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# LOGARITHM OF THE GREATEST PRIME FACTOR

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ABSTRACT. In this article we prove asymptotic formulae for sums where appear $\log^k P(n)$, where $P(n)$ is the greatest prime factor of $n$. In this way we generalize a formula of N. G. De Bruijn. Our methods are very elementary, we only use the concept of integral of a continuous function. We do not use the theory of complex functions.

## 1. INTRODUCTION

Let $P(n)$ be the greatest prime factor in the prime factorization of $n$. For example, $P(2^3.5.11^3) = 11$, $P(5) = 5$.

Let $0 < \lambda \leq 1$ a fixed real number. Let $\epsilon_\lambda(x)$ be the set of positive integers not exceeding $x$ such that in their prime factorization only appear primes $p$ pertaining to the interval $\left[0, x^\lambda\right]$. That is, $\epsilon_\lambda(x)$ is the set of positive integers not exceeding $x$ such that the greatest prime factor of these positive integers pertain to the interval $\left[0, x^\lambda\right]$. These numbers are called smooth numbers. The number of positive integers pertaining to the set $\epsilon_\lambda(x)$ we denote $N(\lambda, x)$. It is well-known [3] the following formula

$$N(\lambda, x) = \phi(\lambda)x + f(x)x \tag{1.1}$$

where $f(x) \to 0$. Note that $f(x)$ depend of $\lambda$. Therefore the positive integers in the set $\epsilon_\lambda(x)$ have positive density $\phi(\lambda)$ (this function of $\lambda$ is called Dickman's function). It is a positive, strictly increasing and continuous function on the interval $(0, 1]$. Besides clearly $\phi(1) = 1$. On the other hand, the following limit holds

$$\lim_{\lambda \to 0} \phi(\lambda) = 0 \tag{1.2}$$

Let $B(n, p)$ be the number of positive integers $s$ not exceeding $n$ such that the prime $p$ is the greatest prime factor of $s$ (that is, $P(s) = p$). Then

$$\sum_{2 \leq p \leq n} B(n, p) = n - 1$$

and

$$N(\lambda, n) = \sum_{2 \leq p \leq n^\lambda} B(n, p)$$

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N. G. De Bruijn [1] proved the following asymptotic formula

$$\sum_{2\leq p\leq n} \log pB(n,p) = \sum_{i=2}^{n} \log P(i) = an\log n + O(n)$$

where the constant $a$ is defined in [1]. Therefore we have

$$\sum_{2\leq p\leq n} \log pB(n,p) = \sum_{i=2}^{n} \log P(i) \sim an\log n \tag{1.3}$$

In this article we study the sum

$$\sum_{n^\alpha<p\leq n} \log^k pB(n,p)$$

where $0<\alpha<1$ is a fixed real number and the sum

$$\sum_{2\leq p\leq n} \log^k pB(n,p) = \sum_{i=2}^{n} \log^k P(i)$$

We shall need the following lemmas.

**Lemma 1.1.** *Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be two series of positive terms such that $\lim_{i\to\infty}\frac{a_i}{b_i}=1$. If $\sum_{i=1}^{\infty} b_i$ is divergent then we have the following limit*

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}=1.$$

*Therefore $\sum_{i=1}^{\infty} a_i$ is also divergent.*

*Proof.* See [4, page 332] . □

**Lemma 1.2.** *Let $k$ be a positive integer. We have the following asymptotic formula*

$$\sum_{p\leq x}\log^k p \sim x\log^{k-1} x \tag{1.4}$$

*Proof.* We have

$$\sum_{i=1}^{n}\log^k i = \int_1^n \log^k x\ dx + O(\log^k n) \tag{1.5}$$

since the left side is a sum of rectangles of basis 1 and height $\log^k i$ and the function $\log^k x$ is strictly increasing.

On the other hand (L'Hospital rule) we have

$$\lim_{x\to\infty}\frac{\int_1^x \log^k t\ dt}{x\log^k x}=1 \tag{1.6}$$

Equations (1.5) and (1.6) give

$$\sum_{i=1}^{n}\log^k i \sim n\log^k n \tag{1.7}$$

Equation (1.7), Lemma 1.1 and the prime number Theorem ($p_n \sim n \log n$) give

$$\sum_{p \leq p_n} \log^k p = \sum_{i=1}^{n} \log^k p_i \sim \sum_{i=1}^{n} \log^k i \sim n \log^k n \sim p_n \log^{k-1} p_n \tag{1.8}$$

The function $x \log^{k-1} x$ is strictly increasing. Therefore if $x \in [p_n, p_{n+1})$ we have (see (1.8))

$$\frac{\sum_{p \leq p_n} \log^k p}{p_{n+1} \log^{k-1} p_{n+1}} \leq \frac{\sum_{p \leq x} \log^k p}{x \log^{k-1} x} \leq \frac{\sum_{p \leq p_n} \log^k p}{p_n \log^{k-1} p_n} \tag{1.9}$$

Now, the left side and right side have limit 1. Therefore (1.9) implies (1.4). □

**Lemma 1.3.** *Let $k$ be a positive integer. We have the following asymptotic formula*

$$\sum_{p \leq x} \frac{\log^k p}{p} \sim \frac{1}{k} \log^k x \tag{1.10}$$

*Proof.* We have

$$\sum_{i=1}^{n} \frac{\log^{k-1} i}{i} = \int_1^n \frac{\log^{k-1} x}{x}\, dx + O(1) \sim \frac{1}{k} \log^k n \tag{1.11}$$

since the left side is a sum of rectangles of basis 1 and height $\frac{\log^{k-1} i}{i}$ and the function $\frac{\log^{k-1} x}{x}$ is strictly decreasing from a certain value of $x$.

Equation (1.11), Lemma 1.1 and the prime number Theorem ($p_n \sim n \log n$) give

$$\sum_{p \leq p_n} \frac{\log^k p}{p} = \sum_{i=1}^{n} \frac{\log^k p_i}{p_i} \sim \sum_{i=1}^{n} \frac{\log^k i}{i \log i} = \sum_{i=1}^{n} \frac{\log^{k-1} i}{i} \sim \frac{1}{k} \log^k n \sim \frac{1}{k} \log^k p_n \tag{1.12}$$

The function $\log^k x$ is strictly increasing. Therefore if $x \in [p_n, p_{n+1})$ we have (see (1.12))

$$\frac{\sum_{p \leq p_n} \frac{\log^k p}{p}}{\frac{1}{k} \log^k p_{n+1}} \leq \frac{\sum_{p \leq x} \frac{\log^k p}{p}}{\frac{1}{k} \log^k x} \leq \frac{\sum_{p \leq p_n} \frac{\log^k p}{p}}{\frac{1}{k} \log^k p_n} \tag{1.13}$$

Now, the left side and right side have limit 1. Therefore (1.13) implies (1.10). □

## 2. Main results

**Theorem 2.1.** *Let $\frac{1}{2} \leq \alpha < 1$ a fixed real number. The following asymptotic formula holds*

$$\sum_{n^\alpha < p \leq n} \log^k p B(n,p) \sim a_{\alpha,k} n \log^k n \tag{2.1}$$

*where*

$$a_{\alpha,k} = \frac{1}{k} \left(1 - \alpha^k\right) = \int_\alpha^1 x^{k-1}\, dx \tag{2.2}$$

*Proof.* The set of multiples of $p$ not exceeding $n$ is

$$\left\{p.1, p.2, p.3, \ldots, p\left\lfloor\frac{n}{p}\right\rfloor\right\}$$

If $p > n^{1/2}$ then $p > \frac{n}{p} \geq \left\lfloor\frac{n}{p}\right\rfloor$. Consequently $B(n,p) = \left\lfloor\frac{n}{p}\right\rfloor$. Therefore

$$\begin{aligned}\sum_{n^\alpha < p \leq n} \log^k p B(n,p) &= \sum_{n^\alpha < p \leq n} \log^k p \left\lfloor\frac{n}{p}\right\rfloor \\ &= n \sum_{n^\alpha < p \leq n} \frac{\log^k p}{p} - \sum_{n^\alpha < p \leq n} \log^k p \left(\frac{n}{p} - \left\lfloor\frac{n}{p}\right\rfloor\right) \qquad (2.3)\end{aligned}$$

We have (see (1.10))

$$\begin{aligned}\sum_{n^\alpha < p \leq n} \frac{\log^k p}{p} &= \sum_{p \leq n} \frac{\log^k p}{p} - \sum_{p \leq n^\alpha} \frac{\log^k p}{p} \\ &= \frac{1}{k} \log^k n - \frac{1}{k} \alpha^k \log^k n + o(\log^k n) = \frac{1}{k}\left(1 - \alpha^k\right) \log^k n + o(\log^k n) \quad (2.4)\end{aligned}$$

and (see (1.4))

$$\sum_{n^\alpha < p \leq n} \log^k p \left(\frac{n}{p} - \left\lfloor\frac{n}{p}\right\rfloor\right) \leq \sum_{p \leq n} \log^k p \leq cn \log^{k-1} n$$

that is

$$\sum_{n^\alpha < p \leq n} \log^k p \left(\frac{n}{p} - \left\lfloor\frac{n}{p}\right\rfloor\right) = O(n \log^{k-1} n) \qquad (2.5)$$

Equations (2.1) and (2.2) are an immediate consequence of (2.3), (2.4) and (2.5). □

**Theorem 2.2.** *Let $0 < \alpha < \frac{1}{2}$ a fixed real number. The following asymptotic formula holds*

$$\sum_{n^\alpha < p \leq n} \log^k p B(n,p) \sim a_{\alpha,k} n \log^k n \qquad (2.6)$$

*where*

$$a_{\alpha,k} = \frac{1}{k}\left(1 - \left(\frac{1}{2}\right)^k\right) + \int_\alpha^{1/2} \phi\left(\frac{x}{1-x}\right) x^{k-1}\, dx \qquad (2.7)$$

*Proof.* Let $\epsilon > 0$. Let us consider the chain of inequalities

$$\alpha = \beta_0 < \beta_1 < \beta_2 < \cdots < \beta_k = \frac{1}{2}$$

where $\beta_{i+1} - \beta_i < \epsilon$, $(i = 0, 1, \ldots, k-1)$. We have

$$\sum_{n^\alpha < p \leq n} \log^k p B(n,p) = \sum_{i=0}^{k-1}\left(\sum_{n^{\beta_i} < p \leq n^{\beta_{i+1}}} \log^k p B(n,p)\right) + \sum_{n^{1/2} < p \leq n} \log^k p B(n,p) \qquad (2.8)$$

where (see Theorem 2.1)

$$\frac{1}{k}\left(1-\left(\frac{1}{2}\right)^k\right)-\epsilon<\frac{\sum_{n^{1/2}<p\le n}\log^k pB(n,p)}{n\log n}<\frac{1}{k}\left(1-\left(\frac{1}{2}\right)^k\right)+\epsilon \tag{2.9}$$

For sake of simplicity we put (see (2.8))

$$A(n)=\sum_{i=0}^{k-1}\left(\sum_{n^{\beta_i}<p\le n^{\beta_{i+1}}}\log^k pB(n,p)\right) \tag{2.10}$$

Let us consider the inequality

$$n^{\beta_i}<p\le n^{\beta_{i+1}} \tag{2.11}$$

Inequality (2.11) implies the following inequality

$$n^{1-\beta_{i+1}}\le\frac{n}{p}<n^{1-\beta_i} \tag{2.12}$$

Inequalities (2.11) and (2.12) imply the following two inequalities

$$p\le n^{\beta_{i+1}}=\left(n^{1-\beta_{i+1}}\right)^{\frac{\beta_{i+1}}{1-\beta_{i+1}}}\le\left(\frac{n}{p}\right)^{\frac{\beta_{i+1}}{1-\beta_{i+1}}} \tag{2.13}$$

$$p>n^{\beta_i}=\left(n^{1-\beta_i}\right)^{\frac{\beta_i}{1-\beta_i}}>\left(\frac{n}{p}\right)^{\frac{\beta_i}{1-\beta_i}} \tag{2.14}$$

The set of multiples of $p$ not exceeding $n$ is

$$\left\{p.1,p.2,p.3,\ldots,p\left\lfloor\frac{n}{p}\right\rfloor\right\}$$

Let us consider the set

$$\left\{1,2,3,\ldots,\left\lfloor\frac{n}{p}\right\rfloor\right\} \tag{2.15}$$

Let $C(n,p)$ be the number of numbers in the set (2.15) such that in their prime factorization appear only primes not exceeding $p$. Consequently

$$B(n,p)=C(n,p) \tag{2.16}$$

Equations (2.16), (2.13), (1.1) and (2.12) give

$$\begin{aligned} B(n,p)=C(n,p)&\le N\left(\frac{\beta_{i+1}}{1-\beta_{i+1}},\frac{n}{p}\right)=\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)\frac{n}{p}\\ +\quad f\left(\frac{n}{p}\right)\frac{n}{p}&=\left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)+\epsilon\right)\frac{n}{p}+\left(f\left(\frac{n}{p}\right)-\epsilon\right)\frac{n}{p}\\ \le\quad&\left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)+\epsilon\right)\frac{n}{p} \end{aligned} \tag{2.17}$$

In the same way, equations (2.16), (2.14), (1.1) and (2.12) give

$$B(n,p)=C(n,p)\ge\left(\phi\left(\frac{\beta_i}{1-\beta_i}\right)-\epsilon\right)\frac{n}{p} \tag{2.18}$$

Note that the function $h(x)=\phi\left(\frac{x}{1-x}\right)$ is strictly increasing and continuous on the interval $[0,1/2]$. Besides $h(0)=0$ and $h(1/2)=1$.

Now, we have (see (1.10))

$$\sum_{n^{\beta_i}<p\leq n^{\beta_{i+1}}}\frac{\log^k p}{p}=\sum_{p\leq n^{\beta_{i+1}}}\frac{\log^k p}{p}-\sum_{p\leq n^{\beta_i}}\frac{\log^k p}{p}=\frac{1}{k}\left(\beta_{i+1}^k-\beta_i^k\right)\log^k n+o\left(\log^k n\right) \tag{2.19}$$

Equations (2.17) and (2.19) give

$$\begin{aligned}\sum_{n^{\beta_i}<p\leq n^{\beta_{i+1}}}\log^k pB(n,p) &\leq \left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)+\epsilon\right)n\sum_{n^{\beta_i}<p\leq n^{\beta_{i+1}}}\frac{\log^k p}{p}\\ =&\ \phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)\frac{1}{k}\left(\beta_{i+1}^k-\beta_i^k\right)n\log^k n+\epsilon\frac{1}{k}\left(\beta_{i+1}^k-\beta_i^k\right)n\log^k n\\ +&\ o\left(n\log^k n\right)\end{aligned} \tag{2.20}$$

Equations (2.18) and (2.19) give

$$\begin{aligned}\sum_{n^{\beta_i}<p\leq n^{\beta_{i+1}}}\log^k pB(n,p) &\geq \left(\phi\left(\frac{\beta_{i}}{1-\beta_{i}}\right)-\epsilon\right)n\sum_{n^{\beta_i}<p\leq n^{\beta_{i+1}}}\frac{\log^k p}{p}\\ =&\ \phi\left(\frac{\beta_{i}}{1-\beta_{i}}\right)\frac{1}{k}\left(\beta_{i+1}^k-\beta_i^k\right)n\log^k n-\epsilon\frac{1}{k}\left(\beta_{i+1}^k-\beta_i^k\right)n\log^k n\\ +&\ o\left(n\log^k n\right)\end{aligned} \tag{2.21}$$

We have (mean value Theorem)

$$\begin{aligned}&\sum_{i=0}^{k-1}\phi\left(\frac{\beta_i}{1-\beta_i}\right)\frac{1}{k}\left(\beta_{i+1}^k-\beta_i^k\right)\geq\sum_{i=0}^{k-1}\phi\left(\frac{\beta_i}{1-\beta_i}\right)\beta_i^{k-1}\left(\beta_{i+1}-\beta_i\right)\\ =&\int_\alpha^{1/2}\phi\left(\frac{x}{1-x}\right)x^{k-1}\,dx\\ -&\sum_{i=0}^{k-1}\int_{\beta_i}^{\beta_{i+1}}\left(\phi\left(\frac{x}{1-x}\right)x^{k-1}-\phi\left(\frac{\beta_i}{1-\beta_i}\right)\beta_i^{k-1}\right)\,dx\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}0\ &<\ \sum_{i=0}^{k-1}\int_{\beta_i}^{\beta_{i+1}}\left(\phi\left(\frac{x}{1-x}\right)x^{k-1}-\phi\left(\frac{\beta_i}{1-\beta_i}\right)\beta_i^{k-1}\right)\,dx\\ &\leq\ \sum_{i=0}^{k-1}\left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)\beta_{i+1}^{k-1}-\phi\left(\frac{\beta_i}{1-\beta_i}\right)\beta_i^{k-1}\right)\left(\beta_{i+1}-\beta_i\right)\\ &\leq\ \epsilon\sum_{i=0}^{k-1}\left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)\beta_{i+1}^{k-1}-\phi\left(\frac{\beta_i}{1-\beta_i}\right)\beta_i^{k-1}\right)\\ &=\ \epsilon\left(\left(\frac{1}{2}\right)^{k-1}-\phi\left(\frac{\alpha}{1-\alpha}\right)\alpha^{k-1}\right)<\epsilon\end{aligned} \tag{2.23}$$

Therefore, equations (2.22) and (2.23) give

$$\sum_{i=0}^{k-1} \phi\left(\frac{\beta_i}{1-\beta_i}\right) \frac{1}{k}\left(\beta_{i+1}^k - \beta_i^k\right) \geq \int_\alpha^{1/2} \phi\left(\frac{x}{1-x}\right) x^{k-1}\, dx - \epsilon \tag{2.24}$$

Note that (mean value Theorem)

$$\begin{aligned} 0 &< \sum_{i=0}^{k-1} \frac{1}{k}\left(\beta_{i+1}^k - \beta_i^k\right) \leq \sum_{i=0}^{k-1} \left(\beta_{i+1} - \beta_i\right) \beta_{i+1}^{k-1} \leq \left(\frac{1}{2}\right)^{k-1} \sum_{i=0}^{k-1} \left(\beta_{i+1} - \beta_i\right) \\ &= \left(\frac{1}{2} - \alpha\right)\left(\frac{1}{2}\right)^{k-1} < 1 \end{aligned} \tag{2.25}$$

Equations (2.10), (2.21), (2.24) and (2.25) give

$$\begin{aligned} A(n) &\geq \left(\sum_{i=0}^{k-1} \phi\left(\frac{\beta_i}{1-\beta_i}\right) \frac{1}{k}\left(\beta_{i+1}^k - \beta_i^k\right)\right) n \log^k n \\ &- \epsilon \left(\sum_{i=0}^{k-1} \left(\beta_{i+1}^k - \beta_i^k\right)\right) n \log^k n + o(n \log^k n) \\ &\geq \left(\int_\alpha^{1/2} \phi\left(\frac{x}{1-x}\right) x^{k-1}\, dx\right) n \log^k n - 2\epsilon n \log^k n + o(n \log^k n) \end{aligned} \tag{2.26}$$

we have (mean value Theorem)

$$\begin{aligned} &\sum_{i=0}^{k-1} \phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right) \frac{1}{k}\left(\beta_{i+1}^k - \beta_i^k\right) \leq \sum_{i=0}^{k-1} \phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right) \beta_{i+1}^{k-1} \left(\beta_{i+1} - \beta_i\right) \\ &= \int_\alpha^{1/2} \phi\left(\frac{x}{1-x}\right) x^{k-1}\, dx \\ &+ \sum_{i=0}^{k-1} \int_{\beta_i}^{\beta_{i+1}} \left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right) \beta_{i+1}^{k-1} - \phi\left(\frac{x}{1-x}\right) x^{k-1}\right)\, dx \end{aligned} \tag{2.27}$$

where

$$\begin{aligned} 0 &< \sum_{i=0}^{k-1} \int_{\beta_i}^{\beta_{i+1}} \left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right) \beta_{i+1}^{k-1} - \phi\left(\frac{x}{1-x}\right) x^{k-1}\right)\, dx \\ &\leq \sum_{i=0}^{k-1} \left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right) \beta_{i+1}^{k-1} - \phi\left(\frac{\beta_i}{1-\beta_i}\right) \beta_i^{k-1}\right) \left(\beta_{i+1} - \beta_i\right) \\ &\leq \epsilon \sum_{i=0}^{k-1} \left(\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right) \beta_{i+1}^{k-1} - \phi\left(\frac{\beta_i}{1-\beta_i}\right) \beta_i^{k-1}\right) \\ &= \epsilon \left(\left(\frac{1}{2}\right)^{k-1} - \phi\left(\frac{\alpha}{1-\alpha}\right) \alpha^{k-1}\right) < \epsilon \end{aligned} \tag{2.28}$$

Therefore, equations (2.27) and (2.28) give

$$\sum_{i=0}^{k-1}\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)\frac{1}{k}\left(\beta_{i+1}^k-\beta_i^k\right)\leq\int_{\alpha}^{1/2}\phi\left(\frac{x}{1-x}\right)x^{k-1}\,dx+\epsilon \tag{2.29}$$

Equations (2.10), (2.20), (2.29) and (2.25) give

$$\begin{aligned}A(n) &\leq \left(\sum_{i=0}^{k-1}\phi\left(\frac{\beta_{i+1}}{1-\beta_{i+1}}\right)\frac{1}{k}\left(\beta_{i+1}^k-\beta_i^k\right)\right)n\log^k n\\ &+\ \epsilon\left(\sum_{i=0}^{k-1}\left(\beta_{i+1}^k-\beta_i^k\right)\right)n\log^k n+o(n\log^k n)\\ &\leq\ \left(\int_{\alpha}^{1/2}\phi\left(\frac{x}{1-x}\right)x^{k-1}\,dx\right)n\log^k n+2\epsilon n\log^k n+o(n\log^k n) \quad (2.30)\end{aligned}$$

Equations (2.26) and (2.30) give

$$\begin{aligned}&\int_{\alpha}^{1/2}\phi\left(\frac{x}{1-x}\right)x^{k-1}\,dx-3\epsilon\leq\int_{\alpha}^{1/2}\phi\left(\frac{x}{1-x}\right)x^{k-1}\,dx-2\epsilon+o(1)\\ \leq\ &\frac{A(n)}{n\log^k n}\leq\int_{\alpha}^{1/2}\phi\left(\frac{x}{1-x}\right)x^{k-1}\,dx+2\epsilon+o(1)\\ \leq\ &\int_{\alpha}^{1/2}\phi\left(\frac{x}{1-x}\right)x^{k-1}\,dx+3\epsilon\end{aligned}$$

This last equation and equations (2.8), (2.9) and (2.10) give equations (2.6) and (2.7), since $\epsilon$ is arbitrarily small. □

**Corollary 2.3.** *The following asymptotic formula holds*

$$\sum_{2\leq p\leq n}\log^k pB(n,p)=\sum_{i=2}^{n}\log^k P(i)\sim a_k n\log^k n \quad (k\geq 1) \tag{2.31}$$

*where*

$$a_k=\frac{1}{k}\left(1-\left(\frac{1}{2}\right)^k\right)+\int_0^{1/2}\phi\left(\frac{x}{1-x}\right)x^{k-1}\,dx \quad (k\geq 1)$$

*Proof.* In this proof, for sake of simplicity, we put $\phi\left(\frac{x}{1-x}\right)x^{k-1}=\rho$.

Let $\epsilon>0$. We choose $\alpha<\epsilon$.

The set of multiples of $p$ not exceeding $n$ is $\left\{p.1,p.2,p.3,\ldots,p\left\lfloor\frac{n}{p}\right\rfloor\right\}$ Therefore we have the inequality $B(n,p)\leq\frac{n}{p}$, and consequently we have (see (1.10))

$$\sum_{2\leq p\leq n^{\alpha}}\log^k pB(n,p)\leq n\sum_{2\leq p\leq n^{\alpha}}\frac{\log^k p}{p}=\frac{\alpha^k}{k}n\log^k n+o(n\log^k n)$$

Hence

$$0\leq\frac{\sum_{2\leq p\leq n^{\alpha}}\log^k pB(n,p)}{n\log^k n}\leq\frac{\alpha^k}{k}+o(1)\leq\alpha+o(1)\leq 2\alpha \tag{2.32}$$

On the other hand, we have (see Theorem 2.2)

$$\begin{aligned}
& \frac{1}{k}\left(1-\left(\frac{1}{2}\right)^k\right)+\int_0^{1/2}\rho\,dx-\int_0^{\alpha}\rho\,dx-\epsilon \\
= & \frac{1}{k}\left(1-\left(\frac{1}{2}\right)^k\right)+\int_{\alpha}^{1/2}\rho\,dx-\epsilon \\
\leq & \frac{\sum_{n^\alpha<p\leq n}\log^k pB(n,p)}{n\log^k n} \\
\leq & \frac{1}{k}\left(1-\left(\frac{1}{2}\right)^k\right)+\int_{\alpha}^{1/2}\rho\,dx+\epsilon \\
= & \frac{1}{k}\left(1-\left(\frac{1}{2}\right)^k\right)+\int_0^{1/2}\rho\,dx-\int_0^{\alpha}\rho\,dx+\epsilon \qquad (2.33)
\end{aligned}$$

If we sum (2.32) and (2.33) then we find that

$$\begin{aligned}
& \frac{1}{k}\left(1-\left(\frac{1}{2}\right)^k\right)+\int_0^{1/2}\rho\,dx-3\epsilon \\
\leq & \frac{1}{k}\left(1-\left(\frac{1}{2}\right)^k\right)+\int_0^{1/2}\rho\,dx-\int_0^{\alpha}\rho\,dx-\epsilon \\
\leq & \frac{\sum_{2\leq p\leq n}\log^k pB(n,p)}{n\log^k n} \\
\leq & \frac{1}{k}\left(1-\left(\frac{1}{2}\right)^k\right)+\int_0^{1/2}\rho\,dx+\left(\alpha-\int_0^{\alpha}\rho\,dx\right)+\alpha+\epsilon \\
\leq & \frac{1}{k}\left(1-\left(\frac{1}{2}\right)^k\right)+\int_0^{1/2}\rho\,dx+3\epsilon
\end{aligned}$$

Since

$$\int_0^{\alpha}\phi\left(\frac{x}{1-x}\right)x^{k-1}\,dx\leq\alpha\phi\left(\frac{\alpha}{1-\alpha}\right)\alpha^{k-1}\leq\alpha^k\phi\left(\frac{\alpha}{1-\alpha}\right)\leq\alpha$$

□

Note that if $k=1$ equation (2.31) becomes equation (1.3), where

$$a=a_1=\frac{1}{2}+\int_0^{1/2}\phi\left(\frac{x}{1-x}\right)\,dx$$

Let us consider the following function $F(x)$. $F(x)=\phi\left(\frac{x}{1-x}\right)x^{k-1}$ if $(0\leq x\leq 1/2)$ and $F(x)=x^{k-1}$ if $(1/2\leq x\leq 1)$. Using this function we can write Theorem 2.1 and Theorem 2.2 in the following unique theorem.

**Theorem 2.4.** *Let $0 < \alpha < 1$ a fixed real number. We have the following asymptotic formula*

$$\sum_{n^\alpha < p \leq n} \log^k p B(n,p) \sim a_{\alpha,k} n \log^k n$$

*where $a_{\alpha,k} = \int_\alpha^1 F(x)\ dx$.*

In the same way we can write Corollary 2.3 as follows

**Corollary 2.5.** *We have the following asymptotic formula*

$$\sum_{i=2}^{n} \log^k P(i) \sim a_k n \log^k n$$

*where $a_k = \int_0^1 F(x)\ dx$.*

## 3. On the logarithm of the least prime factor

Let $p(n)$ be the least prime factor of the positive integer $n$.
In a previous article [2] we prove the following theorem.

**Theorem 3.1.** *We have*

$$\sum_{i=2}^{n} \log p(i) = n g(n)$$

*where $g(n) \to \infty$ and $g(n) = o(\log n)$. Therefore*

$$\sum_{i=2}^{n} \log p(i) = o(n \log n)$$

In exactly the same way (see [2]) can be proved the following more general theorem.

**Theorem 3.2.** *We have*

$$\sum_{i=2}^{n} \log^k p(i) = n g(n)$$

*where $g(n) \to \infty$ and $g(n) = o(\log^k n)$. Therefore*

$$\sum_{i=2}^{n} \log^k p(i) = o(n \log^k n)$$

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