_\infty^+(E; X) = \mathcal{L}(E; X)$. O.I. Zhukova [24], gives the Pietsch Domination Theorem concerning this class. The operator T belongs to $\Pi_p^+(E; X)$ if and only if there exist a Radon probability measure μ on the set $B_{E^*}^+$ and a positive constant C such that for every $x \in E^+$

$$\|T(x)\| \leq C \left(\int_{B_{E^*}^+} \langle x, x^* \rangle^p d\mu(x^*) \right)^{\frac{1}{p}}. \quad (1.3)$$

Positive operator ideal: We provide the definition of the positive ideal introduced and studied in [15]: A positive left ideal, denoted by $\mathcal{B}_L^+$, is a subclass of all continuous linear operators from a Banach space into a Banach lattice such that for every Banach space X and Banach lattice E , the components

$$\mathcal{B}_L^+(X; E) := \mathcal{L}(X; E) \cap \mathcal{B}_L^+$$

satisfy:

- (i) $\mathcal{B}_L^+(X; E)$ is a linear subspace of $\mathcal{L}(X; E)$ containing the linear mappings of finite rank.
- (ii) The positive ideal property: If $T \in \mathcal{B}_L^+(X; E)$, $u \in \mathcal{L}(Y; X)$ and $v \in \mathcal{L}^+(E; F)$, then $v \circ T \circ u$ is in $\mathcal{B}_L^+(Y; F)$.

If $\|\cdot\|_{\mathcal{B}_L^+} : \mathcal{B}_L^+ \rightarrow \mathbb{R}^+$ satisfies:

- a) $(\mathcal{B}_L^+(X; E), \|\cdot\|_{\mathcal{B}_L^+})$ is a Banach space for all Banach space X and Banach lattice E .
- b) The linear form $T : \mathbb{K} \rightarrow \mathbb{K}$ given by $T(\lambda) = \lambda$ satisfies $\|T\|_{\mathcal{B}_L^+} = 1$,
- c) $T \in \mathcal{B}_L^+(X; E)$, $u \in \mathcal{L}(Y; X)$ and $v \in \mathcal{L}^+(E; F)$ then

$$\|v \circ T \circ u\|_{\mathcal{B}_L^+} \leq \|v\| \|T\|_{\mathcal{B}_L^+} \|u\|.$$

The class $(\mathcal{B}_L^+, \|\cdot\|_{\mathcal{B}_L^+})$ is a positive Banach ideal.

The *positive right ideal*, denoted $\mathcal{B}_R^+$, is obtained by reversing the roles of the operators u and v . In this case, we consider compositions with positive linear operators on the right and arbitrary linear operators on the left. Similarly, the *positive ideal*, denoted $\mathcal{B}^+$, is obtained by restricting to positive linear operators on both sides of the composition. Note that every positive right or left ideal is automatically a positive ideal.

Polynomial ideal: An ideal of m -homogeneous polynomials (or polynomial ideal) $\mathcal{Q}$ is a subclass of the class of all continuous homogeneous polynomials between Banach spaces such that for all $m \in \mathbb{N}^*$, and Banach spaces X and Y , the components $\mathcal{Q}(^m X; Y) = \mathcal{P}(^m X; Y) \cap \mathcal{Q}$ satisfy:

(i) $\mathcal{Q}(^m X; Y)$ is a linear subspace of $\mathcal{P}(^m X; Y)$ which contains the m -homogeneous polynomials of finite type.

(ii) The positive ideal property: If $u \in \mathcal{L}(X_0; X)$, $P \in \mathcal{P}(^m X; Y)$ and $v \in \mathcal{L}(Y; Y_0)$, then $v \circ P \circ u$ is in $\mathcal{Q}(^m X_0; Y_0)$.

If $\|\cdot\|_{\mathcal{Q}} : \mathcal{Q} \rightarrow \mathbb{R}^+$ satisfies:

a) $(\mathcal{Q}(^m X; Y), \|\cdot\|_{\mathcal{Q}})$ is a normed (Banach) space for all Banach spaces X, Y and m .

b) The canonical polynomial $P^m : \mathbb{K} \rightarrow \mathbb{K}$ given by $P^m(\lambda) = \lambda^m$ satisfies $\|P^m\|_{\mathcal{Q}} = 1$,

c) If $u \in \mathcal{L}(X_0; X)$, $P \in \mathcal{P}(^m X; Y)$ and $v \in \mathcal{L}(Y; Y_0)$, then

$$\|v \circ P \circ u\|_{\mathcal{Q}} \leq \|v\| \|P\|_{\mathcal{Q}} \|u\|^m.$$

The class $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is called a normed (Banach) polynomial ideal. The case $m = 1$ recovers the classical theory of normed and Banach operator ideals. For further details on linear operator ideals, we refer to [13].

2. POSITIVE POLYNOMIAL IDEALS

In this section, we introduce positive polynomial ideals and develop abstract methods for their construction in the setting of m -homogeneous polynomials. We analyze the positive ideal property through the behavior of the associated linear operators, establishing a natural link between the linear and polynomial frameworks. These polynomial ideals extend and complement the theory of positive multilinear ideals studied in detail in [15].

Definition 2.1. Fix $m \in \mathbb{N}^*$. A positive left polynomial ideal (or positive left ideal of m -homogeneous polynomials), denoted by $\mathcal{P}_L^+$, is a subclass of all continuous m -homogeneous polynomials from a Banach space into a Banach lattice such that for all Banach space X and Banach lattice E , the components

$$\mathcal{P}_L^+(^m X; E) := \mathcal{P}(^m X; E) \cap \mathcal{P}_L^+,$$

satisfy:

(i) $\mathcal{P}_L^+(^m X; E)$ is a linear subspace of $\mathcal{P}(^m X; E)$ containing the polynomials of finite rank.

(ii) The positive ideal property: If $P \in \mathcal{P}_L^+(^m X; E)$, $u \in \mathcal{L}(Y; X)$ and $v \in \mathcal{L}^+(E; F)$, then $v \circ P \circ u$ is in $\mathcal{P}_L^+(^m Y; F)$.

If $\|\cdot\|_{\mathcal{P}_L^+} : \mathcal{P}_L^+ \rightarrow \mathbb{R}^+$ satisfies:

a) $(\mathcal{P}_L^+(^m X; E), \|\cdot\|_{\mathcal{P}_L^+})$ is a Banach (respectively, quasi-Banach) space for all Banach space X and Banach lattice E ,

b) The canonical polynomial $P^m : \mathbb{K} \rightarrow \mathbb{K}$ given by $P^m(\lambda) = \lambda^m$ satisfies $\|P^m\|_{\mathcal{P}_L^+} = 1$,

c) If $P \in \mathcal{P}_L^+(^m X; E)$, $u \in \mathcal{L}(Y; X)$ and $v \in \mathcal{L}^+(E; F)$, then

$$\|v \circ P \circ u\|_{\mathcal{P}_L^+} \leq \|v\| \|P\|_{\mathcal{P}_L^+} \|u\|^m.$$

The class $(\mathcal{P}_L^+, \|\cdot\|_{\mathcal{P}_L^+})$ is a positive left Banach (respectively, quasi-Banach) polynomial ideal.

Remark 2.2. In condition (ii), because every regular operator is a difference of positive ones, the set $\mathcal{L}^+(E; F)$ can be replaced by the space $\mathcal{L}^r(E; F)$, and condition (ii) remains the same.

Analogous to the previous approach, we introduce the *positive right polynomial ideal*, denoted $\mathcal{P}_R^+$, by swapping the roles of the operators u and v . In doing so, we examine the composition of positive linear operators on the right-hand side and linear operators on the left-hand side. Similarly, we define the *positive polynomial ideal*, denoted $\mathcal{P}^+$, by considering only the positive linear operators, with composition occurring on both the right and left sides.

Remark 2.3. 1) It is evident that every polynomial ideal is indeed positive polynomial ideal.

2) Every positive right or left polynomial ideal is positive polynomial ideal.

Proposition 2.4. *Let $\mathcal{B}_R^+$ be a positive right operator ideal and $\mathcal{P}_L^+$ a positive left polynomial ideal. The composition ideal $\mathcal{P}_L^+ \circ \mathcal{B}_R^+$ is defined as the space of polynomials P that admit a factorization $P = Q \circ u$, with $u \in \mathcal{B}_R^+(E; X)$ and $Q \in \mathcal{P}_L^+({}^m X; F)$. This construction yields a positive polynomial ideal.*

Proof. Let E and F be Banach lattices. We will verify that $\mathcal{P}_L^+ \circ \mathcal{B}_R^+({}^m E; F)$ is a linear subspace. Let $\lambda \in \mathbb{K}$ and $P \in \mathcal{P}_L^+ \circ \mathcal{B}_R^+({}^m E; F)$. There exist a Banach space X and elements $u_0 \in \mathcal{B}_R^+(E; X)$, $Q_0 \in \mathcal{P}_L^+({}^m X; F)$ such that $P = Q_0 \circ u_0$. Then, $\lambda P = (\lambda Q_0) \circ u_0 \in \mathcal{P}_L^+ \circ \mathcal{B}_R^+({}^m E; F)$. Now, let $P_1, P_2 \in \mathcal{P}_L^+ \circ \mathcal{B}_R^+({}^m E; F)$ such that there exist Banach spaces X, Y and elements $u_1 \in \mathcal{B}_R^+(E; X)$, $u_2 \in \mathcal{B}_R^+(E; Y)$, $Q_1 \in \mathcal{P}_L^+({}^m X; F)$, and $Q_2 \in \mathcal{P}_L^+({}^m Y; F)$ with the following commutative diagrams:

$$\begin{array}{ccc} E & \xrightarrow{P_1} & F \\ u_1 \downarrow & \nearrow Q_1 & \\ X & & \end{array} \quad \text{and} \quad \begin{array}{ccc} E & \xrightarrow{P_2} & F \\ u_2 \downarrow & \nearrow Q_2 & \\ Y & & \end{array}$$

We define $A = i_1 \circ u_1 + i_2 \circ u_2$, where $i_1 : X \rightarrow X \times Y$ and $i_2 : Y \rightarrow X \times Y$ are given by $i_1(x) = (x, 0)$ and $i_2(y) = (0, y)$ with $x \in X$ and $y \in Y$. Since $u_1 \in \mathcal{B}_R^+(E; X)$ and $u_2 \in \mathcal{B}_R^+(E; Y)$, we have $i_1 \circ u_1, i_2 \circ u_2 \in \mathcal{B}_R^+(E; X \times Y)$. Consequently,

$$A = i_1 \circ u_1 + i_2 \circ u_2 \in \mathcal{B}_R^+(E; X \times Y).$$

On the other hand, we define $B = Q_1 \circ \pi_1 + Q_2 \circ \pi_2$, where $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ are given by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. We have $B : (X \times Y) \rightarrow F$ is an m -homogeneous polynomial, where $\widehat{B} : (X \times Y) \times \overset{(m)}{\cdots} \times (X \times Y) \rightarrow F$, its associated symmetric multilinear, is defined by

$$\widehat{B} = \widehat{Q}_1 \left(\pi_1, \overset{(m)}{\cdots}, \pi_1 \right) + \widehat{Q}_2 \left(\pi_2, \overset{(m)}{\cdots}, \pi_2 \right).$$

Indeed, let $(x, y) \in X \times Y$, we have

$$\begin{aligned} & \widehat{B} \left((x, y), \overset{(m)}{\cdots}, (x, y) \right) \\ &= \widehat{Q}_1 \left(\pi_1(x, y), \overset{(m)}{\cdots}, \pi_1(x, y) \right) + \widehat{Q}_2 \left(\pi_2(x, y), \overset{(m)}{\cdots}, \pi_2(x, y) \right) \\ &= Q_1 \circ \pi_1(x, y) + Q_2 \circ \pi_2(x, y) = B(x, y). \end{aligned}$$

Similarly, since $Q_1 \in \mathcal{P}_L^+({}^m X; F)$ and $Q_2 \in \mathcal{P}_L^+({}^m Y; F)$, we have $Q_1 \circ \pi_1, Q_2 \circ \pi_2 \in \mathcal{P}_L^+({}^m X \times Y; F)$. Consequently,

$$B = Q_1 \circ \pi_1 + Q_2 \circ \pi_2 \in \mathcal{P}_L^+({}^m X \times Y; F).$$

A simple calculation shows that

$$P_1 + P_2 = B \circ A,$$

which implies that

$$P_1 + P_2 \in \mathcal{P}_L^+ \circ \mathcal{B}_R^+({}^m E; F).$$

Let $P_f \in \mathcal{P}({}^m E; F)$ be a finite rank operator. It can be expressed as a combination of operators of the form $\varphi^m b$ where $\varphi \in E^*$ and $b \in F$. Let $u = \varphi^m b$. Define $B : \mathbb{K} \rightarrow F$ by $B(\lambda) = \lambda^m b = (id_{\mathbb{K}}(\lambda))^m b$. Clearly, $B \in \mathcal{P}_L^+({}^m \mathbb{K}; F)$ and define $A : E \rightarrow \mathbb{K}$ by $A(x) = \varphi(x)$ which belongs to $\mathcal{B}_R^+(E; \mathbb{K})$. Then, we have

$$u(x) = B \circ A(x) \in \mathcal{P}_L^+ \circ \mathcal{B}_R^+({}^m E; F).$$

By the vector space structure of $\mathcal{P}_L^+ \circ \mathcal{B}_R^+({}^m E; F)$ it follows that $P_f \in \mathcal{P}_L^+ \circ \mathcal{B}_R^+({}^m E; F)$. Finally, we verify the positive ideal property. Let $P = Q_0 \circ u_0 \in \mathcal{P}_L^+ \circ \mathcal{B}_R^+({}^m E; F)$, $u \in \mathcal{L}^+(G; E)$ and $v \in \mathcal{L}^+(F; H)$. Then

$$v \circ P \circ u = (v \circ Q_0) \circ (u_0 \circ u).$$

Since $v \circ Q_0 \in \mathcal{P}_L^+({}^m X; H)$ and $u_0 \circ u \in \mathcal{B}_R^+(G; X)$, we obtain

$$v \circ P \circ u \in \mathcal{P}_L^+ \circ \mathcal{B}_R^+({}^m E; F).$$

□

Let $\mathcal{B}_R^+$ be a positive right Banach ideal and $\mathcal{P}_L^+$ a positive left Banach polynomial ideal. If E and F are Banach lattices and $P \in \mathcal{P}_L^+ \circ \mathcal{B}_R^+({}^m E; F)$, we define

$$\|P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} = \inf \left\{ \|Q\|_{\mathcal{P}_L^+} \|u\|_{\mathcal{B}_R^+}^m : P = Q \circ u \right\}. \quad (2.1)$$

Proposition 2.5. 1) For every $P \in \mathcal{P}_L^+ \circ \mathcal{B}_R^+({}^m E; F)$, $u \in \mathcal{L}^+(G; E)$ and $v \in \mathcal{L}^+(F; H)$, we have

$$\|v \circ P \circ u\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \leq \|v\| \|P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \|u\|^m.$$

2) Let $P^m : \mathbb{K} \rightarrow \mathbb{K}$ given by $P^m(\lambda) = \lambda^m$. We have $\|P^m\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} = 1$.

3) For every $P \in \mathcal{P}_L^+ \circ \mathcal{B}_R^+({}^m E; F)$, we have

$$\|P\| \leq \|P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+}. \quad (2.2)$$

Proof. 1) Let $P \in \mathcal{P}_L^+ \circ \mathcal{B}_R^+({}^m E; F)$, $u \in \mathcal{L}^+(G; E)$ and $v \in \mathcal{L}^+(F; H)$. Take a representation $P = Q_0 \circ u_0$. Then

$$\begin{aligned} \|v \circ P \circ u\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} &= \|(v \circ Q_0) \circ (u_0 \circ u)\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \\ &\leq \|v \circ Q_0\|_{\mathcal{P}_L^+} \|u_0 \circ u\|_{\mathcal{B}_R^+}^m \\ &\leq \|v\| \|Q_0\|_{\mathcal{P}_L^+} \|u_0\|_{\mathcal{B}_R^+}^m \|u\|^m. \end{aligned}$$

Taking the infimum over all representations of P , we obtain

$$\|v \circ P \circ u\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \leq \|v\| \|P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \|u\|^m.$$

2) Note that $P^m = (id_{\mathbb{K}})^m \circ id_{\mathbb{K}}$. Hence

$$\|P^m\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} = \|(id_{\mathbb{K}})^m \circ id_{\mathbb{K}}\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \leq \|(id_{\mathbb{K}})^m\|_{\mathcal{P}_L^+} \|id_{\mathbb{K}}\|_{\mathcal{B}_R^+}^m = 1.$$

On the other hand, let $Q_0 \circ u_0$ be a factorization of P^m with $u_0 : \mathbb{K} \rightarrow X$ and $Q_0 : X \rightarrow \mathbb{K}$. Then, there exists $x_0 \in X$ such that

$$u_0(\lambda) = \lambda x_0 \text{ and } Q_0(x_0) = 1.$$

Moreover,

$$\|u_0\| = \|x_0\| \text{ and } \|Q_0\| \geq \left\| Q_0 \left(\frac{x_0}{\|x_0\|} \right) \right\| = \frac{1}{\|x_0\|^m}$$

Hence,

$$\|Q_0\|_{\mathcal{P}_L^+} \|u_0\|_{\mathcal{B}_R^+}^m \geq \|Q_0\| \|u_0\|^m \geq 1$$

Taking the infimum over all possible factorizations of P^m , we conclude that

$$\|P^m\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \geq 1.$$

3) Let $\varphi \in F^*$ and $x \in E$. Consider $u_0 : \mathbb{K} \rightarrow E$ defined by $u_0(\lambda) = \lambda x$. We have $\|u_0\| = \|x\|$ and

$$\varphi \circ P \circ u_0(\lambda) = \lambda^m \langle \varphi, P(x) \rangle.$$

Thus $\varphi \circ P \circ u_0 = \langle \varphi, P(x) \rangle P^m$. We have

$$\begin{aligned} |\langle \varphi, P(x) \rangle| &= |\langle \varphi, P(x) \rangle| \|P^m\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} = \|\langle \varphi, P(x) \rangle P^m\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \\ &= \|\varphi \circ P \circ u_0\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \\ &\leq \|\varphi\| \|P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \|u_0\|^m = \|\varphi\| \|P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \|x\|^m. \end{aligned}$$

Then

$$\begin{aligned} \|P(x)\| &= \sup_{\varphi \in B_{F^*}} |\langle \varphi, P(x) \rangle| \leq \sup_{\varphi \in B_{F^*}} \|\varphi\| \|P\|_{\mathcal{B}_L^+ \circ \mathcal{B}_R^+} \|x\|^m \\ &\leq \|P\|_{\mathcal{B}_L^+ \circ \mathcal{B}_R^+} \|x\|^m. \end{aligned}$$

Consequently, $\|P\| \leq \|P\|_{\mathcal{B}_L^+ \circ \mathcal{B}_R^+}$. □

Lemma 2.6. *The quantity $\|\cdot\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+}$ defined in (2.1) can equivalently be expressed as*

$$\|P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} = \inf \left\{ \|Q\|_{\mathcal{P}_L^+} : P = Q \circ u \text{ and } \|u\|_{\mathcal{B}_R^+} = 1 \right\}.$$

Proof. First, it is clear that

$$\|P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \leq \inf \left\{ \|Q\|_{\mathcal{P}_L^+} : P = Q \circ u \text{ and } \|u\|_{\mathcal{B}_R^+} = 1 \right\}.$$

Consider a representation of P as $Q_0 \circ u_0$. We can rewrite it as $P = (\|u_0\|_{\mathcal{B}_R^+}^m Q_0) \circ (\frac{u_0}{\|u_0\|_{\mathcal{B}_R^+}})$. Hence

$$\left\| \|u_0\|_{\mathcal{B}_R^+}^m Q_0 \right\|_{\mathcal{P}_L^+} \geq \inf \left\{ \|Q\|_{\mathcal{P}_L^+} : P = Q \circ u \text{ and } \|u\|_{\mathcal{B}_R^+} = 1 \right\}.$$

This implies

$$\|u_0\|_{\mathcal{B}_R^+}^m \|Q_0\|_{\mathcal{P}_L^+} \geq \inf \left\{ \|Q\|_{\mathcal{P}_L^+} : P = Q \circ u \text{ and } \|u\|_{\mathcal{B}_R^+} = 1 \right\}$$

Taking the infimum over all factorizations P , we get

$$\|P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \geq \inf \left\{ \|Q\|_{\mathcal{P}_L^+} : P = Q \circ u \text{ and } \|u\|_{\mathcal{B}_R^+} = 1 \right\}.$$

□

The following result establishes that $\|\cdot\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+}$, as defined in (2.1), is a quasi-norm.

Theorem 2.7. *Let $\mathcal{B}_R^+$ be a positive right Banach ideal and $\mathcal{P}_L^+$ a positive left Banach polynomial ideal. Then*

$$\left(\mathcal{P}_L^+ \circ \mathcal{B}_R^+, \|\cdot\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \right),$$

is a positive quasi-Banach polynomial ideal.

Proof. We show that $\|\cdot\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+}$ defines a quasi-norm, the remaining properties follow from Proposition 2.5. Let $\lambda \in \mathbb{K}$ and $P \in \mathcal{P}_L^+ \circ \mathcal{B}_R^+(^m E; F)$. There exist a Banach space X and elements $u_0 \in \mathcal{B}_R^+(E; X)$, $Q_0 \in \mathcal{P}_L^+(^m X; F)$ such that $P = Q_0 \circ u_0$. Then

$$\|\lambda P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \leq \|\lambda Q_0\|_{\mathcal{P}_L^+} \|u_0\|_{\mathcal{B}_R^+}^m = |\lambda| \|Q_0\|_{\mathcal{P}_L^+} \|u_0\|_{\mathcal{B}_R^+}^m.$$

Taking the infimum over all factorizations of P , we get

$$\|\lambda P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \leq |\lambda| \|P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+}.$$

For the reverse inequality, assume $\lambda \neq 0$. If $Q_0 \circ u_0$ is a representation of λP , then $P = \frac{Q_0}{\lambda} \circ u_0$, giving

$$\|P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \leq \left\| \frac{Q_0}{\lambda} \right\|_{\mathcal{P}_L^+} \|u_0\|_{\mathcal{B}_R^+}^m \leq \frac{1}{|\lambda|} \|Q_0\|_{\mathcal{P}_L^+} \|u_0\|_{\mathcal{B}_R^+}^m.$$

Taking the infimum over all factorizations of λP , we obtain

$$|\lambda| \|P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \leq \|\lambda P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+}.$$

By (2.2) if $\|P\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} = 0$, then $P = 0$. Let $P_1, P_2 \in \mathcal{P}_L^+ \circ \mathcal{B}_R^+(^m E; F)$. Following a similar approach to the proof of Proposition 2.4, $P_1 + P_2 = B \circ A$. We can then establish the following inequalities

$$\begin{aligned} \|A\|_{\mathcal{B}_R^+} &\leq \|i_1 \circ u_1\|_{\mathcal{B}_R^+} + \|i_2 \circ u_2\|_{\mathcal{B}_R^+} \\ &\leq \|i_1\| \|u_1\|_{\mathcal{B}_R^+} + \|i_2\| \|u_2\|_{\mathcal{B}_R^+} = \|u_1\|_{\mathcal{B}_R^+} + \|u_2\|_{\mathcal{B}_R^+}. \end{aligned}$$

Similarly,

$$\begin{aligned} \|B\|_{\mathcal{P}_L^+} &\leq \|Q_1 \circ \pi_1\|_{\mathcal{P}_L^+} + \|Q_2 \circ \pi_2\|_{\mathcal{P}_L^+} \\ &\leq \|Q_1\|_{\mathcal{P}_L^+} \|\pi_1\|^m + \|Q_2\|_{\mathcal{P}_L^+} \|\pi_2\|^m = \|Q_1\|_{\mathcal{P}_L^+} + \|Q_2\|_{\mathcal{P}_L^+}. \end{aligned}$$

Now, for each $\varepsilon > 0$ we can choose u_1, u_2, Q_1, Q_2 such that

$$\|Q_j\|_{\mathcal{P}_L^+} \leq \|P_j\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} + \varepsilon \text{ and } \|u_j\|_{\mathcal{B}_R^+} = 1 \text{ for } j = 1, 2.$$

A simple calculation shows that

$$\begin{aligned} \|P_1 + P_2\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} &\leq \|B\|_{\mathcal{P}_L^+} \|A\|_{\mathcal{B}_R^+} \\ &\leq \left(\|Q_1\|_{\mathcal{P}_L^+} + \|Q_2\|_{\mathcal{P}_L^+} \right) \left(\|u_1\|_{\mathcal{B}_R^+} + \|u_2\|_{\mathcal{B}_R^+} \right) \\ &\leq 2 \left(\|P_1\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} + \|P_2\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} + 2\varepsilon \right). \end{aligned}$$

Since ε is arbitrary, it follows that

$$\|P_1 + P_2\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \leq 2 \left(\|P_1\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} + \|P_2\|_{\mathcal{P}_L^+ \circ \mathcal{B}_R^+} \right).$$

□

The composition method. Let $\mathcal{B}_L^+$ be a positive left ideal. Let X be a Banach space, and E a Banach lattice. A polynomial $P \in \mathcal{P}(^m X; E)$ belongs to $\mathcal{B}_L^+ \circ \mathcal{P}$ if there exist a Banach space Y , a polynomial $Q \in \mathcal{P}(^m X; Y)$, and an operator $u \in \mathcal{B}_L^+(Y; E)$ such that

$$\begin{array}{ccc} X & \xrightarrow{P} & E \\ Q \downarrow & \nearrow u & \\ Y & & \end{array}$$

that is, $P = u \circ Q$. In this case, we write $P \in \mathcal{B}_L^+ \circ \mathcal{P}(^m X; E)$.

Remark 2.8. By an argument analogous to that used in [10, Proposition 3.3], the class $\mathcal{B}_L^+ \circ \mathcal{P}$ forms a positive left polynomial ideal.

If $\mathcal{B}_L^+$ is a positive left Banach ideal, we define

$$\|P\|_{\mathcal{B}_L^+ \circ \mathcal{P}} = \inf \{ \|u\|_{\mathcal{B}_L^+} \|Q\| \}, \quad (2.3)$$

where the infimum is taken over all possible factorizations of P as described above. Similarly to [10, Proposition 3.7], if $\mathcal{B}_L^+$ is a positive left Banach ideal, the pair $(\mathcal{B}_L^+ \circ \mathcal{P}, \|\cdot\|_{\mathcal{B}_L^+ \circ \mathcal{P}})$ forms a positive left Banach polynomial ideal. We have the following result.

Proposition 2.9. *Let $\mathcal{B}_L^+$ be a positive left ideal. Let X be a Banach space, and E a Banach lattice. For $P \in \mathcal{P}(^m X; E)$, the following statements are equivalent:*

- 1) *The polynomial P belongs to $\mathcal{B}_L^+ \circ \mathcal{P}(^m X; E)$.*
- 2) *The linearization P_L belongs to $\mathcal{B}_L^+(\widehat{\otimes}_{\pi,s}^m X; E)$.*

In this case, we have

$$\|P\|_{\mathcal{B}_L^+ \circ \mathcal{P}} = \|P_L\|_{\mathcal{B}_L^+}.$$

Consequently, we obtain the following isometric identification

$$\mathcal{B}_L^+ \circ \mathcal{P}(^m X; E) = \mathcal{B}_L^+(\widehat{\otimes}_{\pi,s}^m X; E).$$

Proof. 1) $\implies$ 2) : Let $P \in \mathcal{B}_L^+ \circ \mathcal{P}(^m X; E)$. Then there exist $Q \in \mathcal{P}(^m X; Y)$ and $u \in \mathcal{B}_L^+(Y; E)$ such that $P = u \circ Q$. Since $P_L = u \circ Q_L$, the positive ideal property implies $P_L \in \mathcal{B}_L^+(\widehat{\otimes}_{\pi,s}^m X; E)$. Moreover, by (1.1), we have $P = P_L \circ \delta_m$ with $\delta_m \in \mathcal{P}(^m X; \widehat{\otimes}_{\pi,s}^m X)$ and $\|\delta_m\| = 1$. By (2.3)

$$\begin{aligned} \|P_L\|_{\mathcal{B}_L^+} &= \|u \circ Q_L\|_{\mathcal{B}_L^+} \\ &\leq \|u\|_{\mathcal{B}_L^+} \|Q_L\| = \|u\|_{\mathcal{B}_L^+} \|Q\|. \end{aligned}$$

Taking the infimum over all such representations of P , it follows that

$$\|P_L\|_{\mathcal{B}_L^+} \leq \|P\|_{\mathcal{B}_L^+ \circ \mathcal{P}}.$$

2) $\implies$ 1) : Suppose that $P_L \in \mathcal{B}_L^+(\widehat{\otimes}_{\pi,s}^m X; E)$. Since $P = P_L \circ \delta_m$, it follows that

$$P \in \mathcal{B}_L^+ \circ \mathcal{P}(^m X; E).$$

Moreover, because $\|\delta_m\| = 1$, we obtain

$$\|P_L\|_{\mathcal{B}_L^+} = \|P_L\|_{\mathcal{B}_L^+} \|\delta_m\| \geq \inf \{ \|u\|_{\mathcal{B}_L^+} \|Q\| : P = u \circ Q \} = \|P\|_{\mathcal{B}_L^+ \circ \mathcal{P}}.$$

For the surjectivity, let $R \in \mathcal{B}_L^+(\widehat{\otimes}_{\pi,s}^m X; E)$. Define

$$\begin{aligned} P_R(x) &= R\left(x \otimes \cdots \otimes x\right) \\ &= R \circ \Delta_m\left(x, \dots, x\right), \end{aligned}$$

where $\Delta_m : X \times \cdots \times X \rightarrow \otimes^m X$ is the canonical (symmetric) multilinear defined by

$$\Delta_m(x_1, \dots, x_m) = x_1 \otimes \cdots \otimes x_m.$$

We have $P_R \in \mathcal{P}(^m X; E)$ and

$$\widehat{P}_R = R \circ \Delta_m.$$

We verify that $(P_R)_L = R$. Indeed, for any $\sum_{i=1}^n \lambda_i x_i \otimes \cdots \otimes x_i \in \otimes_{\pi,s}^m X$,

$$\begin{aligned} (P_R)_L\left(\sum_{i=1}^n \lambda_i x_i \otimes \cdots \otimes x_i\right) &= \sum_{i=1}^n \lambda_i (P_R)_L\left(x_i \otimes \cdots \otimes x_i\right) \\ &= \sum_{i=1}^n \lambda_i P_R(x_i) = \sum_{i=1}^n \lambda_i R\left(x_i \otimes \cdots \otimes x_i\right) \\ &= R\left(\sum_{i=1}^n \lambda_i x_i \otimes \cdots \otimes x_i\right). \end{aligned}$$

Thus $(P_R)_L$ and R coincide on $\otimes_{\pi,s}^m X$, and by density they coincide on the whole space $\widehat{\otimes}_{\pi,s}^m X$. $\square$

The factorization method. Let $\mathcal{B}_R^+$ be a positive right Banach ideal. We define the class $\mathcal{P}(\mathcal{B}_R^+)$ as follows: Let Y be a Banach space, and E a Banach lattice. A polynomial P belongs to $\mathcal{P}(\mathcal{B}_R^+)(^m E; Y)$ if there exist Banach space X , an operator $u \in \mathcal{B}_R^+(E; X)$, and a polynomial $Q \in \mathcal{P}(^m X; Y)$ such that

$$\begin{array}{ccc} E & \xrightarrow{P} & Y \\ u \downarrow & \nearrow Q & \\ X & & \end{array}$$

that is, $P = Q \circ u$. In this case, for every $P \in \mathcal{P}(\mathcal{B}_R^+)$, we define

$$\|P\|_{\mathcal{P}(\mathcal{B}_R^+)} = \inf\{\|Q\| \|u\|_{\mathcal{B}_R^+}^m\},$$

where the infimum is taken over all possible factorizations of P as described above.

Proposition 2.10. *Let $\mathcal{B}_R^+$ be a positive right Banach ideal. Then, the pair $(\mathcal{P}(\mathcal{B}_R^+), \|\cdot\|_{\mathcal{P}(\mathcal{B}_R^+)})$ forms a positive right quasi-Banach polynomial ideal.*

Proof. We first verify the positive ideal property. Let $P \in \mathcal{P}(\mathcal{B}_R^+)(^m E; Y)$, $u \in \mathcal{L}^+(G; E)$ and $v \in \mathcal{L}(Y; Z)$. Suppose $P = Q_0 \circ u_0$ is a factorization of P . Then

$$v \circ P \circ u = v \circ (Q_0 \circ u_0) \circ u$$

Since $(u_0 \circ u) \in \mathcal{B}_R^+(G; X)$ and $(v \circ Q_0) \in \mathcal{P}(^m X; Z)$, we get

$$(v \circ Q_0) \circ (u_0 \circ u) \in \mathcal{P}(\mathcal{B}_R^+)(^m G; Z).$$

The remaining steps proceed exactly as in the proof of Proposition 2.4. For the quasi-norm, the argument follows Theorem 2.7, which shows that $\|\cdot\|_{\mathcal{P}(\mathcal{B}_R^+)}$ indeed defines a quasi-norm. $\square$

3. EXAMPLES OF POSITIVE POLYNOMIAL IDEALS

In this section, we present several classes of polynomials that provide concrete examples of positive polynomial ideals. These classes illustrate the notions and properties discussed in the previous section and include Cohen positive strongly p -summing polynomials, Cohen positive p -nuclear polynomials, positive p -dominated polynomials. We also introduce a new class, referred to as positive $(q; r)$ -dominated polynomials. For this class, we establish both the Pietsch Domination Theorem and the Kwapien Factorization Theorem, and we show that it forms an important example of a positive polynomial ideal.

3.1. Cohen positive strongly p -summing polynomials. Hamdi et al. [17], introduced the notion of Cohen positive strongly p -summing polynomials. Let X be a Banach space and E a Banach lattice. A polynomial $P \in \mathcal{P}(^m X; E)$ is said to be Cohen positive strongly p -summing ($1 < p \leq \infty$) if there exists a constant $C > 0$ such that for any $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset E^{*+}$, the following inequality holds:

$$\sum_{i=1}^n |\langle P(x_i), y_i^* \rangle| \leq C \left(\sum_{i=1}^n \|x_i\|^{mp} \right)^{\frac{1}{p}} \|(y_i^*)_{i=1}^n\|_{p^*, w}. \quad (3.1)$$

The space consisting of all such mappings is denoted by $\mathcal{P}_{Coh, p}^+(^m X; E)$. In this case, we define

$$d_p^{m+}(P) = \inf\{C > 0 : C \text{ satisfies (3.1)}\}.$$

Proposition 3.1. *The class $\mathcal{P}_{Coh, p}^+$ is a positive left Banach polynomial ideal, obtained by the composition method from the positive left ideal $\mathcal{D}_p^+$. More precisely, for every Banach space X and Banach lattice E*

$$\mathcal{P}_{Coh, p}^+(^m X; E) = \mathcal{D}_p^+ \circ \mathcal{P}(^m X; E).$$

Proof. Directly by Proposition 2.9 and [17, Proposition 6]. $\square$

3.2. Positive Cohen p -nuclear polynomials. Achour and Alouani [2] introduced the notion of Cohen p -nuclear multilinear operators as a natural extension of the linear concept originally proposed by Cohen [12]. The polynomial counterpart was later introduced and studied in [6]. Hammou et al. [18] subsequently developed the positive version of this notion.

Definition 3.2. Let $m \in \mathbb{N}^*$. Let $1 \leq p \leq \infty$. Let E and F be Banach lattices. A polynomial $P \in \mathcal{P}(^m E; F)$ is said to be positive Cohen p -nuclear if there exists a constant $C > 0$ such that for any $(x_i)_{i=1}^n \subset E^+$ and $(y_i^*)_{i=1}^n \subset F^{*+}$, the following inequality holds:

$$\sum_{i=1}^n |\langle P(x_i), y_i^* \rangle| \leq C \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^n \langle x^*, x_i \rangle^{mp} \right)^{\frac{1}{p}} \|(y_i^*)_{i=1}^n\|_{p^*, w}. \quad (3.2)$$

The space consisting of all such mappings is denoted by $\mathcal{P}_{N-p}^{c+}(^m E; F)$. In this case, we define

$$n_p^{m+}(P) = \inf\{C > 0 : C \text{ satisfies (3.2)}\}.$$

It is easy to verify that every Cohen p -nuclear is positive Cohen p -nuclear, i.e.,

$$\mathcal{P}_{p,N}^c({}^m E; F) \subset \mathcal{P}_{N-p}^{c+}({}^m E; F),$$

where $\mathcal{P}_{p,N}^c({}^m E; F)$ is the Banach space of all Cohen p -nuclear polynomials.

Proposition 3.3. *The class $\mathcal{P}_{N-p}^{c+}$ is a positive polynomial ideal defined by, $\mathcal{P}_{N-p}^{c+} = \mathcal{P}_{Coh,p}^+ \circ \Pi_{mp}^+$. That is, for every pair of Banach lattices E and F*

$$\mathcal{P}_{N-p}^{c+}({}^m E; F) = \mathcal{P}_{Coh,p}^+ \circ \Pi_{mp}^+({}^m E; F).$$

Proof. Directly by [18, Theorem 9]. $\square$

3.3. Positive p -dominated polynomials. The concept of positive p -dominated polynomials has been introduced by Hamdi et al. [17]. Let E be a Banach lattice and Y a Banach space. A polynomial $P \in \mathcal{P}({}^m E; Y)$ is said to be positive p -dominated ($1 \leq p < \infty$) if there exists a constant $C > 0$ such that for any $(x_i)_{i=1}^n \subset E^+$, the following inequality holds:

$$\left(\sum_{i=1}^n \|P(x_i)\|^{\frac{p}{m}} \right)^{\frac{m}{p}} \leq C \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^n |x^*(x_i)|^p \right)^{\frac{m}{p}}. \quad (3.3)$$

The space consisting of all such mappings is denoted by $\mathcal{P}_{d,p}^+({}^m E; Y)$. In this case, we define

$$\delta_p^+(P) = \inf\{C > 0 : C \text{ satisfies (3.3)}\}.$$

We note that if $p \geq m$, $\delta_p^+(\cdot)$ is a norm, but for $p < m$, it is only a quasi-norm.

Theorem 3.4. [17, Theorem 6] *Let $1 \leq p < \infty$; an m -homogeneous polynomial $P : E \rightarrow Y$ is positive p -dominated if there are $C > 0$ and a probability measure μ on $B_{E^*}^+$ such that for every $x \in E^+$*

$$\|P(x)\| \leq C \left(\int_{B_{E^*}^+} \langle x^*, x \rangle^p d\mu \right)^{\frac{m}{p}}.$$

Moreover, the smallest C is $\delta_p^+(P)$.

The authors in [17] did not provide a factorization result for this class. In what follows, we present a version of Kwapień's theorem concerning the class of positive p -dominated polynomials. This allows us to establish that this class can be interpreted through the factorization method from the positive class Π_p^+ .

Theorem 3.5. *Let $m \in \mathbb{N}^*$ and $1 \leq p < \infty$. An m -homogeneous polynomial $P : E \rightarrow Y$ is positive p -dominated if and only if, there exist a Banach space X , a positive p -summing operator $u : E \rightarrow X$, and a polynomial $Q \in \mathcal{P}({}^m X; Y)$ such that*

$$P = Q \circ u.$$

Moreover,

$$\delta_p^+(P) = \inf \{ \|Q\| \pi_p^+(u)^m : P = Q \circ u \}.$$

Proof. Let $P : E \rightarrow Y$ be an m -homogeneous polynomial such that $P = Q \circ u$ where $u \in \Pi_p^+(E; X)$ and $Q \in \mathcal{P}({}^m X; Y)$ with X is a Banach space. Let $x \in E^+$. We have

$$\begin{aligned} \|P(x)\| &= \|Q \circ u(x)\| \\ &\leq \|Q\| \|u(x)\|^m. \end{aligned}$$

Since u is positive p -summing, by (1.3) we obtain

$$\|P(x)\| \leq \|Q\| \pi_p^+(u)^m \left(\int_{B_{E^*}^+} \langle x^*, x \rangle^p d\mu \right)^{\frac{m}{p}}.$$

Then, P is positive p -dominated and

$$\delta_p^+(P) \leq \|Q\| \pi_p^+(u)^m.$$

Taking the infimum over all representation of P , we get

$$\delta_p^+(P) \leq \inf \{ \|Q\| \pi_p^+(u)^m : P = Q \circ u \}.$$

To prove the first implication. Let $P \in \mathcal{P}_{d,p}^+({}^m E; Y)$. By Theorem 3.4, there is a probability measure μ on $B_{E^*}^+$ such that for all $x \in E^+$ we have

$$\|P(x)\| \leq \delta_p^+(P) \left(\int_{B_{E^*}^+} \langle x^*, x \rangle^p d\mu \right)^{\frac{m}{p}}.$$

We now consider the operator $u_0 : E \rightarrow L_p(B_{E^*}^+, \mu)$ which is given by $u_0(x)(x^*) = x^*(x)$. Notice that for all $x \in E^+$, we have

$$\begin{aligned} \|u_0(x)\| &= \left(\int_{B_{E^*}^+} \langle x, x^* \rangle^p d\mu \right)^{\frac{1}{p}} \\ &\leq \|x\|. \end{aligned}$$

Let $X = \overline{u_0(E)}^{L_p(B_{E^*}^+, \mu)}$ be the closure in $L_p(B_{E^*}^+, \mu)$ of the range of u_0 , and let $u : E \rightarrow X$ be the induced operator. Note that u is positive p -summing with $\pi_p^+(u) = 1$. Let $Q_0 : u_0(E) \rightarrow Y$ be the polynomial operator defined on $u_0(E)$ by

$$Q_0(u_0(x)) = P(x).$$

This is well defined since

$$\|Q_0(u_0(x))\| \leq \delta_p^+(P) \|u_0(x)\|^m.$$

Hence Q_0 is continuous on $u_0(E)$ and extends uniquely to a continuous polynomial $Q : X \rightarrow Y$. We thus have $P = Q \circ u$, with $u \in \Pi_p^+(E; X)$ and $Q \in \mathcal{P}({}^m X; Y)$, and

$$\begin{aligned} \|Q\| \pi_p^+(u)^m &= \|Q\| \\ &\leq \delta_p^+(P). \end{aligned}$$

That is,

$$\inf \{ \|Q\| \pi_p^+(u)^m : P = Q \circ u \} \leq \delta_p^+(P).$$

This completes the proof. $\square$

Corollary 3.6. *Let E be a Banach lattice and Y a Banach space. The class $\mathcal{P}_{d,p}^+$ is a positive right polynomial ideal obtained through the factorization method from the positive right ideal Π_p^+ . Specifically, for every Banach lattice E and Banach space Y ,*

$$\mathcal{P}_{d,p}^+({}^m E; Y) = \mathcal{P}(\Pi_p^+)({}^m E; Y).$$

3.4. Positive $(q; r)$ -dominated polynomials. The concept of absolutely (p, q, r) -summing operators was first introduced by Pietsch [22]. It was later extended to the multilinear setting by Achour [1], and to the polynomial setting by Achour and Bernardino [4]. A positive counterpart of this notion was subsequently introduced and investigated in [15]. In this section, we develop and analyze the corresponding positive polynomial version, which serves as a natural example of a positive polynomial ideal.

Definition 3.7. Let $m \in \mathbb{N}^*$. Let $1 \leq r, p, q \leq \infty$ with $\frac{1}{p} = \frac{m}{q} + \frac{1}{r}$. Let E and F be Banach lattices. A polynomial $P \in \mathcal{P}(^m E; F)$ is called positive $(q; r)$ -dominated if there exists a constant $C > 0$ such that for any $(x_i)_{i=1}^n \subset E^+$ and $(y_i^*)_{i=1}^n \subset F^{*+}$, the following inequality holds:

$$\|(\langle P(x_i), y_i^* \rangle)_{i=1}^n\|_p \leq C \|(x_i)_{i=1}^n\|_{q,w}^m \|(y_i^*)_{i=1}^n\|_{r,w}. \quad (3.4)$$

The space of all such polynomials is denoted by $\mathcal{P}_{d,(q;r)}^+(^m E; F)$. Its norm is given by

$$d_{d,(q;r)}^{m+}(P) = \inf\{C > 0 : C \text{ satisfies (3.4)}\}.$$

An equivalent formulation of (3.4) is

$$\|(\langle P(x_i), y_i^* \rangle)_{i=1}^n\|_p \leq C \|(|x_i|)_{i=1}^n\|_{q,w}^m \|(|y_i^*|)_{i=1}^n\|_{r,w}$$

for every $(x_i)_{i=1}^n \subset E$ and $(y_i^*)_{i=1}^n \subset F^*$. It is straightforward to check that every $(q; r)$ -dominated polynomial is positive $(q; r)$ -dominated. Hence, by [4, Proposition 3.10]

$$\mathcal{P}_f(^m E; F) \subset \mathcal{P}_{d,(q;r)}^+(^m E; F). \quad (3.5)$$

Proposition 3.8. Let $P \in \mathcal{P}_{d,(q;r)}^+(^m E; F)$, $u \in \mathcal{L}^+(G; E)$ and $v \in \mathcal{L}^+(F; H)$. Then $v \circ P \circ u \in \mathcal{P}_{d,(q;r)}^+(^m G; H)$ and we have

$$d_{d,(q;r)}^{m+}(v \circ P \circ u) \leq \|v\| d_{d,(q;r)}^{m+}(P) \|u\|^m.$$

Proof. Let $(x_i)_{i=1}^n \subset E^+$ and $(y_i^*)_{i=1}^n \subset F^{*+}$. Then

$$\begin{aligned} \left(\sum_{i=1}^n |\langle v \circ P \circ u(x_i), y_i^* \rangle|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n |\langle P \circ u(x_i), v^* \circ y_i^* \rangle|^p\right)^{\frac{1}{p}} \\ &\leq d_{d,(q;r)}^{m+}(P) \|(u(x_i))_{i=1}^n\|_{q,w}^m \|(v^* \circ y_i^*)_{i=1}^n\|_{r,w} \\ &\leq d_{d,(q;r)}^{m+}(P) \|u\|^m \|(x_i)_{i=1}^n\|_{q,w}^m \|v^*\| \|(y_i^*)_{i=1}^n\|_{r,w} \\ &\leq \|v\| d_{d,(q;r)}^{m+}(P) \|u\|^m \|(x_i)_{i=1}^n\|_{q,w}^m \|(y_i^*)_{i=1}^n\|_{r,w} \end{aligned}$$

thus $v \circ P \circ u$ is positive $(q; r)$ -dominated and

$$d_{d,(q;r)}^{m+}(v \circ P \circ u) \leq \|v\| d_{d,(q;r)}^{m+}(P) \|u\|^m.$$

□

The pair $(\mathcal{P}_{d,(q;r)}^+, d_{d,(q;r)}^+)$ defines a positive Banach polynomial ideal. The proof follows directly from the previous proposition and the inclusion (3.5), while the remaining details are straightforward. We now turn to the characterization of positive $(q; r)$ -dominated polynomials through a Pietsch Domination Theorem. To this end, we apply the General Pietsch Domination Theorem established by Pellegrino et al in [21, Theorem 4.6].

Theorem 3.9 (Pietsch domination theorem). *Let $m \in \mathbb{N}^*$. Let $1 \leq r, p, q \leq \infty$ with $\frac{1}{p} = \frac{m}{q} + \frac{1}{r}$. Let E and F be Banach lattices. The following statements are equivalent:*

- 1) *The polynomial $P \in \mathcal{P}(^m E; F)$ is positive $(q; r)$ -dominated.*
- 2) *There exist a constant $C > 0$, Borel probability measures μ on $B_{E^*}^+$ and η on $B_{F^{**}}^+$ such that*

$$|\langle P(x), y^* \rangle| \leq C \left(\int_{B_{E^*}^+} \langle |x|, x^* \rangle^q d\mu \right)^{\frac{m}{q}} \left(\int_{B_{F^{**}}^+} \langle |y^*|, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}}, \quad (3.6)$$

for all $(x, y^*) \in E \times F^*$. Therefore, we have

$$d_{d,(q;r)}^{m+}(P) = \inf\{C > 0 : C \text{ satisfies (3.6)}\}.$$

- 3) *There exist a constant $C > 0$, Borel probability measures μ on $B_{E^*}^+$ and η on $B_{F^{**}}^+$ such that*

$$|\langle P(x), y^* \rangle| \leq C \left(\int_{B_{E^*}^+} \langle x, x^* \rangle^q d\mu \right)^{\frac{m}{q}} \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}}, \quad (3.7)$$

for all $(x, y^*) \in E^+ \times F^{*+}$. Therefore, we have

$$d_{d,(q;r)}^{m+}(P) = \inf\{C > 0 : C \text{ satisfies (3.7)}\}.$$

Proof. 1) $\Leftrightarrow$ 2) : We will choose the parameters as specified in [21, Theorem 4.6]

$$\begin{cases} S : \mathcal{P}(^m E; F) \times (E \times F^*) \times \mathbb{K} \times \mathbb{K} \rightarrow \mathbb{R}^+ : \\ S(P, (x, y^*), \lambda_1, \lambda_2) = |\lambda_2| |\langle P(x), y^* \rangle| \\ R_1 : B_{E^*}^+ \times (E \times F^*) \times \mathbb{K} \rightarrow \mathbb{R}^+ : R_1(x^*, (x, y^*), \lambda_1) = \langle |x|, x^* \rangle^m \\ R_2 : B_{F^{**}}^+ \times (E \times F^*) \times \mathbb{K} \rightarrow \mathbb{R}^+ : R_2(y^{**}, (x, y^*), \lambda_2) = |\lambda_2| \langle |y^*|, y^{**} \rangle. \end{cases}$$

These maps satisfy conditions (1) and (2) from [21, Theorem 4.6], allowing us to conclude that $P : E \rightarrow F$ is positive $(q; r)$ -dominated if, and only if,

$$\begin{aligned} & \left(\sum_{i=1}^n S(P, (x_i, y_i^*), \lambda_{i,1}, \lambda_{i,2})^p \right)^{\frac{1}{p}} \\ & \leq C \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^n R_1(x^*, (x_i, y_i^*), \lambda_{i,1})^{\frac{q}{m}} \right)^{\frac{m}{q}} \sup_{y^{**} \in B_{F^{**}}^+} \left(\sum_{i=1}^n R_2(y^{**}, (x_i, y_i^*), \lambda_{i,2})^r \right)^{\frac{1}{r}}, \end{aligned}$$

i.e., P is R_1, R_2 - S -abstract $(\frac{q}{m}; r)$ -summing. As outlined in [21, Theorem 4.6], this implies that P is R_1, R_2 - S -abstract $(\frac{q}{m}; r)$ -summing if, and only if, there exist a positive constant C , probability measures μ on $B_{E^*}^+$ and η on $B_{F^{**}}^+$, such that

$$\begin{aligned} & S(P, (x, y^*), \lambda_1, \lambda_2) \\ & \leq C \left(\int_{B_{E^*}^+} R_1(x^*, (x, y^*), \lambda_1)^{\frac{q}{m}} d\mu \right)^{\frac{m}{q}} \left(\int_{B_{F^{**}}^+} R_2(y^{**}, (x, y^*), \lambda_2)^r d\eta \right)^{\frac{1}{r}}. \end{aligned}$$

Consequently

$$|\langle P(x), y^* \rangle| \leq C \left(\int_{B_{E^*}^+} \langle |x|, x^* \rangle^q d\mu \right)^{\frac{m}{q}} \left(\int_{B_{F^{**}}^+} \langle |y^*|, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}}.$$

The implications 2) $\Rightarrow$ 3) and 3) $\Rightarrow$ 1) are immediate. $\square$

As an immediate consequence of Theorem 3.9, we can show that if $q_1 \leq q_2$ and $r \leq s$ then

$$\mathcal{P}_{d,(q_1;r)}^+(^m E; F) \subset \mathcal{P}_{d,(q_2;s)}^+(^m E; F).$$

The following result shows that the class of positive $(q; r)$ -dominated polynomials can be represented as the composition of the class of Cohen positive strongly r^* -summing polynomials $\mathcal{P}_{Coh, r^*}^+$ with the class of positive p -summing operators Π_p^+ . This provides a positive analogue of the Kwapien factorization.

Theorem 3.10. *Let $m \in \mathbb{N}^*$. Let $1 \leq r, p, q \leq \infty$ with $\frac{1}{p} = \frac{m}{q} + \frac{1}{r}$. Then, $P \in \mathcal{P}^m(E; F)$ is positive $(q; r)$ -dominated if and only if there exist Banach space X , a Cohen positive strongly r^* -summing polynomial $Q : X \rightarrow F$ and a positive q -summing operator $u \in \Pi_q^+(E; X)$ so that $T = Q \circ u$, i.e.,*

$$\mathcal{P}_{d, (q; r)}^+({}^m E; F) = \mathcal{P}_{Coh, r^*}^+ \circ \Pi_q^+({}^m E; F).$$

Moreover,

$$d_{d, (q; r)}^{m+}(P) = \inf \{ d_{r^*}^{m+}(Q) \pi_q^+(u)^m : P = Q \circ u \}.$$

Proof. First we prove the converse. Suppose that $P = Q \circ u$ where u is positive q -summing and Q is Cohen positive strongly r^* -summing polynomial. By [7, Theorem 2.5], there exists η on $B_{F^{**}}^+$ such that, for all $x \in E^+$ and $y^* \in F^{*+}$, we have

$$\begin{aligned} |\langle P(x), y^* \rangle| &= |\langle Q(u(x)), y^* \rangle| \\ &\leq d_{r^*}^{m+}(Q) \|u(x)\|^m \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}}. \end{aligned}$$

Since u is positive q -summing then, by (1.3) there is a probability measure μ on $B_{E^*}^+$ such that

$$\|u(x)\| \leq \pi_q^+(u) \left(\int_{B_{E^*}^+} \langle x, x^* \rangle^q d\mu \right)^{\frac{1}{q}}.$$

Consequently,

$$|\langle P(x), y^* \rangle| \leq d_{r^*}^{m+}(Q) \pi_q^+(u)^m \left(\int_{B_{E^*}^+} \langle x, x^* \rangle^q d\mu \right)^{\frac{m}{q}} \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}}.$$

Thus, P is positive $(q; r)$ -dominated by Theorem 3.9 and

$$d_{d, (q; r)}^{m+}(P) \leq d_{r^*}^{m+}(Q) \pi_q^+(u)^m.$$

Taking the infimum over all representations P , we get

$$d_{d, (q; r)}^{m+}(P) \leq \inf \{ d_{r^*}^{m+}(Q) \pi_q^+(u)^m : P = Q \circ u \}.$$

We now prove the direct implication. Let $P \in \mathcal{P}_{d, (q; r)}^+({}^m E; F)$. By Theorem 3.9, there are probability measures μ on $B_{E^*}^+$ and η on $B_{F^{**}}^+$ such that for all $x \in E^+$ and $y^* \in F^{*+}$ we have

$$|\langle P(x), y^* \rangle| \leq d_{d, (q; r)}^{m+}(P) \left(\int_{B_{E^*}^+} \langle x, x^* \rangle^q d\mu \right)^{\frac{m}{q}} \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}}.$$

Define the operator $u_0 : E \rightarrow L_q(B_{E^*}^+, \mu)$ by $u_0(x)(x^*) = x^*(x)$. For every $x \in E^+$, we have

$$\|u_0(x)\| = \left(\int_{B_{E^*}^+} \langle x, x^* \rangle^q d\mu \right)^{\frac{1}{q}} \leq \|x\|.$$

Let $X = \overline{u_0(E)}^{L_q(B_{E^*}^+, \mu)}$, and denote by $u : E \rightarrow X$ the induced operator. Then, u is positive q -summing with $\pi_q^+(u) = 1$. Now define the polynomial operator Q_0 on $u_0(E)$ by

$$Q_0(u_0(x)) = P(x).$$

This definition is consistent because

$$|\langle Q_0(u_0(x)), y^* \rangle| \leq d_{d,(q;r)}^{m+}(P) \|u_0(x)\|^m \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^r d\eta \right)^{\frac{1}{r}}.$$

Hence Q_0 is continuous on $u_0(E)$, and extends uniquely to a bounded polynomial Q on X . Moreover, Q is Cohen positive strongly r^* -summing polynomial and

$$d_{r^*}^{m+}(Q) \leq d_{d,(q;r)}^{m+}(P).$$

Finally, we obtain $P = Q \circ u$ where $u \in \Pi_q^+(E; X)$, $Q \in \mathcal{P}_{Coh, r^*}^+({}^m X; F)$ and

$$\begin{aligned} \inf \{ d_{r^*}^{m+}(Q) \pi_q^+(u)^m : P = Q \circ u \} &\leq d_{r^*}^{m+}(Q) \pi_q^+(u)^m \\ &\leq d_{d,(q;r)}^{m+}(P). \end{aligned}$$

Which completes the proof. $\square$

Every positive $(q; r)$ -dominated polynomial can be factored through a Cohen positive strongly r^* -summing polynomial and positive q -summing linear operator. Consequently, the class $\mathcal{P}_{d,(q;r)}^+$ forms a positive polynomial ideal of type $\mathcal{P}_L^+ \circ \mathcal{B}_R^+$ where

$$\mathcal{P}_L^+ = \mathcal{P}_{Coh, r^*}^+ \text{ and } \mathcal{B}_R^+ = \Pi_q^+.$$

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Declarations

Conflict of interest. The authors declare that they have no conflicts of interest.

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^{1,2} LABORATORY OF FUNCTIONAL ANALYSIS AND GEOMETRY OF SPACES, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF MOHAMED BOUDIAF-M'SILA, PO BOX 166, ICHEBILIA, 28000, M'SILA, ALGERIA.

Email address: adel.bounabab@univ-msila.dz; khalil.saadi@univ-msila.dz