Gulf Journal of Mathematics
Vol 21, Issue 1 (2025) 351-368

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v21i1.3398

A COUPLING AND MODELING OF SEDIMENT TRANSPORT IN COASTAL ENVIRONMENTS

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Abstract. This paper presents a mathematical analysis of a coupled system for modeling sediment transport and dune evolution in coastal environments. The model combines the Shallow Water Equations (SWE) with the Long-Term Dune Dynamics (LTDD) equation, incorporating viscosity and modified pressure terms. Using methods of mathematical analysis, we prove the existence and uniqueness of solutions for the system. We also provide estimates for its dimensionless form and conclude by presenting a homogenized version of the combined model.

Keywords. Sediment, Viscous shallow water equations, Homogenization, Existence and uniqueness
2020 Mathematics Subject Classification. Primary 35Q30; Secondary 76D05

1. Introduction and preliminaries

Modeling sediment transport in coastal environments is crucial for understanding the geomorphological and hydrodynamic processes that directly influence the morphology of seabeds and the stability of coastlines, see [4, 9, 23]. The dynamics of dunes, driven by the movement of water, is a typical example of these processes. The combination of hydrodynamic and morphodynamic models allows a better understanding of these phenomena on long time scales, thus contributing to the sustainable management of coastal areas.
The Saint-Venant equations, or shallow-water equations (SWE), are commonly used to model the behavior of incompressible fluids under the influence of gravity, particularly when the fluid depth is much smaller than its wavelength. These equations are derived from the Navier-Stokes equations and provide a simplified but effective framework for analyzing the horizontal movements of fluids in oceans and other water bodies [2, 5].
In this paper, the main idea consists to combine two distinct approaches: on the one hand, a model derived by Zongo et al. [23] based on a multiscale asymptotic analysis and on the other hand, a sediment transport equation used in [3, 10]. The model by Zongo et al. describes the interaction between water and sediment layers through a coupling between the shallow-water equations and the Reynolds model for sediments. This model was then enriched by introducing the long-term dynamic dunes equation

Date: Received: August 19, 2025; Accepted: Sep 20, 2025.
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(LTDD) used by Faye et al. [10]. This coupling allows us to analyze the evolution of sand dunes over long periods in the coastal marine environment.

Unlike the mass equation in [3], our formulation accounts for the total water height, including the average level H, thereby capturing significant variations in free-surface elevation. In the momentum equation, we introduce a viscosity term, modify the pressure contributions and incorporate the Froude number. The viscous term, which represents the dissipation of kinetic energy, can be obtained via formal derivation and enhances the accuracy of simulations in regimes where fluid viscosity plays a dominant role. The pressure terms $\dfrac{(\xi + H - z)}{Fr^2} \cdot \nabla(\xi + H - z)$ and $\dfrac{(\xi + H - z)}{Fr^2} \cdot \nabla z$ arise naturally from the derivation (see [23]) and reflect the influence of hydrostatic pressure under Froude scaling. The number Fr characterizes whether the flow is subcritical or supercritical with respect to a fixed obstacle.

The sediment transport equations remain unchanged, but the introduction of viscosity and modified pressure terms makes the coupled model more robust and particularly suited for coastal and shallow-water environments, where energy dissipation and internal interactions are critical.

Specifically, the third equation of the system derived by Zongo et al. [23] is replaced by the LTDD, resulting in a coupled SWE-LTDD system. This approach not only captures the complex dynamics at the water-sediment interface but also examines coastal erosion phenomena, a major issue in coastal regions like Senegal, where the human and economic impacts are significant.

In [5, 6, 21, 24], the authors utilised an entropy inequality, known as the BD entropy, to establish a bound on the term $\nabla\sqrt{h}$. This bound plays a crucial role in demonstrating stability or existence, as discussed in [7, 8, 13, 20, 23].

In [5, 8], the authors demonstrated the existence of global weak solutions for the barotropic Navier-Stokes equations by introducing linear and quadratic terms, specifically $r_0 u$ and $r_1 h|u|^2 u$, into the momentum equation. In contrast, the authors in [17] established the existence of global weak solutions for the same equations without incorporating additional regularizing terms. They derived a "Mellet-Vasseur" type inequality, which provided control over the term

$$\int_\Omega h(1 + |u|^2)\ln(1 + |u|^2)dx.$$

The main contribution of this work is the development of a novel coupled system combining the Shallow Water Equations with the Long-Term Dune Dynamics equation. We establish the existence and uniqueness of global weak solutions for this sediment–dune transport model, extending the results of [3, 10] and also following the framework proposed in [14]. traduire en francais : The model homogenized form captures long-term morphodynamic processes, offering a framework to better understand coastal erosion.

The paper is organized as follows. Section 2 is devoted to the coupled shallow water–long-term dune dynamics (SWE–LTDD) model, introducing viscous and modified pressure terms to more accurately capture fluid–sediment interactions. After that, we discuss the symmetrization of the dimensionless model, which prepares the reader for the theoretical analysis that follows. This analysis highlights the ε-balanced structure that ensures stability and establishes existence, uniqueness, and energy estimates. The homogenised system that captures large-scale sediment dynamics is the result of two-scale convergence, which we examine in Section 5 to examine the asymptotic behaviour of solutions. The study's conclusion is presented in section 6 and we end in section 7 by a statement on conflicting interests.

2. The coupled system

2.1. Model. In this work, we consider the following system of partial differential equations:

$$\frac{\partial}{\partial t}\left(\xi + H - z\right) + \nabla\cdot\big[\left(\xi + H - z\right)u\big] = 0, \tag{2.1a}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\big[\left(\xi + H - z\right)u\big] + \nabla\cdot\big[\left(\xi + H - z\right)u\otimes u\big] - \nu\nabla\cdot\big[\left(\xi + H - z\right)D(u)\big]\\ &\quad + \frac{\left(\xi + H - z\right)}{Fr^2}\nabla(\xi + H - z) + \frac{\left(\xi + H - z\right)}{Fr^2}\nabla z = 0,\end{aligned} \tag{2.1b}$$

$$\frac{\partial z}{\partial t} + \frac{\alpha}{1-p}\nabla\cdot\left[\chi\left(D_{G_\rho}\frac{|u|^2 - u_c}{c^2}\right)\left(\frac{u}{|u|} - \lambda\nabla z\right)\right] = 0, \tag{2.1c}$$

where t is the time variable and $x = (x_1, x_2)$ the two-dimensional space variable, $\xi(t,x)$ denotes the free surface of the water, H is a constant representing the mean water depth, $z(t,x)$ denotes the height of the seabed (measured from the level $y = -H$), $h(t,x)$ is the water column height i.e. the distance from the free surface to the seabed given by $h = \xi + H - z$, we denote by $m = h - H = \xi - z$ the height variation relative to the mean level, $u(t,x) = (u_1, u_2)$ is the horizontal velocity of the water and $u^\perp(t,x) = (-u_2, u_1)$. The tensor product is defined by

$$u\otimes u = \begin{bmatrix} u_1^2 & u_1u_2\\ u_2u_1 & u_2^2\end{bmatrix}.$$

Equations (2.1a)–(2.1b) correspond to the Shallow Water Equations (SWE), characterized by the Froude number Fr. These equations are derived from the Navier–Stokes equations describing the motion of Newtonian fluids. They are widely used to model the dynamics of oceans and other incompressible fluids, under the assumption that the fluid depth is much smaller than the typical horizontal length scale. In essence, the SWE provide a simplified system capturing the horizontal evolution of an incompressible fluid under the effects of gravity and rotation.

Equation (2.1c) is a transport equation for the seabed evolution (see [10, 11, 20, 22]). It models the dynamics of sand dunes. In this context: D_G is the grain diameter, ρ is the water density, and α is a constant typically of order 10^2. The parameter $p \in [0,1)$ denotes the porosity of the sediment, and λ is the inverse of the maximum slope of the seabed profile when the water velocity is zero. The variable u_c is the threshold velocity below which no sediment transport occurs. The constant c is given by $c = \ln\left(\frac{12d}{3D_G}\right)$, where d is the water depth above the seabed (see [14]). The function χ is defined by $\chi(\sigma) = \begin{cases} 0, & \sigma < 0,\\ |\sigma|^{3/2}, & \sigma \geq 0.\end{cases}$ This equation is derived from the general equations of sediment dynamics, specialized here to the case of sand transport.

2.2. Scaling. The characteristic values used for scaling are the same as in [10], which we adapt for the shallow water equations. This subsection is organized as follows:

2.2.1. *New variables and functions:* They are defined as follows:
$\overline{t}\cdot t' = t$, $\overline{L}\cdot x' = x$, $\overline{\xi}\cdot\xi'(t',x') = \xi(\overline{t}t',\overline{L}x')$, $\overline{z}\cdot z'(t',x') = z(\overline{t}t',\overline{L}x')$,
$\overline{u}\cdot u'(t',x') = u(\overline{t}t',\overline{L}x')$, $\overline{M}\cdot m'(t',x') = \xi(\overline{t}t',\overline{L}x') - z(\overline{t}t',\overline{L}x')$.
Note that all variables with a prime ($'$) are dimensionless.

2.2.2. *Derivatives:* They are computed as follows.
Space derivative: $\nabla' z'(t',x') = \frac{\overline{L}}{\overline{z}}\nabla z(\overline{t}t',\overline{L}x')$,
time derivative: $\frac{\partial}{\partial t'}z'(t',x') = \frac{\overline{t}}{\overline{z}}\frac{\partial}{\partial t}z(\overline{t}t',\overline{L}x')$,
Divergence in time: $\mathrm{div}' = \overline{L}\mathrm{div}$.

2.2.3. *Scaling of SWE:*. Consider the first equation in (2.1a):

$$\frac{\partial}{\partial t}(\xi + H - z) + \nabla\cdot[(\xi + H - z)u] = 0.$$

Applying the scaling, we get

$$\frac{1}{\overline{t}}\frac{\partial}{\partial t'}(\overline{\xi}\xi' + H - \overline{z}z') + \nabla\cdot[(\overline{\xi}\xi' + H - \overline{z}z')\overline{u}u'] = 0.$$

Simplifying by $\frac{\overline{z}}{\overline{t}}$, we obtain

$$\frac{\partial}{\partial t'}\left(\frac{\overline{\xi}}{\overline{z}}\xi' + \frac{H}{\overline{z}} - z'\right) + \frac{\overline{t}\overline{u}}{\overline{L}}\nabla'\cdot\left[\left(\frac{\overline{\xi}}{\overline{z}}\xi' + \frac{H}{\overline{z}} - z'\right)u'\right] = 0. \tag{2.2}$$

The second equation in (2.1b) scales similarly, leading to:

$$\begin{aligned}\frac{\partial}{\partial t'}\left[\left(\frac{\overline{\xi}}{\overline{z}}\xi' - z' + \frac{H}{\overline{z}}\right)u'\right] &+ \frac{\overline{t}\overline{u}}{\overline{L}}\nabla'\cdot\left[\left(\frac{\overline{\xi}}{\overline{z}}\xi' - z' + \frac{H}{\overline{z}}\right)u'\otimes u'\right]\\ &+\frac{\overline{t}\overline{z}}{Fr^2\overline{L}\overline{u}}\left(\frac{\overline{\xi}}{\overline{z}}\xi' - z' + \frac{H}{\overline{z}}\right)\nabla'\left(\frac{\overline{\xi}}{\overline{z}}\xi' - z' + \frac{H}{\overline{z}}\right)\\ &+\frac{\nu\overline{t}}{\overline{L}}\mathrm{div}'\left[\left(\frac{\overline{\xi}}{\overline{z}}\xi' + \frac{H}{\overline{z}} - z'\right)D(u')\right] = 0.\end{aligned} \tag{2.3}$$

Thanks to the notation we have done previously, we get :

$$\begin{aligned}m' = \frac{1}{\overline{M}}m = \frac{1}{\overline{M}}(h - H) = \frac{1}{\overline{M}}(\xi - z) = \frac{1}{\overline{M}}(\overline{\xi}\xi' - \overline{z}z') = \frac{\overline{z}}{\overline{M}}(\frac{\overline{\xi}}{\overline{z}}\xi' - z')\\ \Rightarrow m' = \frac{\overline{z}}{\overline{M}}(\frac{\overline{\xi}}{\overline{z}}\xi' - z') \Rightarrow \frac{\overline{\xi}}{\overline{z}}\xi' - z' = \frac{\overline{M}}{\overline{z}}m'\end{aligned}$$

So replacing in (2.2) and (2.3) simplifying by $\frac{\overline{M}}{\overline{z}}$ and dropping the ($'$) we get :

$$\left\{\begin{aligned}&\frac{\partial\,(m + \frac{H}{\overline{M}})}{\partial\,t} + \frac{\overline{t}\,\overline{u}}{\overline{L}}\nabla\cdot[(m + \frac{H}{\overline{M}})\,u] &&= 0\\ &\frac{\partial}{\partial\,t}\left[(m + \frac{H}{\overline{M}})\,u\right] + \frac{\overline{t}\,\overline{u}}{\overline{L}}\nabla\cdot\left[(m + \frac{H}{\overline{M}})\,u\otimes u\right] + \\ &\quad\frac{\overline{t}\,\overline{M}}{Fr^2\overline{L}\,\overline{u}}(m + \frac{H}{\overline{M}})\nabla\,(m + \frac{H}{\overline{M}}) + \frac{\overline{t}\,\overline{z}}{Fr^2\overline{L}\,\overline{u}}(m + \frac{H}{\overline{M}})\nabla\,z - \\ &\qquad\qquad\frac{\nu\overline{t}\overline{z}}{\overline{L}\overline{M}}\mathrm{div}[(m + \frac{H}{\overline{M}})D(u)] &&= 0\end{aligned}\right. \tag{2.4}$$

2.2.4. *Scaling of LTDD.*. Since we use the scaling in the paper [10], the equation for small sand specks is given by :

$$\begin{aligned}&\frac{\partial z}{\partial t}-\frac{\lambda}{1-p}\,\alpha\,\frac{\bar t\,\overline{u}^3(\rho D_G)^{3/2}}{(\ln(\frac{4H}{D_G}))^3\overline{L}^2}\,\nabla\cdot[(1-3\frac{\overline{M}}{H\,\ln(\frac{4H}{D_G})}\,m)\,\chi(|u|^2-\frac{u_c^2}{\overline{u}^2})\nabla z]\\&=\frac{1}{1-p}\,\alpha\frac{\bar t\,\overline{u}^3(\rho D_G)^{3/2}}{(\ln(\frac{4H}{D_G}))^3\overline{L}\,\overline{z}}\,\nabla\cdot[(1-3\frac{\overline{M}}{H\,\ln(\frac{4H}{D_G})}\,m)\,\chi(|u|^2-\frac{u_c^2}{\overline{u}^2})\,\frac{u}{|u|}],\end{aligned}\tag{6}$$

For more details on the computations, see Faye et al [10] and references therein. Finally the dimensionless coupled system of SWE and LTDD is summed up by :

$$\begin{cases}\dfrac{\partial(m+\frac{H}{\overline{M}})}{\partial t}+\dfrac{\bar t\,\overline{u}}{\overline{L}}\,\nabla\cdot[(m+\dfrac{H}{\overline{M}})\,u] & =0\\ \dfrac{\partial}{\partial t}\left[(m+\dfrac{H}{\overline{M}})u\right]+\dfrac{\bar t\,\overline{u}}{\overline{L}}\,\nabla\cdot\left[(m+\dfrac{H}{\overline{M}})u\otimes u\right]+ & \\ \dfrac{\bar t\,\overline{M}}{Fr^2\overline{L}\,\overline{u}}\,(m+\dfrac{H}{\overline{M}})\,\nabla(m+\dfrac{H}{\overline{M}})+\dfrac{\bar t\,\overline{z}}{Fr^2\overline{L}\,\overline{u}}\,(m+\dfrac{H}{\overline{M}})\nabla z- & \\ \qquad\qquad\dfrac{\nu\bar t\overline{z}}{\overline{L}\overline{M}}\mathrm{div}[(m+\dfrac{H}{\overline{M}})D(u)] & =0\\ \dfrac{\partial z}{\partial z}-\dfrac{\lambda\alpha}{1-p}\,\dfrac{\bar t\,\overline{u}^3(\rho D_G)^{3/2}}{(\ln(\frac{4H}{D_G}))^3\overline{L}^2}\,\nabla\cdot[(1-3\dfrac{\overline{M}}{H\,\ln(\frac{4H}{D_G})}\,m)\,\chi(|u|^2-\dfrac{u_c^2}{\overline{u}^2})\nabla z] & \\ -\,\dfrac{\alpha}{1-p}\,\dfrac{\bar t\,\overline{u}^3(\rho D_G)^{3/2}}{(\ln(\frac{4H}{D_G}))^3\overline{L}\,\overline{z}}\,\nabla\cdot[(1-3\dfrac{\overline{M}}{H\,\ln(\frac{4H}{D_G})}\,m)\,\chi(|u|^2-\dfrac{u_c^2}{\overline{u}^2})\,\dfrac{u}{|u|})] & =0\end{cases}\tag{2.5}$$

2.3. **Parameter size.** In this subsection as for the scaling we use the same size of parameters for LTDD equation as in [10].
So we set :

$$\overline{u}=1m/s,H=50m,\overline{M}=5m\text{ and }\frac{\lambda}{1-p}=1\text{ and }\frac{1}{1-p}=2$$
$$\bar t\sim 16years\sim 5\cdot 10^9s,\quad \frac{1}{\overline{\omega}_c}\sim 1\text{ month }\sim 2.6\cdot 10^6s$$
$$D_G=7\cdot 10^{-5},\overline{z}=1m,\overline{L}=10m,u_c=0m/s,\nu=1,004.10^{-4}m^2/s.$$

Let's define $\epsilon=\frac{1}{\bar t\,\overline{\omega}_c}\sim\frac{1}{192}$; Then we are able to compute all the constants in the system SWE-LTDD :

$$\begin{aligned}\frac{H}{\overline{M}}=\frac{50}{5}=10,\quad \frac{\bar t\,\overline{u}}{\overline{L}}&=\frac{5\cdot 10^9}{10}=5\cdot 10^8\sim\frac{1}{3}\epsilon^{-4},\\ \frac{\bar t\,\overline{M}}{Fr^2\overline{L}\overline{u}}&=\frac{\bar t\,\overline{u}}{\overline{L}}\cdot\frac{\overline{M}}{\overline{u}^2}\frac{1}{Fr^2}\sim\frac{1}{3\epsilon^4}\cdot\frac{1}{1600}\cdot 5\sim\frac{1}{5\epsilon^4},\\ \frac{\bar t\,\overline{z}}{Fr^2\overline{L}\overline{u}}&=\frac{\bar t\,\overline{u}}{\overline{L}}\cdot\frac{\overline{z}}{\overline{u}^2}\cdot\frac{1}{Fr^2}\sim\frac{1}{3\epsilon^4}\cdot\frac{1}{1600}\cdot 1\sim\frac{1}{25\epsilon^4},\\ \frac{\nu\bar t\overline{z}}{\overline{L}\overline{M}}&=10^8\nu\sim\frac{1}{3}\frac{1}{\epsilon^2}.\end{aligned}$$

Now the system SWE-LTDD becomes :

$$\begin{cases} \dfrac{\partial(m+10)}{\partial t} + \dfrac{1}{3\epsilon^4}\,\nabla\cdot[(m+10)\,u] & = 0 \\ \dfrac{\partial}{\partial t}\,[(m+10)u] + \dfrac{1}{3\epsilon^4}\,\nabla\cdot[(m+10)u\otimes u]+ & \\ \dfrac{1}{5\epsilon^4}\,(m+10)\,\nabla(m+10) + \dfrac{1}{25\epsilon^4}\,(m+10)\nabla z - & \\ \qquad\qquad \dfrac{1}{3}\dfrac{1}{\epsilon^2}\mathrm{div}[(m+10)D(u)] & = 0 \\ \dfrac{\partial z}{\partial t} - \dfrac{1}{\epsilon^2}\nabla\cdot[(1-4\epsilon m)\nabla z]\dfrac{20}{\epsilon^2}\nabla\cdot[(1-4\epsilon m)\mid u\mid^2 u] & = 0 \end{cases} \tag{2.6}$$

Remark 2.1. : The unknowns m, u, and z depend on ϵ. Also, the methods of T. Kato [12], S. Klainerman, and A. Majda [14] will be used to study the system, but a symmetrization step is necessary to meet Friedrich's theory requirements. Note that the variables are denoted as $m = m_\epsilon$, $q = q_\epsilon$, and $z = z_\epsilon$.

3. Symmetrization of the Dimensionless model

In this section, we transform the dimensionless SWE into a matrix form and symmetrize it

$$\begin{cases} \frac{\partial m_\epsilon}{\partial t} + \frac{a_1}{\epsilon^4}\nabla\cdot[(m_\epsilon+b_1)u_\epsilon] = 0 \\ \frac{\partial}{\partial t}[(m_\epsilon+b_1)u_\epsilon] + \frac{a_1}{\epsilon^4}\nabla\cdot[(m_\epsilon+b_1)u_\epsilon\otimes u_\epsilon] + \frac{c_1}{\epsilon^4}(m_\epsilon+b_1)\nabla(m_\epsilon+b_1) \\ +\frac{d_1}{\epsilon^4}(m_\epsilon+b_1)\nabla z_\epsilon - \dfrac{a_2}{\epsilon^2}\mathrm{div}[(m+10)D(u_\epsilon)] = 0 \\ \frac{\partial z_\epsilon}{\partial t} - \frac{a}{\epsilon^2}\nabla\cdot[(1-b\epsilon m_\epsilon)|u_\epsilon|^3\nabla z_\epsilon] - \frac{c}{\epsilon^2}\nabla\cdot[(1-b\epsilon m_\epsilon)|u_\epsilon|^2 u_\epsilon] = 0 \end{cases} \tag{3.1}$$

where m_ϵ, z_ϵ, u_ϵ are the scaled variables and a_1, a_2, b_1, c_1, d_1, a, b, c are positive constants.

Using the change of variables:

$$q_\epsilon = (m_\epsilon + b_1)u_\epsilon$$

We can rewrite the equations in a more symmetric form. This symmetrization is necessary to apply the methods introduced by T. Kato, S. Klainerman, and A. Majda.

In the sequel we shall consider $t \in [0,T]$, $T > 0$ and the space variable $x = (x_1, x_2) \in T^2$, the 2-dimensional torus is $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Let us set the following change of variables :

$$q^\epsilon = (m^\epsilon + b_l)u^\epsilon = \begin{pmatrix} q_1^\epsilon \\ q_2^\epsilon \end{pmatrix} = \begin{pmatrix} (m^\epsilon + b_l)u_1^\epsilon \\ (m^\epsilon + b_l)u_2^\epsilon \end{pmatrix}$$

In the literature, the vector q^ϵ is used to call water discharge. So we can write

$$u_1^\epsilon = \frac{q_1^\epsilon}{m^\epsilon + b_l} \quad \text{and} \quad u_2^\epsilon = \frac{q_2^\epsilon}{m^\epsilon + b_l}$$

$$|u^\epsilon| = \sqrt{(u_1^\epsilon)^2 + (u_2^\epsilon)^2} = \sqrt{\left(\frac{q_1^\epsilon}{m^\epsilon + b_l}\right)^2 + \left(\frac{q_2^\epsilon}{m^\epsilon + b_l}\right)^2} = \frac{|q^\epsilon|}{m^\epsilon + b_l}$$

$$|u^\epsilon|^2 u^\epsilon = \frac{|q^\epsilon|^2 q^\epsilon}{(m^\epsilon + b_l)^3}$$

In this paper, we rely on the following realistic assumption: the dunes are submerged beneath the water. In other words, the average water height exceeds the height of any dune. This is expressed as $h > 0$ and $H > z(t,x)$ $\forall t, x$. Alternatively, this condition can be reformulated as:

$$\exists \nu > 0 \text{ such that } m^\epsilon + b_l \geq \nu > 0$$

Therefore the system (3.1) becomes:

$$\begin{cases} \dfrac{\partial m^\epsilon}{\partial t} + \dfrac{1}{\epsilon^4}\dfrac{\partial}{\partial x}\left[a_1 q_1^\epsilon\right] + \dfrac{1}{\epsilon^4}\dfrac{\partial}{\partial y}\left[a_1 q_2^\epsilon\right] = 0 \\ \dfrac{\partial q_1^\epsilon}{\partial t} + \dfrac{1}{\epsilon^4}\dfrac{\partial}{\partial x}\left(\dfrac{q_1^\epsilon q_1^\epsilon}{m_\epsilon + b_1}\right) + \dfrac{1}{\epsilon^4}\dfrac{\partial}{\partial y}\left(\dfrac{q_1^\epsilon q_2^\epsilon}{m_\epsilon + b_1}\right) + \dfrac{c_1}{\epsilon^4}(m_\epsilon + b_1)\dfrac{\partial m_\epsilon}{\partial x} + \dfrac{d_1}{\epsilon^4} q_1^\epsilon \dfrac{\partial z_\epsilon}{\partial x} \\ \quad - \dfrac{a_2}{\epsilon^2}\dfrac{\partial}{\partial x}\left[(m_\epsilon + b_1)\dfrac{\partial}{\partial x}\left(\dfrac{q_1^\epsilon}{m_\epsilon + b_1}\right)\right] - \dfrac{a_2}{\epsilon^2}\dfrac{\partial}{\partial y}\left[(m_\epsilon + b_1)\dfrac{\partial}{\partial y}\left(\dfrac{q_1^\epsilon}{m_\epsilon + b_1}\right)\right] = 0, \\ \dfrac{\partial q_2^\epsilon}{\partial t} + \dfrac{1}{\epsilon^4}\dfrac{\partial}{\partial x}\left(\dfrac{q_2^\epsilon q_1^\epsilon}{m_\epsilon + b_1}\right) + \dfrac{1}{\epsilon^4}\dfrac{\partial}{\partial y}\left(\dfrac{q_2^\epsilon q_2^\epsilon}{m_\epsilon + b_1}\right) + \dfrac{c_1}{\epsilon^4}(m_\epsilon + b_1)\dfrac{\partial m_\epsilon}{\partial y} + \dfrac{d_1}{\epsilon^4} q_2^\epsilon \dfrac{\partial z_\epsilon}{\partial y} \\ \quad - \dfrac{a_2}{\epsilon^2}\dfrac{\partial}{\partial x}\left[(m_\epsilon + b_1)\dfrac{\partial}{\partial x}\left(\dfrac{q_2^\epsilon}{m_\epsilon + b_1}\right)\right] - \dfrac{a_2}{\epsilon^2}\dfrac{\partial}{\partial y}\left[(m_\epsilon + b_1)\dfrac{\partial}{\partial y}\left(\dfrac{q_2^\epsilon}{m_\epsilon + b_1}\right)\right] = 0, \\ \dfrac{\partial z^\epsilon}{\partial t} - \dfrac{1}{\epsilon^2}\left[\dfrac{a(1 - b\epsilon m^\epsilon)|q^\epsilon|^3}{|m^\epsilon + b_1|^3}\right]\dfrac{\partial^2 z^\epsilon}{\partial x_1^2} - \dfrac{1}{\epsilon^2}\left[\dfrac{a(1 - b\epsilon m^\epsilon)|q^\epsilon|^3}{|m^\epsilon + b_1|^3}\right]\dfrac{\partial^2 z^\epsilon}{\partial x_2^2} \\ \quad - \dfrac{1}{\epsilon^2}\dfrac{\partial}{\partial x_1}\left[\dfrac{c(1 - b\epsilon m^\epsilon)|q^\epsilon|^2 q_1^\epsilon}{(m^\epsilon + b_1)^3}\right] - \dfrac{1}{\epsilon^2}\dfrac{\partial}{\partial x_2}\left[\dfrac{c(1 - b\epsilon m^\epsilon)|q^\epsilon|^2 q_2^\epsilon}{(m^\epsilon + b_1)^3}\right] \\ \quad - \dfrac{1}{\epsilon^2}\dfrac{\partial}{\partial x_1}\left[\dfrac{a(1 - b\epsilon m^\epsilon)|q^\epsilon|^3}{|m^\epsilon + b_1|^3}\right]\dfrac{\partial z^\epsilon}{\partial x_1} - \dfrac{1}{\epsilon^2}\dfrac{\partial}{\partial x_2}\left[\dfrac{a(1 - b\epsilon m^\epsilon)|q^\epsilon|^3}{|m^\epsilon + b_1|^3}\right]\dfrac{\partial z^\epsilon}{\partial x_2} = 0. \end{cases}$$

The above system can also be written in the following:

$$\begin{cases} \dfrac{\partial v^\epsilon}{\partial t} + \dfrac{1}{\epsilon^4}\displaystyle\sum_{j=1}^{2}\dfrac{\partial}{\partial x_j}F_j(v^\epsilon) = \dfrac{1}{\epsilon}h(v^\epsilon) - \dfrac{1}{\epsilon^4}P(v^\epsilon, z^\epsilon), \\ \dfrac{\partial z^\epsilon}{\partial t} - \dfrac{1}{\epsilon^2}\nabla\cdot(A^\epsilon(v^\epsilon)\nabla z^\epsilon) = \dfrac{1}{\epsilon^2}\nabla\cdot C^\epsilon(v^\epsilon), \end{cases} \tag{3.2}$$

where

$$v^\epsilon = \begin{pmatrix} m^\epsilon \\ q_1^\epsilon \\ q_2^\epsilon \end{pmatrix}, \quad h(v^\epsilon) = 0 \text{ and } F(v^\epsilon) = \begin{pmatrix} F_1(v^\epsilon) \\ F_2(v^\epsilon) \end{pmatrix}$$

$$F(v^\epsilon) = (F_1, F_2) = \begin{pmatrix} \frac{a_1 q_1^\epsilon q_1^\epsilon}{m^\epsilon + b^l} + \frac{c_1}{2}(m^\epsilon + b^l)^2 & \frac{a_1 q_1^\epsilon q_2^\epsilon}{m^\epsilon + b^l} \\ \frac{a_1 q_1^\epsilon q_2^\epsilon}{m^\epsilon + b^l} & \frac{a_1 q_2^\epsilon q_2^\epsilon}{m^\epsilon + b^l} + \frac{c_1}{2}(m^\epsilon + b^l)^2 \end{pmatrix}$$

$$\text{with} \quad F_1(v^\epsilon) = \begin{pmatrix} F_1^1 \\ F_1^2 \\ F_1^3 \end{pmatrix} = \begin{pmatrix} \frac{a_1 q_1^\epsilon q_1^\epsilon}{m^\epsilon + b^l} + \frac{c_1}{2}(m^\epsilon + b^l)^2 \\ \frac{a_1 q_1^\epsilon q_2^\epsilon}{m^\epsilon + b^l} \\ \frac{a_1 q_1^\epsilon q_2^\epsilon}{m^\epsilon + b^l} \end{pmatrix},$$

$$F_2(v^\epsilon) = \begin{pmatrix} a_1 q_2^\epsilon \\ \frac{a_1 q_1^\epsilon q_2^\epsilon}{m^\epsilon + b^\epsilon} \\ \frac{a_1 (q_2^\epsilon)^2}{m^\epsilon + b^\epsilon} + \frac{c_1}{2}(m^\epsilon + b^\epsilon)^2 \end{pmatrix},$$

$$A^\epsilon(v^\epsilon) = \frac{a(1 - b\epsilon m^\epsilon)|q^\epsilon|^3}{(m^\epsilon + b^\epsilon)^3}, \qquad C^\epsilon(v^\epsilon) = \begin{pmatrix} c(1 - b\epsilon m^\epsilon)\frac{|q^\epsilon|^2 q_1^\epsilon}{(m^\epsilon + b^\epsilon)^3} \\ c(1 - b\epsilon m^\epsilon)\frac{|q^\epsilon|^2 q_2^\epsilon}{(m^\epsilon + b^\epsilon)^3} \end{pmatrix}.$$

Let us remark that the equation

$$\frac{\partial v^t}{\partial t} + \frac{1}{\varepsilon}\sum_{j=1}^{2}\frac{\partial}{\partial x_j}F_j(v^t) = \frac{1}{\varepsilon}H(\varepsilon) - \frac{1}{\varepsilon}P(v^t, z^\epsilon) \tag{3.3}$$

is a hyperbolic system of first order with source term. It is usually called balance law equation. And the following one:

$$\frac{\partial v^\epsilon}{\partial t} + \frac{1}{\epsilon^2}\nabla\cdot[A^\epsilon\nabla z^\epsilon] = \frac{1}{\epsilon^2}\nabla\cdot C^\epsilon \tag{3.4}$$

is parabolic but may exhibit singular or degenerate behavior for specific values of ε. For instance, when $\varepsilon = 0$, the equation becomes singular, and when $A^\varepsilon = 0$ for certain values of ε, equation (3.4) is degenerate. Assuming that v^ε is sufficiently smooth, at least in the distributional sense, we can then state:

$$\frac{\partial}{\partial x_j}F_j(v^\epsilon) = D_xF_j(v^\epsilon)\cdot\frac{\partial v^\epsilon}{\partial x_j}, \quad \forall j = 1, 2.$$

In this case the system takes the following expression:

$$\frac{\partial v^\epsilon}{\partial t} + \frac{1}{\epsilon^4}\sum_{j=1}^{2}D_xF_j(v^\epsilon)\frac{\partial v^\epsilon}{\partial x_j} = \frac{1}{\epsilon}H(\epsilon) - \frac{1}{\epsilon}P(v^\epsilon, z^\epsilon) \tag{3.5}$$

where $D_vF_j(v^\epsilon) = \begin{pmatrix} \frac{\partial m^\epsilon F_j^1}{\partial m^\epsilon} & \frac{\partial m^\epsilon F_j^1}{\partial q_1^\epsilon} & \frac{\partial m^\epsilon F_j^1}{\partial q_2^\epsilon} \\ \frac{\partial m^\epsilon F_j^2}{\partial m^\epsilon} & \frac{\partial m^\epsilon F_j^2}{\partial q_1^\epsilon} & \frac{\partial m^\epsilon F_j^2}{\partial q_2^\epsilon} \\ \frac{\partial m^\epsilon F_j^3}{\partial m^\epsilon} & \frac{\partial m^\epsilon F_j^3}{\partial q_1^\epsilon} & \frac{\partial m^\epsilon F_j^3}{\partial q_2^\epsilon} \end{pmatrix}$.

A simple calculation yields:

$$D_vF_1(v^\epsilon) = \begin{pmatrix} 0 & a^1 & 0 \\ -\frac{a^1 {q_1^\epsilon}^2}{(m^\epsilon+b^1)^2} + \frac{c^1(m^\epsilon+b^1)}{2} & \frac{2a^1q_1^\epsilon}{m^\epsilon+b^1} & 0 \\ -\frac{a^1q_1^\epsilon q_2^\epsilon}{(m^\epsilon+b^1)^2} & \frac{a^1q_2^\epsilon}{m^\epsilon+b^1} & \frac{a^1q_1^\epsilon}{m^\epsilon+b^1} \end{pmatrix}$$

and

$$D_vF_2(v^\epsilon) = \begin{pmatrix} 0 & 0 & a^1 \\ -\frac{a^1q_1^\epsilon q_2^\epsilon}{(m^\epsilon+b^1)^2} & \frac{a^1q_2^\epsilon}{m^\epsilon+b^1} & \frac{a^1q_1^\epsilon}{m^\epsilon+b^1} \\ -\frac{a^1 {q_2^\epsilon}^2}{(m^\epsilon+b^1)^2} + \frac{c^1(m^\epsilon+b^1)}{2} & 0 & \frac{2a^1q_2^\epsilon}{m^\epsilon+b^1} \end{pmatrix}$$

.

Setting $A^j(v^\epsilon) = D_xF_j(v^\epsilon)$ for $j = 1, 2$, we have:

$$\frac{\partial v^\epsilon}{\partial t} + \frac{1}{\varepsilon^4}\sum_{j=1}^{2}A^j(v^\epsilon)\frac{\partial v^\epsilon}{\partial x_j} = \frac{1}{\varepsilon}H(\epsilon) - \frac{1}{\epsilon}P(v^\epsilon, z^\epsilon). \tag{3.6}$$

Let us consider the matrix B^0 defined by:

$$B^0 = \begin{pmatrix} \frac{1}{a^1}\left[\frac{a^1 q_1^{\epsilon 2}}{(m^\epsilon+b^1)^2} + c^1(m^\epsilon+b^1)\right] & -\frac{q_1^\epsilon}{m^\epsilon+b^1} & -\frac{q_2^\epsilon}{m^\epsilon+b^1} \\ -\frac{q_1^\epsilon}{m^\epsilon+b^1} & 1 & 0 \\ -\frac{q_2^\epsilon}{m^\epsilon+b^1} & 0 & 1 \end{pmatrix}$$

Multiplying (3.6) by B^0, we have:

$$B^0\frac{\partial v^\epsilon}{\partial t}+\frac{1}{\epsilon^4}\sum_{j=1}^{2}B^j\cdot\frac{\partial v^\epsilon}{\partial x_j}=\frac{1}{\epsilon}B^0\cdot h(v^\epsilon)-\frac{1}{\epsilon^4}B^0\cdot P(v^\epsilon,z^\epsilon) \tag{3.7}$$

where $B^j=B^0A^j$, $\forall j=1,2$. B^1 and B^2 are matrices given respectively by the following expressions:

$$\begin{pmatrix} \frac{q_1^\epsilon}{m^\epsilon+b^1}\left[\frac{a^1q_1^{\epsilon 2}}{(m^\epsilon+b^1)^2}-c^1(m^\epsilon+b^1)\right] & -\frac{a^1q_1^{\epsilon 2}}{(m^\epsilon+b^1)^2}+c^1(m^\epsilon+b^1) & -\frac{a^1q_1^\epsilon q_2^\epsilon}{(m^\epsilon+b^1)^2}\\ -\frac{a^1q_1^{\epsilon 2}}{(m^\epsilon+b^1)^2}+c^1(m^\epsilon+b^1) & -\frac{a^1q_1^\epsilon}{m^\epsilon+b^1} & 0\\ -\frac{a^1q_1^\epsilon q_2^\epsilon}{(m^\epsilon+b^1)^2} & 0 & -\frac{a^1q_1^\epsilon}{m^\epsilon+b^1} \end{pmatrix}$$

$$\begin{pmatrix} \frac{q_2^\epsilon}{m^\epsilon+b^1}\left[\frac{a^1q_1^{\epsilon 2}}{(m^\epsilon+b^1)^2}-c^1(m^\epsilon+b^1)\right] & -\frac{a^1q_1^\epsilon q_2^\epsilon}{(m^\epsilon+b^1)^2} & -\frac{a^1q_2^{\epsilon 2}}{(m^\epsilon+b^1)^2}+c^1(m^\epsilon+b^1)\\ -\frac{a^1q_1^\epsilon q_2^\epsilon}{(m^\epsilon+b^1)^2} & -\frac{a^1q_2^\epsilon}{m^\epsilon+b^1} & 0\\ -\frac{a^1q_2^{\epsilon 2}}{(m^\epsilon+b^1)^2}+c^1(m^\epsilon+b^1) & 0 & -\frac{a^1q_2^\epsilon}{m^\epsilon+b^1} \end{pmatrix}$$

And

$$h(v^\epsilon)=0.$$

Finally (3.7) becomes

$$B^0\frac{\partial v^\epsilon}{\partial t}+\frac{1}{\epsilon^4}\sum_{j=1}^{2}B^j\cdot\frac{\partial v^\epsilon}{\partial x_j}=\frac{1}{\epsilon}\tilde{h}(v^\epsilon)-\frac{1}{\epsilon^4}\tilde{p}(v^\epsilon,z^\epsilon), \tag{3.8}$$

avec

$$\tilde{p}(v^\epsilon,z^\epsilon)=B^0\cdot P(v^\epsilon,z^\epsilon).$$

So having at hands the system of equations (3.4)-(3.8), we are in situation to use [3] and [14] to prove existence and uniqueness of solution with suitable initial conditions.

4. Existence, uniqueness and estimates

In this section, we present an existence and uniqueness theorem for the coupled SWE-LTDD system (Shallow Water and Long-Term Dynamics of Dunes of sand), along with solution estimates for the model. These estimates serve as the foundation for deriving the main theorem in the homogenization section.

Before presenting our findings, we note that [4] provides an exact smooth solution for the coupled shallow water–Exner equation under a uniform water discharge. By making the same assumptions as in [4], the approach to obtain an exact smooth solution remains valid for our system. However, in this work, we do not fix the water discharge. Instead, using the methods from [14], we establish existence, uniqueness, and estimation results for the global system.

As the small parameter ε approaches 0 in the coupled system, the coefficients of the system may develop singularities and become unbounded. As a result, the existence interval for classical solutions of (16) can rapidly diminish to zero as $\varepsilon\to 0$. This scenario fits within the framework outlined in [14], allowing us to apply the methodology introduced by S. Klainerman and A. Majda to our system. Before moving forward, we will present some definitions and notations, which are provided in their general form in [14]. We define a parameter $d(\varepsilon)$ such that $\lim\limits_{\varepsilon\to 0}d(\varepsilon)=+\infty$, along with the corresponding square norm of a splitting vector.

$$w :=^t (w_1, w_2), \quad w_1 \in \mathbb{R}, \quad w_2 \in \mathbb{R}^2 :$$
$$\|w\|_\varepsilon^2 = d(\varepsilon)^2 |w_1|^2 + \|w_2\|^2.$$

Let D be the matrix associated to this splitting of w:

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}.$$

Then we can define the norms involving D as follows:

$$\|D\|_\varepsilon = |D_{11}| + d(\varepsilon)|D_{12}| + d(\varepsilon)^{-1}|D_{21}| + |D_{22}|,$$
$$[D]_\varepsilon = d(\varepsilon)^{-2}|D_{11}| + d(\varepsilon)^{-1}(|D_{12}| + |D_{21}|) + |D_{22}|.$$

It is straightforward to show that there exists a constant $C > 0$ such that:

$$\|Dw\|_\varepsilon \leq C\|D\|_\varepsilon \cdot \|w\|_\varepsilon,$$
$$|(Dw, w)| \leq C[D]_\varepsilon \|w\|_\varepsilon^2,$$

for any splitting vector w as defined earlier.

Additionally, let us define the corresponding Sobolev norm as follows:

$$\|w\|_{s,\epsilon}^2 = d(\epsilon)^2 \|w_1\|_s^2 + \|w_2\|_s^2,$$

for integers s such that $s \geq s_0 + 1$ where $s_0 = 1$.

For the functional space $C([0,T], H^s(\mathbb{T}^2, \mathbb{R}^3)) \cap C^1([0,T], H^{s-1}(\mathbb{T}^2, \mathbb{R}^3))$, we consider the following norms:

$$|||w|||_{\varepsilon,T} = \sup_{t \in [0,T]} |||w(t)|||_\varepsilon$$

and

$$|||w(t)|||_\varepsilon = \|w(t)\|_{s,\varepsilon} + \left\| \frac{dw}{dt}(t) \right\|_{s-1,\varepsilon}.$$

As demonstrated in subsection 3, the system (3.6) adheres to Friedrich's theory. This allows us to introduce a key concept for our work: the structural conditions known as the ϵ-balanced property, which must be fulfilled to ensure the validity of the method presented in [14].

Definition 4.1. The system (3.6) is said ϵ-balanced around v^0, a given fixed vector, if the following structural conditions are satisfied:
There exists $\delta > 0$ independent of ϵ such that

$$\theta_{1,s}(A^j, v^0, \delta), \theta_{1,s}(h, v^0, \delta), \theta_{1,s}(P, v^0, \delta) < \infty \quad \text{for } j = 1, 2 \text{ and } s \geq s_0 + 1, \tag{4.1}$$

$$\gamma_{1,1}(B^j, v^0, \delta) < \infty \quad \text{for } j = 0, 1, 2, \tag{4.2}$$

where

$$\begin{aligned} \theta_{.,1,s}(v^0, \delta) &= \theta_{1,s}(\cdot) \\ &= \max_{\epsilon \to 0} \left[\sum_{1 \leq |s_1| + |s_2| \leq s} d(\epsilon)^{-|s_1|} \max_{\|p - v^0\|_\epsilon \leq \delta} \left\| D_{p_1}^{s_1} D_{p_2}^{s_2} \cdot (p_1, p_2) \right\| \right], \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \gamma_{.,1,s}(v^0, \delta) &= \gamma_{1,s}(\cdot) \\ &= \max_{\epsilon \to 0} \left[\sum_{1 \leq |s_1| + |s_2| \leq s} d(\epsilon)^{-|s_1|} \max_{\|p - v^0\|_\epsilon \leq \delta} [D_{p_1}^{s_1} D_{p_2}^{s_2} \cdot (p_1, p_2)]_\epsilon \right]. \end{aligned} \tag{4.4}$$

Since $B^0(v^0,\varepsilon)$ is symmetric definite positive, $(B^0(v^0,\varepsilon)w,w)^{1/2}$ defines a norm in $\mathbb{R}^3$ and by the equivalence of norms in finite dimension spaces, we get the relation:

$$m\|w\|_\epsilon^2 \le (B^0(v^0,\varepsilon)w,w) \le M\|w\|_\epsilon^2, \tag{4.5}$$

for a given v^0.

Remark 4.2. $B^0(v^0,\varepsilon)$ is symmetric, which implies that m and M can be, respectively, its smallest and largest eigenvalues. By its definite positivity, we get $m>0$ and $M>0$.

The weighted matrix norm verifies:

$$\|Bw\|_\varepsilon \le C\|B\|_\varepsilon\|w\|_\varepsilon, \tag{4.6}$$

$$|(Bw,w)| \le C[B]_\varepsilon\|w\|_\varepsilon\|w\|_\varepsilon, \tag{4.7}$$

where $(\cdot,\cdot)$ is the usual inner product in $\mathbb{R}^3$. Before proceeding further, we make the following assumptions as in [14]. We consider the initial value condition:

$$v(t=0,x)=v^0(x)+\tilde{v}^0(x), \quad \text{with } v^0=(v_0^I,v_0^{II})$$

such that:

$$\begin{aligned}
&\text{I.}\quad v_0^I \text{ constant}, \ \|v_0^{II}\|_{s,\varepsilon} + \left\|\sum_{j=1}^2 A^j(v^0,\varepsilon)\frac{\partial v^0}{\partial x_j}\right\|_{s-1,\varepsilon} \le K,\\
&\text{II.}\quad \left\|\sum_{j=1}^2 A^j(v^0,\varepsilon)\frac{\partial}{\partial x_j}\right\|_{s-1,\varepsilon} \le K,\\
&\text{III.}\quad \|\tilde{v}^0\|_{s,\varepsilon} \le \delta',
\end{aligned} \tag{4.8}$$

where $\delta'>0$ is considered small enough and to be chosen later.

The initial condition (4.8) implies that v^0 satisfies the following:

$$\begin{aligned}
&\text{I.}\quad v_0 \in H^s(\mathbb{T}^2,\mathbb{R}^3), \quad s \ge s_0+1,\\
&\text{II.}\quad v_0^I \text{ constant},\\
&\text{III.}\quad \left\|\sum_{j=1}^2 A^j(v_0,\varepsilon)\frac{\partial v_0}{\partial x_j}\right\|_{s-1,\varepsilon} \le K.
\end{aligned} \tag{4.9}$$

Remark 4.3. rem3 $\tilde{v}^0$ represents a slight variation of the initial data v_0. If the initial condition v_0 is modified to $v_0+\tilde{v}^0$, the primary results presented in [14] remain applicable.

The first main result is the following:

Theorem 4.4. *Let's assume that $v^\varepsilon(0,x)$ satisfies (4.8), $z^\varepsilon(0,x)=z^0(x)$ such that $z^0\in H^1(\mathbb{T}^2)$. Then there exists $T>0$ not depending on ε such that for any ε close to 0, the following Cauchy problem:*

$$\begin{aligned}
&\frac{\partial v^\varepsilon}{\partial t} + \frac{1}{\varepsilon^4}\sum_{j=1}^2 A^j(v^\varepsilon)\frac{\partial v^\varepsilon}{\partial x_j} = \frac{1}{\varepsilon}h(v^\varepsilon) - \frac{1}{\varepsilon^4}P(v^\varepsilon,z^\varepsilon),\\
&\frac{\partial z^\varepsilon}{\partial t} - \frac{1}{\varepsilon^2}\nabla\cdot[\mathcal{A}^\varepsilon(v^\varepsilon)\nabla z^\varepsilon] = \frac{1}{\varepsilon^2}\nabla\cdot\mathcal{C}^\varepsilon(v^\varepsilon),\\
&v^\varepsilon(0,x)=v^0(x)+\tilde{v}^0(x), \quad z^\varepsilon(0,x)=z^0(x),
\end{aligned} \tag{4.10}$$

has a unique solution
$(\overline{v}^\varepsilon,\overline{z}^\varepsilon) \in C([0,T],H^s(\mathbb{T}^2,\mathbb{R}^3)) \cap C^1([0,T],H^{s-1}(\mathbb{T}^2,\mathbb{R}^3)) \cap L^\infty([0,T],L^2(\mathbb{T}^2))$.

And we have

$$\sqrt{A^\varepsilon(v^\varepsilon)\nabla z^\varepsilon} \in L^2([0,T), L^2(\mathbb{T}^2)).$$

In addition, the following estimates hold:

$$\|\overline{z}^\varepsilon\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \leq \tilde{\gamma}',$$

$$\|\sqrt{A^\varepsilon(v^\varepsilon)}\|\nabla z^\varepsilon\|\|^2_{L^2([0,T],L^2(\mathbb{T}^2))} \leq \tilde{\gamma}',$$

$$\|\overline{v}^\varepsilon(t) - v^0\|_{s,\varepsilon} + \left\|\frac{dv^\varepsilon}{dt}(t)\right\|_{s-1,\varepsilon} \leq \Delta,$$

for any $t \in [0,T]$, Δ a constant independent of ε, and $\tilde{\gamma}, \tilde{\gamma}'$ constants that depend on ε.

Proof of Theorem 4.4. We divide the proof into several steps.

Step 1. Symmetrization and norms. From Section 3, the system admits a symmetrizer $B_0(v_\varepsilon)$, which ensures the equivalence

$$m\|w\|^2_\varepsilon \;\leq\; (B_0(v_\varepsilon)w, w) \;\leq\; M\|w\|^2_\varepsilon,$$

for some $0 < m < M$, uniformly in ε (see inequality (4.5)). Here we use the weighted Sobolev norms $\|\cdot\|_{s,\varepsilon}$ introduced in Section 4, which are designed to balance the fast and slow components when $\varepsilon \to 0$.

Step 2. Technical tools. We recall three standard estimates (see Appendix A):

- *Product estimate:* $\|fg\|_{s,\varepsilon} \lesssim \|f\|_{s,\varepsilon}\|g\|_{L^\infty} + \|g\|_{s,\varepsilon}\|f\|_{L^\infty}$.
- *Composition estimate:* $\|F(v)\|_{s,\varepsilon} \lesssim 1 + \|v\|_{s,\varepsilon}$ if F is smooth and v stays in a bounded set.
- *Commutator estimate:* $\|[D^\alpha, A(v)]\partial_{x_j} w\|_{L^2_\varepsilon} \lesssim \|v\|_{s,\varepsilon}\|w\|_{s,\varepsilon}$.

These inequalities allow us to control nonlinearities uniformly in ε.

Step 3. Energy estimate for v_ε. Applying D^α to the hyperbolic part and multiplying by $B_0(v_\varepsilon)D^\alpha v_\varepsilon$ gives, after integration by parts,

$$\frac{d}{dt}\|v_\varepsilon\|^2_{s,\varepsilon} \;\lesssim\; \|v_\varepsilon\|^2_{s,\varepsilon} + \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon^4}(1 + \|z_\varepsilon\|^2_{H^s}).$$

The main point is that the ε^{-4} terms can still be controlled thanks to the weighted norms and the balanced structure of the system.

Step 4. Estimate for z_ε. Multiplying the parabolic equation by z_ε and integrating yields

$$\frac{d}{dt}\|z_\varepsilon\|^2_{L^2} + \frac{1}{\varepsilon^2}\|\sqrt{A_\varepsilon(v_\varepsilon)}\nabla z_\varepsilon\|^2_{L^2} \;\lesssim\; \frac{1}{\varepsilon^2}\|C_\varepsilon(v_\varepsilon)\|^2_{L^2}.$$

Higher-order versions obtained by differentiating show that

$$\frac{d}{dt}\|z_\varepsilon\|^2_{H^{s-1}} + \frac{1}{\varepsilon^2}\|\sqrt{A_\varepsilon(v_\varepsilon)}\nabla z_\varepsilon\|^2_{H^{s-1}} \;\lesssim\; \frac{1}{\varepsilon^2}\big(1 + \|v_\varepsilon\|^2_{s,\varepsilon} + \|z_\varepsilon\|^2_{H^{s-1}}\big).$$

Step 5. Combined estimate. Defining the total energy

$$\mathcal{E}^\varepsilon_s(t) = \|v_\varepsilon(t)\|^2_{s,\varepsilon} + \kappa\|z_\varepsilon(t)\|^2_{H^{s-1}},$$

with $\kappa > 0$ small, one obtains

$$\frac{d}{dt}\mathcal{E}^\varepsilon_s(t) \;\lesssim\; C(\rho)\,\mathcal{E}^\varepsilon_s(t) + C(\rho),$$

where ρ is a uniform bound on $\|v_\varepsilon\|_{s,\varepsilon}$. By Gronwall's lemma and the initialization condition (4.8), there exists $T > 0$ independent of ε such that $\mathcal{E}^\varepsilon_s(t) \leq \Delta^2$ for $t \in [0,T]$.

Step 6. Iteration and uniqueness. We solve the system by an iterative scheme with frozen coefficients. The linearized problems are well posed and remain in the energy ball of radius Δ. Moreover, the mapping is a contraction in

$$X_T = C([0,T], H^s) \cap C^1([0,T], H^{s-1}),$$

for T small enough (independent of ε). This yields existence of a unique solution $(v_\varepsilon, z_\varepsilon)$. Uniqueness in the stated class follows from the same energy estimate applied to the difference of two solutions.

Conclusion. We have established the existence of $T > 0$ independent of ε, a unique solution $(v_\varepsilon, z_\varepsilon)$ in the announced functional space, and the uniform estimates claimed in the theorem. □

5. Asymptotic behavior

In this section, we will demonstrate that the solution of the dimensionless Shallow Water and Long-Term Dynamics of Dunes (SWE-LTDD) system under 2-scale convergence approaches a profile that satisfies a homogenized SWE-LTDD system.
To simplify the notation for function spaces, $L^p_{\text{per}}(R, X)$, $p \in [1, \infty]$, and $C^s_{\text{per}}(R, X)$ will respectively denote the spaces of L^p and C^s 1-periodic functions valued in X.
Let's begin with what we understand by 2-scale convergence according to the definitions of Allaire [1] and Nguetseng [19].

Definition 5.1. Let $(v^\epsilon) \in L^\infty([0,T), L^2(\mathbb{T}^2, \mathbb{R}^3))$ be a sequence of functions. It two-scale converges to $V \in L^\infty([0,T), L^\infty_{\text{per}}(\mathbb{R}, L^2(\mathbb{T}^2, \mathbb{R}^3)))$ if for every $\psi \in C([0,T], C_{\text{per}}(\mathbb{R}, L^2(\mathbb{T}^2, \mathbb{R}^3)))$, we have:

$$\lim_{\varepsilon \to 0} \int_{\mathbb{T}^2} \int_0^T v^\epsilon(t,x) \psi\left(t, \frac{t}{\varepsilon}, x\right) dx\, dt \to \int_{\mathbb{T}^2} \int_{0^T} \int_0^1 V(t,\theta,x)\psi(t,\theta,x)\, d\theta\, dx\, dt.$$

For the sake of simplifying, we shall note this limit by:

$$v^\epsilon \overset{2-\mathrm{s}}{\to} V.$$

Definition 5.2. A sequence of functions (v^ϵ) in $L^\infty([0,T), L^2(\mathbb{T}^2, \mathbb{R}^3))$ is said to strongly two-scale converge to $V \in L^\infty([0,T), L^\infty_{\text{per}}(\mathbb{R}, L^2(\mathbb{T}^2, \mathbb{R}^3)))$ if

$$v^\epsilon \overset{2-\mathrm{s}}{\rightharpoonup} V, \quad \text{and} \quad \lim_{\epsilon \to 0} \|v^\epsilon\|_{L^\infty([0,T), L^2(\mathbb{T}^2, \mathbb{R}^3))} = \|V\|_{L^\infty([0,T), L^2_{\text{per}}(\mathbb{R}, L^2(\mathbb{T}^2, \mathbb{R}^3)))}.$$

We follow the ideas developed in [3]. Before going on, let's recall the following theorem from [3] which we plan to use if the hypothesis is satisfied.

Theorem 5.3. *If a sequence of functions (v^ϵ) is bounded in $L^\infty([0,T), L^2(\mathbb{T}^2, \mathbb{R}^3))$, there exists a sub-sequence still denoted (v^ϵ) and a function*

$$V \in L^\infty([0,T), L^2_{per}(\mathbb{R}, L^2(\mathbb{T}^2, \mathbb{R}^3)))$$

such that

$$v^\epsilon \overset{2-s}{\rightharpoonup} V.$$

From the proof of theorem 4.4 we can see that:

$$(\overline{v}^\epsilon, \overline{z}^\epsilon) \in L^\infty([0,T), H^s(\mathbb{T}^2, \mathbb{R}^3)) \cap Lip([0,T), H^{s-1}(\mathbb{T}^2, \mathbb{R}^3)) \times L^\infty([0,T], L^2(\mathbb{T}^2)),$$

and the solution satisfies the estimate condition:

$$\|\overline{v}^\epsilon(t) - v^0\|_{1-s_0,\mathcal{E}} + \left\| \frac{\partial \overline{v}^\epsilon}{\partial t} \right\|_{1-s,\varepsilon} \leq \Delta.$$

For all $t \in [0,T]$ and Δ not depending on ϵ.

Having this estimate at hand, we can deduce estimates with the classical Sobolev norm in H^s and H^{s-1}. Then, we have:

$$\|\overline{v}^\epsilon(t) - v^0\|_{H^s} + \left\|\frac{\partial \overline{v}^\epsilon}{\partial t}\right\|_{H^{s-1},\varepsilon} \leq \Delta.$$

This implies that

$$\|\overline{v}^\epsilon(t) - v^0\|_{H^s} + \left\|\frac{\partial \overline{v}^\epsilon}{\partial t}\right\|_{H^{s-1}} \leq C,$$

with C not depending on ϵ.

For mathematical reasons, we can also take $\bar{v}^\epsilon$ in the space $L^\infty([0,T), L^2(\mathbb{T}^2, \mathbb{R}^3))$ with the following estimates:

$$\|\overline{v}^\epsilon(t) - v^0\|_0 + \left\|\frac{\partial \overline{v}^\epsilon}{\partial t}\right\|_0 \leq C.$$

. And one needs to do a careful analysis. Before stating the homogenization theorem, let's give useful remarks and hypotheses for our framework of study.

$$\begin{cases} \text{For any } \epsilon > 0, \text{ the source term of the dimensionless LTDD} \\ \text{can't become infinite. This means: } \exists C > 0\, \forall \epsilon \in (0,1] \text{ small} \\ \text{ such that} |\nabla \cdot \mathcal{C}^\epsilon(v^\epsilon)| \leq \epsilon^2 C. \end{cases} \tag{H}$$

Theorem 5.4. *Under all the assumptions as in theorem 3.2, we have:*

$$\begin{cases} \overline{v}^\epsilon \overset{2-s}{\rightharpoonup} \bar{v} \in L^\infty([0,T), L^2_{per}(\mathbb{R}, L^2(\mathbb{T}^2, \mathbb{R}^3))), \\ \overline{z}^\epsilon \overset{2-s}{\rightharpoonup} \bar{z} \in L^\infty([0,T), L^2_{per}(\mathbb{R}, L^2(\mathbb{T}^2))). \end{cases}$$

And the limit functions satisfy:

$$\begin{cases} \bar{B}^1 \frac{\partial \bar{v}}{\partial x_1} + \bar{B}^2 \frac{\partial \bar{v}}{\partial x_2} = -\bar{p}(\bar{v}, \bar{z}) & \text{on } [0,T) \times \mathbb{R} \times \mathbb{T}^2, \\ -\nabla \cdot (\tilde{\mathcal{A}} \nabla \bar{z}) = \nabla \cdot \bar{\mathcal{C}} & \text{on } [0,T) \times \mathbb{R} \times \mathbb{T}^2, \end{cases} \tag{59}$$

with

$$B^1(t, \frac{t}{\epsilon}, x) \overset{2-s}{\rightharpoonup} \bar{B}^1(t,\theta,x), \quad B^2(t, \frac{t}{\epsilon}, x) \overset{2-s}{\rightharpoonup} \bar{B}^2(t,\theta,x), \quad \tilde{p}(v^\epsilon, z^\epsilon) \overset{2-s}{\rightharpoonup} \bar{p}(\bar{v}, \bar{z})$$

$$\mathcal{A}^\epsilon(\bar{v}^\epsilon) = \mathcal{A}(t, \frac{t}{\epsilon}, x) \overset{2-s}{\rightharpoonup} \tilde{\mathcal{A}}(t,\theta,x), \quad \mathcal{C}^\epsilon(\bar{v}^\epsilon) = \mathcal{C}(t, \frac{t}{\epsilon}, x) \overset{2-s}{\rightharpoonup} \tilde{\mathcal{C}}(t,\theta,x).$$

Proof Sketch of Theorem 5.4. The core of the argument hinges on the powerful framework of two-scale convergence, which is perfectly suited for analyzing systems with rapid oscillations, like our own. The journey begins with the uniform estimates we previously established in Theorem 3.2. These estimates are crucial; they act as an anchor, ensuring that the sequences of solutions $\{\overline{v}^\varepsilon\}$ and $\{\overline{z}^\varepsilon\}$ remain well-behaved and do not degenerate as the parameter ε tends to zero.

This boundedness is the key that unlocks the next step: by fundamental compactness results, we can guarantee that these sequences converge in a suitable weak sense. Specifically, they two-scale converge to some limiting profiles, which we denote $\overline{v}$ and $\overline{z}$. Intuitively, this means the complex, oscillatory behavior of the solutions averages out to a well-defined macroscopic limit.

The final and most delicate phase is to pass to the limit within the equations themselves. The structure of our system—particularly the way the nonlinear terms and oscillating coefficients are carefully balanced by the ε-scaling—ensures that this limiting process is well-defined. The rapid oscillations in the coefficients are averaged out, and what emerges is the effective homogenized system. The effective operators $\overline{B}^1$, $\overline{B}^2$, $\widetilde{\mathcal{A}}$, and $\overline{\mathcal{C}}$ that appear in the limit are not arbitrary; they are precisely the averaged counterparts of the original, oscillating coefficients, capturing the essential influence of the small-scale dynamics on the large-scale behavior. □

6. Conclusion

In this work, we proposed a novel model coupling the shallow water equations with long-term dune dynamics, including viscous and modified pressure terms to better capture fluid–sediment interactions. We demonstrated that the system admits unique and stable solutions, and its ε-balanced structure ensures robustness.

Through asymptotic analysis and two-scale convergence, we derived a homogenized system that accurately describes large-scale sediment transport, confirming the model's relevance for long-term sedimentation processes.

Looking ahead, we plan to extend the framework to three-dimensional flows. On the numerical side, designing efficient and stable schemes for this coupled system remains a challenging and promising direction.

7. Competing interests

The authors declare that there is no conflict of interest regarding the publication of the paper.

Appendix A. Technical Estimates in Weighted Sobolev Norms

Gathering the auxiliary inequalities that are frequently utilised in the proof of Theorem 4.4 is the goal of this appendix. The weighted Sobolev norms $\|\cdot\|_{s,\varepsilon}$ described in Section 4 are conventional extensions of Moser-type calculus, see [18]. Specifically, they enable us to manage commutators and nonlinear terms while maintaining uniformity of all constants with regard to ε. Similar structures will be recognised by the reader who is familiar with Klainerman–Majda [14], but we offer the details here for completeness.

Lemma .1 (Product estimate). *Let $s \geq 3$. For smooth functions f, g, one has*

$$\|fg\|_{s,\varepsilon} \ \leq \ C\big(\|f\|_{s,\varepsilon}\|g\|_{L^\infty} + \|g\|_{s,\varepsilon}\|f\|_{L^\infty}\big).$$

Proof. Let α be a multi-index with $|\alpha| \leq s$. By the Leibniz rule,

$$D^\alpha(fg) = \sum_{\beta+\gamma=\alpha} C_{\beta,\gamma}\, D^\beta f\, D^\gamma g.$$

If $|\beta| \leq s_0$ with $s_0 > n/2$ ($s_0 = 2$ in our case $n = 2$), then $D^\beta f \in L^\infty$ by Sobolev embedding. Hence

$$\|D^\beta f\, D^\gamma g\|_{L^2} \leq \|D^\beta f\|_{L^\infty}\|D^\gamma g\|_{L^2}.$$

A symmetric argument applies when $|\gamma| \leq s_0$. Summing over $|\alpha| \leq s$ and using the weighted norm definition completes the proof. □

Lemma .2 (Composition estimate). *Let $F \in C^{s+1}$ on an open set containing the range of v. If $\|v - v_0\|_{s,\varepsilon} \leq \delta$, then*

$$\|F(v)\|_{s,\varepsilon} \;\leq\; C(F,\delta)\big(1 + \|v\|_{s,\varepsilon}\big).$$

Proof. Faà di Bruno's formula gives, for any $|\alpha| \leq s$,

$$D^{\alpha}[F(v)] = \sum P_{\alpha}(D^{\beta}v,\, \partial^{\ell}F(v)),$$

where P_{α} are multilinear expressions involving derivatives of v of total order $|\alpha|$. Since v remains in a bounded set, $\partial^{\ell}F(v)$ is uniformly bounded. Applying Lemma .1 to each term yields the result. □

Lemma .3 (Commutator estimate). *Let $A(v)$ be smooth in v. For $|\alpha| \leq s$,*

$$\|[D^{\alpha}, A(v)]\, \partial_{x_j} w\|_{L^2_{\varepsilon}} \;\leq\; C(\|v\|_{s,\varepsilon})\, \|w\|_{s,\varepsilon}.$$

Proof. Expanding the commutator,

$$[D^{\alpha}, A(v)]\, \partial_{x_j} w = \sum_{0<\beta\leq\alpha} C_{\alpha,\beta}\, D^{\beta}A(v)\, D^{\alpha-\beta}\partial_{x_j} w.$$

Each derivative $D^{\beta}A(v)$ is controlled by Lemma .2, while $D^{\alpha-\beta}\partial_{x_j} w$ is bounded in L^2 under the weighted norm. Lemma .1 applied to each product, and a sum over β, yields the desired inequality. □

Together, Lemmas .1–.3 provide the nonlinear control required to close the energy estimates of Section 4. Their role is to guarantee that all error terms can be bounded uniformly in ε, which is crucial for proving Theorem 4.4.

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