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A BOUNDARY CONTROL PROBLEM FOR A ONE-DIMENSIONAL PSEUDO-PARABOLIC EQUATION WITH INVOLUTION AND PERIODIC BOUNDARY CONDITIONS

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Abstract. In this study, we investigate a boundary control problem for a one-dimensional pseudo-parabolic equation involving involution and periodic boundary conditions. The main objective of the paper is to determine a boundary control function that ensures the average temperature of a rod reaches a prescribed function over time. By applying separation of variables, the control problem is reduced to a Volterra integral equation of the second kind. The continuity of the kernel and the solvability of the integral equation is established using Laplace transform method. The admissibility of the control function is also proven.

Keywords. pseudo-parabolic equation, Volterra integral equation, admissible control, initial-boundary problem, Laplace transform, involution, periodic
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1. Introduction

It is well known that boundary control problems continue to attract significant attention due to the broad range of applications of partial differential equations (PDEs) in physics and engineering. Consequently, in recent years, the control of heat transfer processes described by PDEs has been extensively studied by numerous researchers.

Pseudo-parabolic equations arise in various scientific and engineering contexts, such as radiation diffusion [28], heat transfer [7], and fluid dynamics [9]. These equations are generally used in place of classical parabolic models when higher-order effects must be incorporated. A boundary control problem for a pseudo-parabolic equation was investigated in [37], where the results were compared with those for the corresponding parabolic case. In [10], classical problems related to the equation such as stability, uniqueness, and existence of solutions were analyzed. The initial study on pointwise control for both parabolic and pseudo-parabolic equations was presented in [38].

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Preliminary results on time-optimal control problems for parabolic-type PDEs have been explored in [18, 21].In their pioneering study, Fattorini and Russell [19] investigated the boundary controllability of a linear parabolic equation on a finite interval, subject to Robin boundary conditions, laying groundwork for subsequent research in this area. Control problems involving parabolic-type equations in infinite-dimensional settings were first studied in [17]. In addition, [8] focused on the evolution of time-optimal control functions for heat conduction equations, emphasizing both theoretical insights and potential applications.

Early work on the control problem considered in our work is studied in detail in [2]. The work presented in [3] investigated boundary-driven heat transfer in cylindrical domains, where the primary aim was to establish a mathematical model of the corresponding heating mechanism. In recent years, control problems for heat conduction equations and pseudo-parabolic equations have been thoroughly investigated in publications [11, 12, 20].

Guo and Littman [23] investigated the problem of achieving null boundary control for a semilinear heat equation subject to Dirichlet conditions in a bounded one-dimensional setting. The optimal time problem in boundary control of the heat conduction equation and its relationship to the "bang-bang" principle were examined in [30]. It was demonstrated that "bang-bang" is the optimal time control with regard to an arbitrary goal temperature distribution in boundary control. In [36], the minimal time control problem for a linear heat conduction equation with memory was investigated. The boundary value control problem for a fourth-order parabolic equation was studied in [13], and an optimal estimate for the control function was found.

Many details regarding optimal control problems can be found in the monographs [22, 27]. In [35], a specific time-optimal control problem was investigated with a closed ball centered at zero as the target. The internally controlled heat conduction equation determined the solution to the problem. Some practical control problems with variable boundary conditions for the linear heat transfer equation were investigated in [26]. The work in [16] is devoted to the analysis of boundary control for a pseudo-parabolic model characterized by spatially dependent permeability within a two-dimensional domain.

The work in [14] demonstrates that boundary control problems involving pseudo-parabolic equations with Neumann boundary conditions in two-dimensional domains are solvable via Laplace transform techniques. Moreover, [15] investigates the inhomogeneous pseudo-parabolic control setting and confirms the admissibility of the boundary control function within a bounded spatial domain.

It is well known that boundary problems pertaining to pseudo-parabolic equations connected to involution have been extensively investigated in recent years due to growing interest in mathematics and physics. Spectral difficulties for involved differential operators have also been the subject of numerous publications (see [25]). In [4], the Fourier approach was used to provide a classical solution to the initial-boundary value problem for the first-order partial differential equation involving involution. An inverse problems for equations of parabolic type with involution are studied in work [33, 34].

The parabolic equation related to involution in a one-dimensional domain is examined in [29] as a boundary value problem. In publications [1, 24], numerous boundary value issues for parabolic and pseudo-parabolic type equations were examined.

The present work investigates a control problem for a one-dimensional pseudo-parabolic equation with an involution and periodic boundary conditions. Previously, control problems involving the heat equation with Dirichlet, Neumann, and Robin boundary conditions have been extensively studied. The main objective of this study is to determine a control function that ensures the rod reaches a prescribed average temperature. By applying the method of separation of variables, we solve the initial-boundary value problem associated with the pseudo-parabolic equation. By applying this approach, the original control problem is reduced to a second-kind Volterra integral equation. The existence of an admissible control is then justified via Laplace transform-based analysis.

2. Statement of Problem

Let $\Omega := (0,\pi) \times (0,\infty)$. In the present article, we consider the following one-dimensional pseudo-parabolic equation:

$$u_t(x,t) - u_{xx}(x,t) - u_{xxt}(x,t) + \varepsilon\, u_{xx}(\pi - x, t) + \varepsilon\, u_{xxt}(\pi - x, t) = 0, \quad (x,t) \in \Omega, \tag{2.1}$$

with the periodic boundary conditions

$$u(0,t) - u(\pi,t) = \nu(t), \quad u_x(0,t) - u_x(\pi,t) = 0, \quad t \geq 0, \tag{2.2}$$

and the initial condition

$$u(x,0) = 0, \quad 0 \leq x \leq \pi, \tag{2.3}$$

where $\nu(t)$ is the control function, and ε is a given nonzero real parameter such that $|\varepsilon| < 1$.

Definition 2.1. If the function $\nu(t)$ is continuously differentiable on the half-line $t \geq 0$ and meets the requirements $\nu(0) = 0$, $|\nu(t)| \leq 1$, we refer to it as admissible control.

Some partial differential equations (PDEs) involve evaluating the unknown function and its derivatives at modified temporal or spatial arguments; such equations are commonly referred to as functional differential equations. Among them, equations that involve involutions are of particular interest [5].

A function $p(y) \not\equiv y$ that bijectively maps a domain D and satisfies $p(p(y)) = y$ or $p^{-1}(y) = p(y)$ is called an involution on D [6, 39].

Equation (2.1) for $\varepsilon = 0$ is evidently a classical one-dimensional pseudo-parabolic type equation. Equation (2.1) becomes a nonlocal equation if $\varepsilon \neq 0$, which connects the values of the second derivatives at two distinct places. It is well known that research [14] have thoroughly examined boundary control problems for the pseudo-parabolic equation in instance $\varepsilon = 0$.

Such boundary conditions may arise in thermal systems where a fixed temperature difference is actively maintained between two ends of a rod, for example, in heat exchangers or devices with thermal regulators. These systems impose a controlled jump in temperature, which is captured by the boundary condition (2.2).

We now examine the subsequent control problem.

Control Problem. Suppose that $\chi(t)$ is a given function. Then find the admissible control function $\nu(t)$ from the equality

$$\int_0^{\pi/2} u(x,t)\,dx = \chi(t), \quad t \geq 0, \tag{2.4}$$

where $u(x,t)$ represents the mixed problem's solution (2.1)-(2.3).

In the above control problem, the physical meaning of equation (2.4) corresponds to the average temperature over the interval $[0, \frac{\pi}{2}]$ in a thin rod of total length π. Our main objective is to determine the control function such that the average temperature in this section of the rod follows a prescribed function $\chi(t)$.

For any constant $M > 0$, we define $W(M)$ to be the set of functions $\chi \in W_2^1(\mathbb{R})$ such that

$$\chi(t) = 0 \quad \text{for } t \leq 0, \quad \text{and} \quad \|\chi\|_{W_2^1(\mathbb{R}_+)} \leq M.$$

We now offer the primary theorem for demonstrating admissible control's existence.

Theorem 2.2. *There exists a positive number M such that for any function $\chi(t) \in W(M)$ there is a solution $\nu(t)$ of equation (2.4) and satisfies the condition $|\nu(t)| \leq 1$.*

We will consider the proof of this theorem step by step in the following sections.

3. Main Integral equation

In this section, we establish the existence of a solution to the underlying mixed problem. Utilizing this solution together with the additional integral condition arising from the control problem, we derive the main integral equation for determining the control function.

We are now considering the following spectral problem:

$$P''(x) - \varepsilon\, P''(\pi - x) + \lambda\, P(x) = 0, \quad 0 < x < \pi,$$

with periodic boundary conditions

$$P(0) = P(\pi), \quad P'(0) = P'(\pi), \quad 0 \leq x \leq \pi,$$

here $|\varepsilon| < 1$, $\varepsilon \in \mathrm{R}\backslash\{0\}$. In [32] it was shown that the solution of the above periodic boundary condition spectral problem can be found with eigenvalues of the following form:

$$\lambda_{m,1} = (1+\varepsilon)\,4m^2, \quad m \in \mathbb{N}, \tag{3.1}$$
$$\lambda_{m,2} = (1-\varepsilon)\,4m^2, \quad m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

and we have the following eigenfunctions:

$$P_0 = 1, \quad P_{m,1} = \sin 2mx, \quad P_{m,2} = \cos 2mx, \quad m \in \mathbb{N}.$$

We introduce the function $U(x,t)$ satisfying the boundary conditions (2.2) as follows:

$$U(x,t) = \frac{\pi - x}{\pi}\nu(t).$$

Now we look for the solution $u(x,t)$ satisfying the homogeneous one-dimensional pseudo-parabolic equation (2.1) the periodic boundary conditions (2.2) and the initial condition (2.3) in the following form

$$u(x,t) = \nu(t)\,\frac{\pi - x}{\pi} - w(x,t), \tag{3.2}$$

here the function $w(x,t) \in C^{2,1}_{x,t}(\Omega) \cap C(\bar{\Omega})$, $w_x \in C(\bar{\Omega})$ is the solution of the following non-homogeneous one-dimensional pseudo-parabolic equation with involution:

$$w_t(x,t) - w_{xx}(x,t) - w_{xxt}(x,t) + \varepsilon\, w_{xx}(\pi - x,t) + \varepsilon\, w_{xxt}(\pi - x,t) = \frac{\pi - x}{\pi}\,\nu'(t),$$

with the homogeneous initial and periodic boundary conditions

$$w(0,t) = w(\pi,t), \quad w_x(0,t) = w_x(\pi,t), \quad w(x,0) = 0.$$

We look for the solution of the above mixed problem by the Fourier method. As a result, we have the following solution (see [31]):

$$w(x,t) = \frac{1}{2}\,\nu(t) + \sum_{m=1}^{\infty} \frac{1}{\pi\, m}\,\frac{1}{1+\lambda_{m,1}} \left(\int_0^t e^{-q_{m,1}(t-\tau)}\,\nu'(\tau)\,d\tau \right) \sin 2mx, \tag{3.3}$$

where $q_{m,1} = \frac{\lambda_{m,1}}{1+\lambda_{m,1}}$ and $\lambda_{m,1}$ is defined by (3.1).

It follows from (3.2) and (3.3) that we can write the solution of equation (2.1) satisfying the periodic boundary conditions (2.2) and the initial condition (2.3) as:

$$\begin{aligned} u(x,t) = {} & \frac{\pi - x}{\pi}\,\nu(t) - \frac{1}{2}\,\nu(t) \\ & - \sum_{m=1}^{\infty} \frac{1}{\pi\, m}\,\frac{1}{1+\lambda_{m,1}} \left(\int_0^t e^{-q_{m,1}(t-\tau)}\,\nu'(\tau)\,d\tau \right) \sin 2mx. \end{aligned} \tag{3.4}$$

Using the average condition (2.4), we integrate $u(x,t)$ over the segment $[0,\pi/2]$:

$$\chi(t) = \int_0^{\pi/2} u(x,t)\,dx.$$

Substituting from (3.4) and integrating term-by-term, we obtain:

$$\begin{aligned} \chi(t) = {} & \nu(t) \int_0^{\pi/2} (\frac{\pi - x}{\pi} - \frac{1}{2})\,dx \\ & - \sum_{m=1}^{\infty} \frac{1}{\pi\, m}\,\frac{1}{1+\lambda_{m,1}} \left(\int_0^t e^{-q_{m,1}(t-\tau)}\,\nu'(\tau)\,d\tau \right) \int_0^{\pi/2} \sin 2mx\,dx. \end{aligned}$$

Then we get

$$\chi(t) = \nu(t) \int_0^{\pi/2} \frac{\pi - 2x}{2\pi}\,dx - \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{1-(-1)^m}{m^2}\,\frac{1}{1+\lambda_{m,1}} \int_0^t e^{-q_{m,1}(t-\tau)}\,\nu'(\tau)\,d\tau,$$

where $\nu(t)$ is the control function, and $\nu(0) = 0$.

We can write using formula for integration by parts and the definition of the admissible control $\nu(t)$

$$\chi(t) = \nu(t) \int_0^{\pi/2} \frac{\pi - 2x}{2\pi}\, dx - \nu(t) \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{1-(-1)^m}{m^2} \frac{1}{1+\lambda_{m,1}}$$
$$+ \frac{2\,(1+\varepsilon)}{\pi} \sum_{m=1}^{\infty} \frac{1-(-1)^m}{(1+\lambda_{m,1})^2} \int_0^t e^{-q_{m,1}(t-\tau)}\, \nu(\tau)\, d\tau. \tag{3.5}$$

The following identities hold:

$$\frac{\pi - 2x}{2\pi} = \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin 2mx,$$

and we derive

$$\int_0^{\pi/2} \frac{\pi - 2x}{2\pi}\, dx = \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{1-(-1)^m}{m^2}. \tag{3.6}$$

By (3.5) and (3.6), we have

$$\chi(t) = \nu(t)\, \frac{2(1+\varepsilon)}{\pi} \sum_{m=1}^{\infty} \frac{1-(-1)^m}{1+\lambda_{m,1}}$$
$$+ \frac{2(1+\varepsilon)}{\pi} \sum_{m=1}^{\infty} \frac{1-(-1)^m}{(1+\lambda_{m,1})^2} \int_0^t e^{-q_{m,1}(t-\tau)}\, \nu(\tau)\, d\tau, \tag{3.7}$$

where $\lambda_{m,1} = (1+\varepsilon)\, 4m^2$ ($|\varepsilon| < 1$) and $q_{m,1} = \frac{\lambda_{m,1}}{1+\lambda_{m,1}} < 1$.

Now we define the following function

$$H(t) = \sum_{m=1}^{\infty} \Phi_m\, e^{-q_{m,1}\, t}, \quad t > 0, \tag{3.8}$$

with coefficients Φ_m given by

$$\Phi_m = \frac{2(1+\varepsilon)}{\pi} \frac{1-(-1)^m}{(1+\lambda_{m,1})^2}.$$

Denote

$$\beta = \frac{2(1+\varepsilon)}{\pi} \sum_{m=1}^{\infty} \frac{1-(-1)^m}{1+\lambda_{m,1}}. \tag{3.9}$$

It is clear that β is a positive constant.

Thus, using (3.7), (3.8) and (3.9), we obtain the following main integral equation, which is a Volterra integral equation of the second kind:

$$\beta\, \nu(t) + \int_0^t H(t-\tau)\, \nu(\tau) d\tau = \chi(t), \quad t > 0. \tag{3.10}$$

The resulting Volterra integral equation (3.10) is the main equation for admissible control $\nu(t)$.

Lemma 3.1. *Let $|\varepsilon| < 1$. Then the kernel $H(t)$ appearing in the integral equation (3.10) is continuous on the half-line $t \geq 0$.*

Proof. From the representation of $H(t)$ given in (3.8), we obtain the estimate:

$$
\begin{aligned}
0 < H(t) &= \sum_{m=1}^{\infty} \Phi_m \, e^{-q_{m,1}t} \leq \sum_{m=1}^{\infty} \Phi_m \\
&= \frac{2(1+\varepsilon)}{\pi} \sum_{m=1}^{\infty} \frac{1-(-1)^m}{(1+\lambda_{m,1})^2} \\
&\leq \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{1}{(1+\lambda_{m,1})^2} < +\infty.
\end{aligned}
$$

Hence, the kernel function $H(t)$ is bounded and defined as a uniformly convergent series of continuous functions on $[0, \infty)$, which implies its continuity. □

4. Proof of Theorem 2.2

It is well known that the solvability of equation (2.4), in conjunction with the validity of Theorem 2.2, ensures the existence of a solution to the integral equation (3.10). Consequently, this section is devoted to first establishing the solvability of the Volterra integral equation. Afterward, we employ geometric estimates to demonstrate that the resulting control function is admissible.

The Laplace transform of the function $\nu(t)$ is given by

$$
\widetilde{\nu}(p) = \int_0^{\infty} e^{-pt} \, \nu(t) \, dt, \tag{4.1}
$$

where $p = \sigma + i\zeta$, with $\sigma > 0$ and $\zeta \in \mathbb{R}$.

Applying the Laplace transform to the integral equation (3.10) yields

$$
\begin{aligned}
\widetilde{\chi}(p) &= \beta \int_0^{\infty} e^{-pt} \nu(t) \, dt + \int_0^{\infty} e^{-pt} \left(\int_0^t H(t-\tau) \, \nu(\tau) \, d\tau \right) dt \\
&= \beta \, \widetilde{\nu}(p) + \widetilde{H}(p) \, \widetilde{\nu}(p),
\end{aligned}
$$

where β is the constant defined in (3.9).

From the previous expression, the Laplace representation of the control function can be rewritten as

$$
\widetilde{\nu}(p) = \frac{\widetilde{\chi}(p)}{\beta + \widetilde{H}(p)}.
$$

Applying the inverse Laplace transform, we recover $\nu(t)$ in the time domain as follows:

$$
\begin{aligned}
\nu(t) &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\widetilde{\chi}(p)}{\beta + \widetilde{H}(p)} \, e^{pt} \, dp \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\widetilde{\chi}(\sigma + i\zeta)}{\beta + \widetilde{H}(\sigma + i\zeta)} \, e^{(\sigma + i\zeta)t} \, d\zeta,
\end{aligned} \tag{4.2}
$$

where $\sigma > 0$ is a real number such that the integration path lies within the region of convergence of the Laplace transform.

Lemma 4.1. *Let $\widetilde{H}(p)$ denote the Laplace transform of the kernel function $H(t)$ and let β be the constant defined in (3.9). Then, the following estimate holds:*

$$\left|\beta + \widetilde{H}(\sigma + i\zeta)\right| \geq \beta.$$

Proof. We begin by applying the Laplace transform to the kernel function $H(t)$ defined in (3.8):

$$\widetilde{H}(p) = \int_0^\infty H(t)\, e^{-pt}\, dt = \sum_{m=1}^\infty \Phi_m \int_0^\infty e^{-(p+q_{m,1})t}\, dt = \sum_{m=1}^\infty \frac{\Phi_m}{p + q_{m,1}}.$$

Evaluating this at $p = \sigma + i\zeta$, where $\sigma > 0$ and $\zeta \in \mathbb{R}$, we obtain

$$\begin{aligned}
\beta + \widetilde{H}(\sigma + i\zeta) &= \beta + \sum_{m=1}^\infty \frac{\Phi_m}{\sigma + q_{m,1} + i\zeta} \\
&= \beta + \sum_{m=1}^\infty \frac{\Phi_m(\sigma + q_{m,1})}{(\sigma + q_{m,1})^2 + \zeta^2} - i\zeta \sum_{m=1}^\infty \frac{\Phi_m}{(\sigma + q_{m,1})^2 + \zeta^2} \\
&= \mathrm{Re}(\beta + \widetilde{H}(\sigma + i\zeta)) + i\, \mathrm{Im}(\beta + \widetilde{H}(\sigma + i\zeta)),
\end{aligned}$$

where

$$\mathrm{Re}(\beta + \widetilde{H}(\sigma + i\zeta)) = \beta + \sum_{m=1}^\infty \frac{\Phi_m(\sigma + q_{m,1})}{(\sigma + q_{m,1})^2 + \zeta^2},$$

and

$$\mathrm{Im}(\beta + \widetilde{H}(\sigma + i\zeta)) = -\zeta \sum_{m=1}^\infty \frac{\Phi_m}{(\sigma + q_{m,1})^2 + \zeta^2}.$$

Noting that

$$(\sigma + q_{m,1})^2 + \zeta^2 \leq \left((\sigma + q_{m,1})^2 + 1\right)(1 + \zeta^2),$$

we derive the inequality

$$\frac{1}{(\sigma + q_{m,1})^2 + \zeta^2} \geq \frac{1}{1 + \zeta^2} \cdot \frac{1}{(\sigma + q_{m,1})^2 + 1}. \tag{4.3}$$

Substituting this bound into the expressions for the real and imaginary parts, we obtain:

$$|\,\mathrm{Re}(\beta + \widetilde{H}(\sigma + i\zeta))| \geq \beta + \frac{1}{1 + \zeta^2} \sum_{m=1}^\infty \frac{\Phi_m(\sigma + q_{m,1})}{(\sigma + q_{m,1})^2 + 1} = \beta + \frac{C_{1,\sigma}}{1 + \zeta^2}, \tag{4.4}$$

and

$$|\,\mathrm{Im}(\beta + \widetilde{H}(\sigma + i\zeta))| \geq \frac{|\zeta|}{1 + \zeta^2} \sum_{m=1}^\infty \frac{\Phi_m}{(\sigma + q_{m,1})^2 + 1} = \frac{C_{2,\sigma}|\zeta|}{1 + \zeta^2}, \tag{4.5}$$

where the constants $C_{1,\sigma}$ and $C_{2,\sigma}$ are defined by

$$C_{1,\sigma} = \sum_{m=1}^\infty \frac{\Phi_m(\sigma + q_{m,1})}{(\sigma + q_{m,1})^2 + 1}, \quad C_{2,\sigma} = \sum_{m=1}^\infty \frac{\Phi_m}{(\sigma + q_{m,1})^2 + 1}.$$

Combining inequalities (4.4) and (4.5), we deduce the estimate

$$|\beta+\widetilde{H}(\sigma+i\zeta)| \geq \beta+\frac{C_\sigma}{\sqrt{1+\zeta^2}} \geq \beta, \tag{4.6}$$

where $C_\sigma = \min(C_{1,\sigma}, C_{2,\sigma})$ is a positive constant depending on σ. □

Assume that the Laplace transform of the function $\chi(t)$ satisfies the following integrability condition:

$$\int_{-\infty}^{+\infty} |\widetilde{\chi}(i\zeta)|\, d\zeta < +\infty.$$

Passing to the limit as $\sigma \to 0$ in equation (4.2), we obtain the inverse Laplace transform representation:

$$\nu(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\widetilde{\chi}(i\zeta)}{\beta+\widetilde{H}(i\zeta)}\, e^{i\zeta t}\, d\zeta. \tag{4.7}$$

The following lemma will be instrumental in proving Theorem 2.2.
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Lemma 4.2. *Let $\chi(t)$ be a function from the class $W(M)$. Then its Laplace transform satisfies the estimate*

$$\int_{-\infty}^{+\infty} |\widetilde{\chi}(i\zeta)|\, d\zeta \leq C_2\, \|\chi\|_{W_2^1(\mathbb{R}_+)},$$

where the constant $C_2 = \sqrt{2}\pi$.

Proof. If we use the formula for integration by pieces in equation (4.1) above, we may write the following equalities:

$$\begin{aligned}\widetilde{\chi}(\sigma+i\zeta) &= \int_0^\infty e^{-(\sigma+i\zeta)t}\chi(t)\, dt \\ &= -\chi(t)\left.\frac{e^{-(\sigma+i\zeta)t}}{\sigma+i\zeta}\right|_{t=0}^{t=\infty} + \frac{1}{\sigma+i\zeta}\int_0^\infty e^{-(\sigma+i\zeta)t}\,\chi'(t)\, dt.\end{aligned}$$

After that, we obtain

$$(\sigma+i\zeta)\,\widetilde{\chi}(\sigma+i\zeta) = \int_0^\infty e^{-(\sigma+i\zeta)t}\,\chi'(t)\, dt.$$

Letting $\sigma \to 0$, we obtain

$$i\zeta\,\widetilde{\chi}(i\zeta) = \int_0^\infty e^{-i\zeta t}\chi'(t)\, dt, \qquad \widetilde{\chi}(i\zeta) = \int_0^\infty e^{-i\zeta t}\chi(t)\, dt.$$

Hence,

$$\int_{-\infty}^{+\infty} |\widetilde{\chi}(i\zeta)|^2(1+\zeta^2)\, d\zeta = 2\pi\, \|\chi\|^2_{W_2^1(\mathbb{R}_+)}.$$

Applying the Cauchy-Schwarz inequality, we obtain the estimate

$$\begin{aligned}\int_{-\infty}^{+\infty} |\widetilde{\chi}(i\zeta)|\, d\zeta &= \int_{-\infty}^{+\infty} \frac{|\widetilde{\chi}(i\zeta)|\sqrt{1+\zeta^2}}{\sqrt{1+\zeta^2}}\, d\zeta \\ &\le \left(\int_{-\infty}^{+\infty} |\widetilde{\chi}(i\zeta)|^2(1+\zeta^2)\, d\zeta\right)^{1/2} \left(\int_{-\infty}^{+\infty} \frac{1}{1+\zeta^2}\, d\zeta\right)^{1/2} \\ &= \sqrt{2\pi}\, \|\chi\|_{W_2^1(\mathbb{R}_+)} \cdot \sqrt{\pi} = C_2\, \|\chi\|_{W_2^1(\mathbb{R}_+)}.\end{aligned}$$

Thus, the integral is finite, and the constant $C_2 = \sqrt{2}\pi$. □

We now present the proof of Theorem 2.2.

Proof of Theorem 2.2. Using (4.6), (4.7), and Lemma 4.2, we arrive at the following estimate:

$$\begin{aligned}|\nu(t)| &\le \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|\widetilde{\chi}(i\zeta)|}{|\beta + \widetilde{H}(i\zeta)|}\, d\zeta \\ &\le \frac{1}{2\pi\beta}\int_{-\infty}^{+\infty} |\widetilde{\chi}(i\zeta)|\, d\zeta \\ &\le \frac{C_2}{2\pi\beta}\|\chi\|_{W_2^1(\mathbb{R}_+)} \\ &\le \frac{C_2\, M}{2\pi\beta} = 1,\end{aligned}$$

where β is defined by (3.9), $C_2 = \pi\sqrt{2}$ is the constant from Lemma 4.2, and the parameter M is chosen as

$$M = \frac{2\pi\beta}{C_2}.$$

Theorem 2.2 is proved.

5. Conclusion

This study addressed a boundary control problem for a pseudo-parabolic equation involving an involution and periodic boundary conditions. By applying the Fourier method, the control problem was reduced to a Volterra integral equation of the second kind. The existence of a solution to this integral equation was established using the Laplace transform method. Furthermore, the admissibility of the control function-i.e., the solution to the integral equation-was verified through appropriate geometric estimates. It should be noted that the existence result established in Theorem 2.2 holds under the assumption that the norm of the target function is sufficiently small. Extending the result to arbitrary target functions remains a challenging problem for future research.

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