

ON COMMON RANDOM FIXED POINT THEOREMS UNDER CONTRACTIVE TYPE CONDITION IN CONE RANDOM METRIC SPACES

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ABSTRACT. The aim of this paper is to establish some common random fixed point theorems under contractive type condition in the framework of cone random metric spaces and also to obtain some classical results as corollaries. Our results extend and generalize many known results from the current existing literature.

1. INTRODUCTION

Random nonlinear analysis is an important branch of probabilistic functional analysis that deals with the solution of classes of random operator equations and related problems. Random fixed point theorems are stochastic generalization of classical or deterministic fixed point theorems and are required for the theory of random equations, random matrices, random partial differential equations and various classes of random operators arising in physical systems (see [15, 28]). The study of random fixed point theory was initiated by the Prague school of Probabilistic in the 1950s [8, 9, 26]. Common random fixed point theorems are stochastic generalization of classical common fixed point theorems. The machinery of random fixed point theory provides a convenient way of modeling many problems arising from economic theory (see [19]) and references mentioned therein. Random methods have revolutionized the financial markets. The survey article by Bharucha-Reid [7] attracted the attention of several mathematicians and gave wings to the theory. Itoh [14] extended Spacek's and Hans's theorem to multivalued contraction mappings. Now this theory has become the full fledged research area and various ideas associated with random fixed point theory are used to obtain the solution of nonlinear random system (see [4, 5, 6, 10, 24]). Papageorgiou [17, 18], Beg [2, 3] studied common random fixed points and random coincidence points of a pair of compatible random operators and proved fixed point theorems for contractive random operators in Polish spaces.

In 2007, Huang and Zhang [11] introduced the concept of cone metric spaces and establish some fixed point theorems for contractive mappings in normal cone

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metric spaces. Subsequently, several other authors [1, 12, 23, 25] studied the existence of fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric space.

In 2008, Rezapour and Hambarani [23] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space. Recently, Mehta et al. [16] introduced the concept of cone random metric space and proved an existence of random fixed point under weak contraction condition in the setting of cone random metric spaces.

The aim of this paper is to establish some common random fixed point theorems under contractive type condition in the framework of cone random metric spaces.

2. PRELIMINARIES

Definition 2.1. (See [16]) Let (E, τ) be a topological vector space. A subset P of E is called a cone whenever the following conditions hold:

(c₁) P is closed, nonempty and $P \neq \{0\}$;

(c₂) $a, b \in R$, $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;

(c₃) If $x \in P$ and $-x \in P$ implies $x = 0$.

For a given cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in P^0$, where P^0 stands for the interior of P .

Definition 2.2. (See [11, 27]) Let X be a nonempty set. Suppose that the mapping $d: X \times X \rightarrow E$ satisfies:

(d₁) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d₂) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d₃) $d(x, y) \leq d(x, z) + d(z, y)$ $x, y, z \in X$.

Then d is called a cone metric [11] or K -metric [27] on X and (X, d) is called a cone metric space [11].

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space.

Example 2.3. (See [11]) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ and $d: X \times X \rightarrow E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space with normal cone P where $K = 1$.

Example 2.4. (See [22]) Let $E = \ell^2$, $P = \{\{x_n\}_{n \geq 1} \in E : x_n \geq 0, \text{ for all } n\}$, (X, ρ) a metric space, and $d: X \times X \rightarrow E$ defined by $d(x, y) = \{\rho(x, y)/2^n\}_{n \geq 1}$. Then (X, d) is a cone metric space.

Clearly, the above examples show that the class of cone metric spaces contains the class of metric spaces.

Definition 2.5. (See [11]) Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

(i) a Cauchy sequence if for every ε in E with $0 \ll \varepsilon$, then there is an N such that for all $n, m > N$, $d(x_n, x_m) \ll \varepsilon$;

(ii) a convergent sequence if for every ε in E with $0 \ll \varepsilon$, then there is an N such that for all $n > N$, $d(x_n, x) \ll \varepsilon$ for some fixed x in X .

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X .

In the following (X, d) will stand for a cone metric space with respect to a cone P with $P^0 \neq \emptyset$ in a real Banach space E and $\leq$ is partial ordering in E with respect to P .

Definition 2.6. (Measurable function) (See [16]) Let (Ω, Σ) be a measurable space with Σ -a sigma algebra of subsets of Ω and M be a nonempty subset of a metric space $X = (X, d)$. Let 2^M be the family of nonempty subsets of M and $C(M)$ the family of all nonempty closed subsets of M . A mapping $G: \Omega \rightarrow 2^M$ is called measurable if for each open subset U of M , $G^{-1}(U) \in \Sigma$, where $G^{-1}(U) = \{\omega \in \Omega : G(\omega) \cap U \neq \emptyset\}$.

Definition 2.7. (Measurable selector) (See [16]) A mapping $\xi: \Omega \rightarrow M$ is called a measurable selector of a measurable mapping $G: \Omega \rightarrow 2^M$ if ξ is measurable and $\xi(\omega) \in G(\omega)$ for each $\omega \in \Omega$.

Definition 2.8. (Random operator) (See [16]) The mapping $T: \Omega \times M \rightarrow X$ is said to be a random operator if and only if for each fixed $x \in M$, the mapping $T(., x): \Omega \rightarrow X$ is measurable.

Definition 2.9. (Continuous random operator) (See [16]) A random operator $T: \Omega \times M \rightarrow X$ is said to be continuous random operator if for each fixed $x \in M$ and $\omega \in \Omega$, the mapping $T(\omega, .): X \rightarrow X$ is continuous.

Definition 2.10. (Random fixed point) (See [16]) A measurable mapping $\xi: \Omega \rightarrow M$ is a random fixed point of a random operator $T: \Omega \times M \rightarrow X$ if and only if $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$.

Definition 2.11. (Cone Random Metric Space) Let M be a nonempty set and the mapping $d: \Omega \times M \rightarrow P$, where P is a cone, $\omega \in \Omega$ be a selector, satisfy the following conditions:

(1) $d(x(\omega), y(\omega)) \geq 0$ and $d(x(\omega), y(\omega)) = 0$ if and only if $x(\omega) = y(\omega)$ for all $x(\omega), y(\omega) \in \Omega \times M$,

(2) $d(x(\omega), y(\omega)) = d(y(\omega), x(\omega))$ for all $x, y \in M$, $\omega \in \Omega$ and $x(\omega), y(\omega) \in \Omega \times M$,

(3) $d(x(\omega), y(\omega)) \leq d(x(\omega), z(\omega)) + d(z(\omega), y(\omega))$ for all $x, y \in M$ and $\omega \in \Omega$ be a selector,

(4) for any $x, y \in X$, $\omega \in \Omega$, $d(x(\omega), y(\omega))$ is non-increasing and left continuous.

Then d is called cone random metric on M and (M, d) is called a cone random metric space.

3. MAIN RESULTS

In this section we shall prove some common random fixed point theorems under contractive type condition in the framework of cone random metric spaces.

Theorem 3.1. *Let (X, d) be a complete cone random metric space with respect to a cone P and let M be a nonempty separable closed subset of X . Let S and T be two continuous random operators defined on M such that for $\omega \in \Omega$, $S(\omega, \cdot), T(\omega, \cdot): \Omega \times M \rightarrow M$ satisfying the condition:*

$$\begin{aligned} d(S(x(\omega)), T(y(\omega))) &\leq a(\omega) [d(x(\omega), Sx(\omega)) + d(y(\omega), Ty(\omega))] \\ &\quad + b(\omega) d(x(\omega), y(\omega)) + c(\omega) \max \left\{ d(x(\omega), T(y(\omega))), \right. \\ &\quad \left. d(y(\omega), S(x(\omega))) \right\} \end{aligned} \quad (3.1)$$

for all $x, y \in M$, $2a(\omega) + b(\omega) + 2c(\omega) < 1$, where $a(\omega), b(\omega), c(\omega) > 0$ and $\omega \in \Omega$. Then S and T have a unique common random fixed point in X .

Proof. For each $x_0(\omega) \in \Omega \times M$ and $n = 0, 1, 2, \dots$, we choose $x_1(\omega), x_2(\omega) \in \Omega \times M$ such that $x_1(\omega) = S(x_0(\omega))$ and $x_2(\omega) = T(x_1(\omega))$. In general we define sequence of elements of X such that $x_{2n+1}(\omega) = S(x_{2n}(\omega))$ and $x_{2n+2}(\omega) = T(x_{2n+1}(\omega))$. Then from (3.1), we have

$$\begin{aligned} d(x_{2n+1}(\omega), x_{2n}(\omega)) &= d(S(x_{2n}(\omega)), T(x_{2n-1}(\omega))) \\ &\leq a(\omega) [d(x_{2n}(\omega), S(x_{2n}(\omega))) + d(x_{2n-1}(\omega), T(x_{2n-1}(\omega)))] \\ &\quad + b(\omega) d(x_{2n}(\omega), x_{2n-1}(\omega)) \\ &\quad + c(\omega) \max \left\{ d(x_{2n}(\omega), T(x_{2n-1}(\omega))), d(x_{2n-1}(\omega), S(x_{2n}(\omega))) \right\} \\ &= a(\omega) [d(x_{2n}(\omega), x_{2n+1}(\omega)) + d(x_{2n-1}(\omega), x_{2n}(\omega))] \\ &\quad + b(\omega) d(x_{2n}(\omega), x_{2n-1}(\omega)) \\ &\quad + c(\omega) \max \left\{ d(x_{2n}(\omega), x_{2n}(\omega)), d(x_{2n-1}(\omega), x_{2n+1}(\omega)) \right\} \\ &= a(\omega) [d(x_{2n}(\omega), x_{2n+1}(\omega)) + d(x_{2n-1}(\omega), x_{2n}(\omega))] \\ &\quad + b(\omega) d(x_{2n}(\omega), x_{2n-1}(\omega)) \\ &\quad + c(\omega) \max \left\{ 0, d(x_{2n-1}(\omega), x_{2n+1}(\omega)) \right\}. \end{aligned} \quad (3.2)$$

Since for non-negative real numbers a and b , we have

$$\max\{a, b\} \leq a + b. \quad (3.3)$$

Using (3.3) in (3.2), we have

$$\begin{aligned}
 d(x_{2n+1}(\omega), x_{2n}(\omega)) &\leq a(\omega) [d(x_{2n}(\omega), x_{2n+1}(\omega)) + d(x_{2n-1}(\omega), x_{2n}(\omega))] \\
 &\quad + b(\omega) d(x_{2n}(\omega), x_{2n-1}(\omega)) + c(\omega) d(x_{2n-1}(\omega), x_{2n+1}(\omega)) \\
 &\leq a(\omega) [d(x_{2n}(\omega), x_{2n+1}(\omega)) + d(x_{2n-1}(\omega), x_{2n}(\omega))] \\
 &\quad + b(\omega) d(x_{2n}(\omega), x_{2n-1}(\omega)) + c(\omega) [d(x_{2n-1}(\omega), x_{2n}(\omega)) \\
 &\quad + d(x_{2n}(\omega), x_{2n+1}(\omega))] \\
 &= (a(\omega) + b(\omega) + c(\omega)) d(x_{2n}(\omega), x_{2n-1}(\omega)) \\
 &\quad + (a(\omega) + c(\omega)) d(x_{2n+1}(\omega), x_{2n}(\omega)).
 \end{aligned} \tag{3.4}$$

The above inequality (3.4) gives,

$$\begin{aligned}
 d(x_{2n+1}(\omega), x_{2n}(\omega)) &\leq \left(\frac{a(\omega) + b(\omega) + c(\omega)}{1 - a(\omega) - c(\omega)} \right) d(x_{2n+1}(\omega), x_{2n}(\omega)) \\
 &= k(\omega) d(x_{2n+1}(\omega), x_{2n}(\omega)),
 \end{aligned} \tag{3.5}$$

where

$$k(\omega) = \left(\frac{a(\omega) + b(\omega) + c(\omega)}{1 - a(\omega) - c(\omega)} \right).$$

By the assumption of the theorem

$$\begin{aligned}
 2a(\omega) + b(\omega) + 2c(\omega) < 1 &\Rightarrow a(\omega) + b(\omega) + c(\omega) < 1 - a(\omega) - c(\omega) \\
 &\Rightarrow k(\omega) = \left(\frac{a(\omega) + b(\omega) + c(\omega)}{1 - a(\omega) - c(\omega)} \right) < 1.
 \end{aligned}$$

Similarly, we have

$$d(x_{2n}(\omega), x_{2n-1}(\omega)) \leq k(\omega) d(x_{2n-1}(\omega), x_{2n-2}(\omega)).$$

Hence

$$d(x_{2n+1}(\omega), x_{2n}(\omega)) \leq k(\omega)^2 d(x_{2n-1}(\omega), x_{2n-2}(\omega)).$$

On continuing in this process, we get

$$d(x_{2n+1}(\omega), x_{2n}(\omega)) \leq k(\omega)^{2n} d(x_1(\omega), x_0(\omega)).$$

Also for $n > m$, we have

$$\begin{aligned}
 d(x_n(\omega), x_m(\omega)) &\leq d(x_n(\omega), x_{n-1}(\omega)) + d(x_{n-1}(\omega), x_{n-2}(\omega)) + \dots \\
 &\quad + d(x_{m+1}(\omega), x_m(\omega)) \\
 &\leq \left(k(\omega)^{n-1} + k(\omega)^{n-2} + \dots + k(\omega)^m \right) d(x_1(\omega), x_0(\omega)) \\
 &\leq \left(\frac{k(\omega)^m}{1 - k(\omega)} \right) d(x_1(\omega), x_0(\omega)).
 \end{aligned}$$

Let $0 \ll \varepsilon$ be given. Choose a natural number N such that $\left(\frac{k(\omega)^m}{1 - k(\omega)} \right) d(x_1(\omega), x_0(\omega)) \ll \varepsilon$ for every $m \geq N$. Thus

$$d(x_n(\omega), x_m(\omega)) \leq \left(\frac{k(\omega)^m}{1 - k(\omega)} \right) d(x_1(\omega), x_0(\omega)) \ll \varepsilon,$$

for every $n > m \geq N$.

This shows that the sequence $\{x_n(\omega)\}$ is a Cauchy sequence in $\Omega \times M$. Since (X, d) is complete, there exists $z(\omega) \in \Omega \times X$ such that $x_n(\omega) \rightarrow z(\omega)$ as $n \rightarrow \infty$. Choose a natural number N_1 such that

$$d(z(\omega), x_{2n+2}(\omega)) \ll \frac{\varepsilon(1 - a(\omega) - c(\omega))}{2(1 + b(\omega) + 2c(\omega))}, \quad (3.6)$$

and

$$d(x_{2n+1}(\omega), x_{2n+2}(\omega)) \ll \frac{\varepsilon(1 - a(\omega) - c(\omega))}{2(a(\omega) + b(\omega) + c(\omega))}, \quad (3.7)$$

for every $n \geq N_1$. Hence for $n \geq N_1$, we have

$$\begin{aligned} d(z(\omega), S(z(\omega))) &\leq d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+2}(\omega), S(z(\omega))) \\ &= d(z(\omega), x_{2n+2}(\omega)) + d(S(z(\omega)), T(x_{2n+1}(\omega))) \\ &\leq d(z(\omega), x_{2n+2}(\omega)) \\ &\quad + a(\omega) [d(z(\omega), S(z(\omega))) + d(x_{2n+1}(\omega), T(x_{2n+1}(\omega)))] \\ &\quad + b(\omega) d(z(\omega), x_{2n+1}(\omega)) \\ &\quad + c(\omega) \max \left\{ d(z(\omega), T(x_{2n+1}(\omega))), d(x_{2n+1}(\omega), S(z(\omega))) \right\} \\ &= d(z(\omega), x_{2n+2}(\omega)) \\ &\quad + a(\omega) [d(z(\omega), S(z(\omega))) + d(x_{2n+1}(\omega), x_{2n+2}(\omega))] \\ &\quad + b(\omega) d(z(\omega), x_{2n+1}(\omega)) \\ &\quad + c(\omega) \max \left\{ d(z(\omega), x_{2n+2}(\omega)), d(x_{2n+1}(\omega), S(z(\omega))) \right\}. \end{aligned}$$

Using (3.3) in the above inequality, we get

$$\begin{aligned} d(z(\omega), S(z(\omega))) &\leq d(z(\omega), x_{2n+2}(\omega)) \\ &\quad + a(\omega) [d(z(\omega), S(z(\omega))) + d(x_{2n+1}(\omega), x_{2n+2}(\omega))] \\ &\quad + b(\omega) d(z(\omega), x_{2n+1}(\omega)) \\ &\quad + c(\omega) [d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+1}(\omega), S(z(\omega)))] \\ &\leq d(z(\omega), x_{2n+2}(\omega)) \\ &\quad + a(\omega) [d(z(\omega), S(z(\omega))) + d(x_{2n+1}(\omega), x_{2n+2}(\omega))] \\ &\quad + b(\omega) [d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+2}(\omega), x_{2n+1}(\omega))] \\ &\quad + c(\omega) [d(z(\omega), x_{2n+2}(\omega)) + d(x_{2n+1}(\omega), x_{2n+2}(\omega)) \\ &\quad + d(x_{2n+2}(\omega), z(\omega)) + d(z(\omega), S(z(\omega)))] \\ &= (1 + b(\omega) + 2c(\omega)) d(z(\omega), x_{2n+2}(\omega)) \\ &\quad + (a(\omega) + b(\omega) + c(\omega)) d(x_{2n+1}(\omega), x_{2n+2}(\omega)) \\ &\quad + (a(\omega) + c(\omega)) d(z(\omega), S(z(\omega))). \end{aligned}$$

The above inequality gives

$$\begin{aligned} d(z(\omega), S(z(\omega))) &\leq \left(\frac{1+b(\omega)+2c(\omega)}{1-a(\omega)-c(\omega)} \right) d(z(\omega), x_{2n+2}(\omega)) \\ &\quad + \left(\frac{a(\omega)+b(\omega)+c(\omega)}{1-a(\omega)-c(\omega)} \right) d(x_{2n+1}(\omega), x_{2n+2}(\omega)). \end{aligned} \quad (3.8)$$

Using (3.6) and (3.7) in (3.8), we get

$$d(z(\omega), S(z(\omega))) \ll \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad (3.9)$$

Thus $d(z(\omega), S(z(\omega))) \ll \frac{\varepsilon}{m}$ for all $m \geq 1$. So $\frac{\varepsilon}{m} - d(z(\omega), S(z(\omega))) \in P$ for all $m \geq 1$. Since $\frac{\varepsilon}{m} \rightarrow 0$ as $m \rightarrow \infty$ and P is closed, we obtain $-d(z(\omega), S(z(\omega))) \in P$. But $d(z(\omega), S(z(\omega))) \in P$. Therefore by **Definition 2.1(c₃)** $d(z(\omega), S(z(\omega))) = 0$ and so $S(z(\omega)) = z(\omega)$.

In an exactly similar way we can prove that for all $\omega \in \Omega$, $T(z(\omega)) = z(\omega)$. Hence $S(z(\omega)) = T(z(\omega)) = z(\omega)$. This shows that $z(\omega)$ is a common random fixed point of S and T .

Uniqueness

Let $v(\omega)$ be another random fixed point common to S and T , that is, for $\omega \in \Omega$, $S(v(\omega)) = T(v(\omega)) = v(\omega)$ such that $z(\omega) \neq v(\omega)$. Then for $\omega \in \Omega$, we have

$$\begin{aligned} d(z(\omega), v(\omega)) &= d(S(z(\omega)), T(v(\omega))) \\ &\leq a(\omega) [d(z(\omega), S(z(\omega))) + d(v(\omega), T(v(\omega)))] + b(\omega) d(z(\omega), v(\omega)) \\ &\quad + c(\omega) \max \left\{ d(z(\omega), T(v(\omega))), d(v(\omega), S(z(\omega))) \right\} \\ &\leq (b(\omega) + c(\omega)) d(z(\omega), v(\omega)) \\ &< d(z(\omega), v(\omega)), \text{ since } 0 < b(\omega) + c(\omega) < 1, \end{aligned}$$

a contradiction. Hence $z(\omega) = v(\omega)$ and so $z(\omega)$ is a unique common random fixed point of S and T . This completes the proof. $\square$

Corollary 3.2. *Let (X, d) be a complete cone random metric space with respect to a cone P and let M be a nonempty separable closed subset of X . Let T be a continuous random operator defined on M such that for $\omega \in \Omega$, $T(\omega, \cdot): \Omega \times M \rightarrow M$ satisfying the condition:*

$$\begin{aligned} d(T(x(\omega)), T(y(\omega))) &\leq a(\omega) [d(x(\omega), Tx(\omega)) + d(y(\omega), Ty(\omega))] \\ &\quad + b(\omega) d(x(\omega), y(\omega)) + c(\omega) \max \left\{ d(x(\omega), T(y(\omega))), \right. \\ &\quad \left. d(y(\omega), T(x(\omega))) \right\} \end{aligned}$$

for all $x, y \in M$, $2a(\omega) + b(\omega) + 2c(\omega) < 1$, where $a(\omega), b(\omega), c(\omega) > 0$ and $\omega \in \Omega$. Then T has a unique random fixed point in X .

Proof. The proof of this corollary immediately follows by putting $S = T$ in Theorem 3.1. This completes the proof. $\square$

If we take $S = T$ and $a(\omega) = c(\omega) = 0$ in Theorem 3.1, then we obtain the following result as corollary.

Corollary 3.3. *Let (X, d) be a complete cone random metric space with respect to a cone P and let M be a nonempty separable closed subset of X . Let T be a random operator defined on M such that for $\omega \in \Omega$, $T(\omega, \cdot): \Omega \times M \rightarrow M$ satisfying the condition:*

$$d(T(x(\omega)), T(y(\omega))) \leq b(\omega) d(x(\omega), y(\omega)),$$

for all $x, y \in M$, $b(\omega) \in (0, 1)$ and $\omega \in \Omega$. Then T has a unique random fixed point in X .

If we take $S = T$ and $b(\omega) = c(\omega) = 0$ in Theorem 3.1, then we obtain the following result as corollary.

Corollary 3.4. ([16], Corollary 3.2) *Let (X, d) be a complete cone random metric space with respect to a cone P and let M be a nonempty separable closed subset of X . Let T be a continuous random operator defined on M such that for $\omega \in \Omega$, $T(\omega, \cdot): \Omega \times M \rightarrow M$ satisfying the condition:*

$$d(T(x(\omega)), T(y(\omega))) \leq a(\omega) [d(x(\omega), T(x(\omega))) + d(y(\omega), T(y(\omega)))]$$

for all $x, y \in M$, $a(\omega) \in (0, \frac{1}{2})$ and $\omega \in \Omega$. Then T has a unique random fixed point in X .

If we take $S = T$ and $a(\omega) = b(\omega) = 0$ in Theorem 3.1, then we obtain the following result as corollary.

Corollary 3.5. *Let (X, d) be a complete cone random metric space with respect to a cone P and let M be a nonempty separable closed subset of X . Let T be a continuous random operator defined on M such that for $\omega \in \Omega$, $T(\omega, \cdot): \Omega \times M \rightarrow M$ satisfying the condition:*

$$d(T(x(\omega)), T(y(\omega))) \leq c(\omega) \max \left\{ d(x(\omega), T(y(\omega))), d(y(\omega), T(x(\omega))) \right\}$$

for all $x, y \in M$, $c(\omega) \in (0, \frac{1}{2})$ and $\omega \in \Omega$. Then T has a unique random fixed point in X .

Theorem 3.6. *Let (X, d) be a complete cone random metric space with respect to a cone P and let M be a nonempty separable closed subset of X . Let S and T be two continuous random operators defined on M such that for $\omega \in \Omega$, $S(\omega, \cdot), T(\omega, \cdot): \Omega \times M \rightarrow M$ satisfying the condition:*

$$\begin{aligned} d(S(x(\omega)), T(y(\omega))) &\leq a(\omega) d(x(\omega), y(\omega)) \\ &\quad + b(\omega) \max \left\{ d(x(\omega), S(x(\omega))), d(y(\omega), T(y(\omega))) \right\} \\ &\quad + c(\omega) \max \left\{ d(x(\omega), T(y(\omega))), d(y(\omega), S(x(\omega))) \right\} \end{aligned} \tag{3.10}$$

for all $x, y \in M$, $a(\omega) + b(\omega) + 2c(\omega) < 1$, where $a(\omega), b(\omega), c(\omega) > 0$ and $\omega \in \Omega$. Then S and T have a unique common random fixed point in X .

Proof. For each $x_0(\omega) \in \Omega \times M$ and $n = 0, 1, 2, \dots$, we choose $x_1(\omega), x_2(\omega) \in \Omega \times M$ such that $x_1(\omega) = S(x_0(\omega))$ and $x_2(\omega) = T(x_1(\omega))$. In general we define sequence of elements of X such that $x_{2n+1}(\omega) = S(x_{2n}(\omega))$ and $x_{2n+2}(\omega) = T(x_{2n+1}(\omega))$. Then from (3.10), we have

$$\begin{aligned}
 d(x_{2n+1}(\omega), x_{2n}(\omega)) &= d(S(x_{2n}(\omega)), T(x_{2n-1}(\omega))) \\
 &\leq a(\omega) d(x_{2n}(\omega), x_{2n-1}(\omega)) \\
 &\quad + b(\omega) \max \left\{ d(x_{2n}(\omega), S(x_{2n}(\omega))), d(x_{2n-1}(\omega), T(x_{2n-1}(\omega))) \right\} \\
 &\quad + c(\omega) \max \left\{ d(x_{2n}(\omega), T(x_{2n-1}(\omega))), d(x_{2n-1}(\omega), S(x_{2n}(\omega))) \right\} \\
 &\leq a(\omega) d(x_{2n}(\omega), x_{2n-1}(\omega)) \\
 &\quad + b(\omega) \max \left\{ d(x_{2n}(\omega), x_{2n+1}(\omega)), d(x_{2n-1}(\omega), x_{2n}(\omega)) \right\} \\
 &\quad + c(\omega) \max \left\{ d(x_{2n}(\omega), x_{2n}(\omega)), d(x_{2n-1}(\omega), x_{2n+1}(\omega)) \right\} \\
 &= a(\omega) d(x_{2n}(\omega), x_{2n-1}(\omega)) \\
 &\quad + b(\omega) \max \left\{ d(x_{2n}(\omega), x_{2n+1}(\omega)), d(x_{2n-1}(\omega), x_{2n}(\omega)) \right\} \\
 &\quad + c(\omega) \max \left\{ 0, d(x_{2n-1}(\omega), x_{2n+1}(\omega)) \right\}. \tag{3.11}
 \end{aligned}$$

Since for non-negative real numbers a and b , we have

$$\max\{a, b\} \leq a + b, \tag{3.12}$$

and taking

$$\max \left\{ d(x_{2n}(\omega), x_{2n+1}(\omega)), d(x_{2n-1}(\omega), x_{2n}(\omega)) \right\} = d(x_{2n-1}(\omega), x_{2n}(\omega)). \tag{3.13}$$

Using (3.12) and (3.13) in (3.11), we have

$$\begin{aligned}
 d(x_{2n+1}(\omega), x_{2n}(\omega)) &\leq a(\omega) d(x_{2n}(\omega), x_{2n-1}(\omega)) + b(\omega) d(x_{2n-1}(\omega), x_{2n}(\omega)) \\
 &\quad + c(\omega) d(x_{2n-1}(\omega), x_{2n+1}(\omega)) \\
 &\leq a(\omega) d(x_{2n}(\omega), x_{2n-1}(\omega)) + b(\omega) d(x_{2n-1}(\omega), x_{2n}(\omega)) \\
 &\quad + c(\omega) [d(x_{2n-1}(\omega), x_{2n}(\omega)) + d(x_{2n}(\omega), x_{2n+1}(\omega))]. \tag{3.14}
 \end{aligned}$$

The above inequality (3.14) gives,

$$\begin{aligned}
 d(x_{2n+1}(\omega), x_{2n}(\omega)) &\leq \left(\frac{a(\omega) + b(\omega) + c(\omega)}{1 - c(\omega)} \right) d(x_{2n+1}(\omega), x_{2n}(\omega)) \\
 &= h(\omega) d(x_{2n+1}(\omega), x_{2n}(\omega)), \tag{3.15}
 \end{aligned}$$

where

$$h(\omega) = \left(\frac{a(\omega) + b(\omega) + c(\omega)}{1 - c(\omega)} \right).$$

By the assumption of the theorem

$$a(\omega) + b(\omega) + 2c(\omega) < 1 \Rightarrow a(\omega) + b(\omega) + c(\omega) < 1 - c(\omega)$$

$$\Rightarrow h(\omega) = \left(\frac{a(\omega) + b(\omega) + c(\omega)}{1 - c(\omega)} \right) < 1.$$

If we take

$$\max \left\{ d(x_{2n}(\omega), x_{2n+1}(\omega)), d(x_{2n-1}(\omega), x_{2n}(\omega)) \right\} = d(x_{2n}(\omega), x_{2n+1}(\omega)). \quad (3.16)$$

in equation (3.11), then we also get the same result as (3.15). The rest of the proof is same as that of Theorem 3.1. This completes the proof. $\square$

Remark 3.7. If we take $S = T$ and $b(\omega) = c(\omega) = 0$ and $a(\omega) = b(\omega)$ in Theorem 3.6, then we obtain Corollary 3.3 of this paper.

Corollary 3.8. *Let (X, d) be a complete cone random metric space with respect to a cone P and let M be a nonempty separable closed subset of X . Let T be a continuous random operators defined on M such that for $\omega \in \Omega$, $T(\omega, \cdot): \Omega \times M \rightarrow M$ satisfying the condition:*

$$\begin{aligned} d(T(x(\omega)), T(y(\omega))) &\leq a(\omega) d(x(\omega), y(\omega)) \\ &\quad + b(\omega) \max \left\{ d(x(\omega), Tx(\omega)), d(y(\omega), Ty(\omega)) \right\} \\ &\quad + c(\omega) \max \left\{ d(x(\omega), T(y(\omega))), d(y(\omega), T(x(\omega))) \right\} \end{aligned}$$

for all $x, y \in M$, $a(\omega) + b(\omega) + 2c(\omega) < 1$, where $a(\omega), b(\omega), c(\omega) > 0$ and $\omega \in \Omega$. Then T has a unique random fixed point in X .

Proof. The proof of this corollary immediately follows by putting $S = T$ in Theorem 3.6. This completes the proof. $\square$

Example 3.9. Let $\Omega = [0, 1]$ and Σ be the sigma algebra of Lebesgue's measurable subset of $[0, 1]$. Take $X = R$ with $d(x, y) = |x - y|$ for $x, y \in R$. Define random mapping T from $\Omega \times X$ to X as $T(\omega, x) = 2\omega - x$. Then a measurable mapping $\xi: \Omega \rightarrow X$ defined as $\xi(\omega) = \omega$ for all $\omega \in \Omega$, serve as a unique random fixed point of T .

Example 3.10. Let $M = R$ and $P = \{x \in M : x \geq 0\}$, also $\Omega = [0, 1]$ and Σ be the sigma algebra of Lebesgue's measurable subset of $[0, 1]$. Let $X = [0, \infty)$ and define a mapping $d: (\Omega \times X) \times (\Omega \times X) \rightarrow M$ by $d(x(\omega), y(\omega)) = |x(\omega) - y(\omega)|$. Then (X, d) is a cone random metric space. Define random operator T form $(\Omega \times X)$ to X as $T(\omega, x) = \frac{1-\omega^2+2x}{3}$. Also sequence of mapping $\xi_n: \Omega \rightarrow X$ is defined by $\xi_n(\omega) = (1 - \omega^2)^{1+(1/n)}$ for every $\omega \in \Omega$ and $n \in N$. Define measurable mapping $\xi: \Omega \rightarrow X$ as $\xi(\omega) = (1 - \omega^2)$ for every $\omega \in \Omega$. Hence $(1 - \omega^2)$ is the random fixed point of the random operator T .

Example 3.11. Let $M = R$ and $P = \{x \in M : x \geq 0\}$, also $\Omega = [0, 1]$ and Σ be the sigma algebra of Lebesgue's measurable subset of $[0, 1]$. Let $X = [0, \infty)$ and define a mapping $d: (\Omega \times X) \times (\Omega \times X) \rightarrow M$ by $d(x(\omega), y(\omega)) = |x(\omega) - y(\omega)|$. Then (X, d) is a cone random metric space. Define random operators S and T form $(\Omega \times X)$ to X as $S(\omega, x) = \frac{1-\omega^2+x}{2}$ and $T(\omega, x) = \frac{1-\omega^2+2x}{3}$. Also sequence of mapping $\xi_n: \Omega \rightarrow X$ is defined by $\xi_n(\omega) = (1 - \omega^2)^{1+(1/n)}$ for every $\omega \in \Omega$

and $n \in N$. Define measurable mapping $\xi: \Omega \rightarrow X$ as $\xi(\omega) = (1 - \omega^2)$ for every $\omega \in \Omega$. Hence $(1 - \omega^2)$ is a common random fixed point of the random operators S and T .

Example 3.12. Let $E = \{0, 1, 2, 3, 4\} \subset R$ with the usual metric d . Consider $\Omega = \{0, 1, 2, 3, 4\}$ and let Σ be the sigma algebra of Lebesgue's measurable subset of Ω . Define $S, T: \Omega \times E \rightarrow E$ by

$$\begin{cases} S(\omega, x) = 4, & \text{where } x = 0 \text{ and } \omega \in \Omega \\ = 3, & \text{otherwise.} \end{cases}$$

and

$$\begin{cases} T(\omega, x) = 2, & \text{where } x = 0 \text{ and } \omega \in \Omega \\ = 3, & \text{otherwise.} \end{cases}$$

Let us take $x(\omega) = 0$ and $y(\omega) = 1$. Then condition (3.1) of Theorem 3.1 is satisfied with $a(\omega) = \frac{1}{11}$, $b(\omega) = \frac{2}{11}$, $c(\omega) = \frac{1}{11}$ and $2a(\omega) + b(\omega) + 2c(\omega) = \frac{6}{11} \in (0, 1)$. The measurable function $\xi: \Omega \rightarrow E$ with $\xi(\omega) = 3$ is a unique common random fixed point of S and T , that is, $S(\omega, x) = T(\omega, x) = 3 = \xi(\omega)$.

Example 3.13. Let $E = \{0, 1, 2, 3, 4\} \subset R$ with the usual metric d . Consider $\Omega = \{0, 1, 2, 3, 4\}$ and let Σ be the sigma algebra of Lebesgue's measurable subset of Ω . Define $S, T: \Omega \times E \rightarrow E$ by

$$\begin{cases} S(\omega, x) = 3, & \text{where } x = 0 \text{ and } \omega \in \Omega \\ = 2, & \text{otherwise.} \end{cases}$$

and

$$\begin{cases} T(\omega, x) = 4, & \text{where } x = 0 \text{ and } \omega \in \Omega \\ = 2, & \text{otherwise.} \end{cases}$$

Let us take $x(\omega) = 0$ and $y(\omega) = 1$. Then condition (3.10) of Theorem 3.6 is satisfied with $a(\omega) = \frac{1}{7}$, $b(\omega) = \frac{1}{7}$, $c(\omega) = \frac{2}{7}$ and $a(\omega) + b(\omega) + 2c(\omega) = \frac{6}{7} \in (0, 1)$. The measurable function $\xi: \Omega \rightarrow E$ with $\xi(\omega) = 2$ is a unique common random fixed point of S and T , that is, $S(\omega, x) = T(\omega, x) = 2 = \xi(\omega)$.

Remark 3.14. Our results extend and generalize many known results from the current existing literature.

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