

ASYMPTOTIC NORMALITY OF ADAPTIVE RECURSIVE KERNEL ESTIMATOR OF THE DENSITY IN A RIGHT CENSORED MODEL

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ABSTRACT. This result focuses on the asymptotic properties of an adaptive recursive kernel density estimator for independent right-censored data. We establish asymptotic normality under regularity condition. As an application, we construct asymptotic confidence intervals for the unknown lifetime density function. The theoretical results are supported by a detailed simulation study, illustrating the estimator's accuracy, efficiency, and robustness in practical settings.

Keywords. Adaptive recursive kernel estimates, density, asymptotic normality, right censored independent data, confidence interval

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1. INTRODUCTION

Nonparametric kernel estimation has been thoroughly studied under a variety of structural assumptions on the data-generating process, including independence, various forms of dependence, and both complete and censored observations. This estimation framework provides a flexible and powerful approach for capturing the uncertainty inherent in forecasting and inference tasks.

The kernel density estimator was first introduced by [16] in the setting of independent and fully observed data. However, in many practical applications such as reliability engineering or biomedical studies, observations are often subject to censoring. For instance, in lifetime data analysis, the exact failure time of a component may not be observable; instead, it may only be known to exceed a given threshold. Handling such incomplete information requires statistical tools capable of accounting for censoring while ensuring consistent and accurate estimation.

Consequently, the theoretical properties of nonparametric estimators under censoring have attracted considerable attention. [7] established the consistency of density and hazard estimators for independent right-censored data. [10] studied strong consistency rates under double censoring, while [15] and [5] investigated

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the asymptotic normality of kernel estimators under right-censored and more general censoring schemes, respectively.

At the same time, recursive estimation methods have gained significant interest due to their computational efficiency and suitability for online or sequential data settings. These methods allow estimates to be updated incrementally as new data arrive, without the need to recompute over the entire sample. In the context of independent and α -mixing complete data, [1] introduced a general class of recursive kernel estimators, which includes several classical recursive forms (see [6, 19, 18]). [13] extended this framework to weakly dependent data, while subsequent work explored the use of recursive techniques in forecasting applications (e.g., [1, 14]). Recently, [17] developed adaptive recursive kernel estimators for conditional density estimation in the presence of censored data.

Motivated by these developments, we propose a new adaptive recursive kernel density estimator for independent right-censored data, building upon the recursive framework introduced by [1]. The proposed estimator preserves the variance-reduction benefits of recursive methods while improving bias properties through the use of an adaptive, data-driven bandwidth. This makes the approach particularly effective in sequential and real-time applications, where computational efficiency must be balanced with statistical precision.

In this paper, we rigorously establish the asymptotic normality of the proposed estimator, providing a theoretical foundation for its use in inferential tasks such as hypothesis testing, confidence interval construction, and dynamic data analysis. The adaptive nature of the bandwidth improves the smoothness and local accuracy of the estimate, offering robustness in regions with sparse or heterogeneous data density.

This study extends the recursive estimation framework of [1] by incorporating adaptive bandwidth selection tailored to right-censored data. The resulting estimator offers notable advantages, including reduced bias, improved mean squared error, and computational efficiency—making it well-suited for applications involving evolving or irregular data structures.

The remainder of the paper is organized as follows. Section 2 introduces the right-censored data model and the construction of the adaptive recursive estimator, based on the Kaplan–Meier estimator [9]. Section 3 presents the regularity conditions and establishes the asymptotic normality of the proposed estimator. Section 4 reports simulation results to assess the estimator’s empirical performance. Finally, Section 5 provides all technical proofs.

2. ESTIMATION FRAME

Consider a positive life time variable T with a continuous distribution function $F(t)$ and an unknown density $f(t)$.

Recall that, in the complete data case, [1] defined the family of recursive estimates as

$$\bar{f}_n^\ell(t) = \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n \frac{1}{h_i^\ell} K\left(\frac{t - T_i}{h_i}\right), \quad t \in \mathbb{R}. \quad (2.1)$$

where h_n is a positive sequence tending to 0, and K is a kernel density function, $\ell \in [0, 1]$ is a smoothing parameter, it is considered here to obtain a more general estimator.

Since the life time variable is not always available, we are attentive here to define the recursive estimator (2.1) in the case of right censored data.

Suppose then that we observe only a sample $(Y_i = \min(T_i, C_i), \delta_i)_{i=1, \dots, n}$ of i.i.d observations of $Y = \min(T, C)$, where C is a right censoring positive variable, having an unknown survival function $G(t) = P(C > t)$ supposed to be continuous, and $\delta = 1_{\{T \leq C\}}$ is the indicator of censorship.

Here, we assume that the censoring variables C_i are independent and identically distributed and they are also independent of the censored variables T_i .

Note that $\mathbb{E}(\delta_i/T_i) = G(T_i)$, so we have

$$\mathbb{E} \left(K \left(\frac{t - Y_i}{h_i} \right) \frac{\delta_i}{G(Y_i)} \right) = \mathbb{E} \left(\frac{K \left(\frac{t - T_i}{h_i} \right)}{G(T_i)} \mathbb{E}(\delta_i/T_i) \right) = \mathbb{E} \left(K \left(\frac{t - T_i}{h_i} \right) \right), \quad i = 1, \dots, n.$$

Consequently from estimator given by (2.1), we first construct the pseudo-estimator

$$\tilde{f}_n^\ell(t) = \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n \frac{1}{h_i^\ell} K \left(\frac{t - Y_i}{h_i} \right) \frac{\delta_i}{G(Y_i)}. \quad (2.2)$$

Since the survival function $G(\cdot)$ is unknown, we use Kaplan and Meier estimator [9] in expression (2.2) given by

$$G_n(t) = \begin{cases} \prod_{i=1}^n \left(1 - \frac{1 - \delta_{(i)}}{n - i + 1} \right)^{\mathbb{I}_{\{Y_{(i)} \leq t\}}}, & \text{if } t < M_n, \\ \lim_{s \rightarrow M_n} G_n(s) & \text{if } t \geq M_n, \end{cases} \quad (2.3)$$

where $M_n = Y_{(n)}$ and the $(Y_{(i)}, \delta_{(i)})_{i=1, \dots, n}$ are the n ordered pairs in $Y_{(i)}$. Here, $\delta_{(i)}$ is the concomitant of $Y_{(i)}$.

We introduce finally our adaptive recursive kernel estimator adapted to right censored data

$$f_n^\ell(t) = \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n \frac{1}{h_i^\ell} K \left(\frac{t - Y_i}{h_i} \right) \frac{\delta_i}{G_n(Y_i)}, \quad t \in \mathbb{R}. \quad (2.4)$$

Note that for a sample of complete data ($\delta_{(i)} = 0, \forall i = \overline{1, n}$). The estimator (2.4) is similar in this case to the estimator introduced by [1].

3. RESULTS

We now establish the normality of the estimator defined in (2.4). The following assumptions will be imposed throughout the analysis.

(H.1): $nh_n^3 \rightarrow \infty, nh_n^5 \rightarrow 0$, as $n \rightarrow \infty$.

(H.2): $\beta_{n,r} = \frac{1}{n} \sum_{i=1}^n \left(\frac{h_i}{h_n} \right)^r \rightarrow \beta_r < \infty$, as $n \rightarrow \infty, r \leq 3$.

(K.1): K is a bounded density function with compact support.

- (K.2): $\int vK(v)dv = 0$.
 (F): f is twice derivable at t .

Roughly speaking, these assumptions are very standard and classically used in the nonparametric estimation. The technical conditions (H.1), (H.2), (K.1) and (K.2) are necessary to maintain the proofs of our results. However, the condition $nh_n^3 \rightarrow \infty$ implies that $nh_n \rightarrow \infty$, while the choice $h_n = n^{-\gamma}$ with $\frac{1}{5} < \alpha < \gamma < \frac{1}{3}$, fulfills the assumption (H.1). The particular recursion assumption of (H.2) previously mentioned in [20] for $r = -1$, [18] for $r = \frac{1}{2}$ and in [8] for $r \leq 0$, are essentials to smooth the complexity of calculations associated with recursive kernels. The assumptions (K.1), (K.2) can be fulfilled for instance by Epachenekov kernel $K(u) = \frac{3}{4}(1-u)^2\mathbb{I}_{\{|u| \leq 1\}}$, quadratic kernel $K(u) = \frac{15}{16}(1-u^2)^2\mathbb{I}_{\{|u| \leq 1\}}$ or cosinus kernel $K(u) = \frac{\pi}{4}\cos(\frac{\pi}{2}u)\mathbb{I}_{\{|u| \leq 1\}}$ as reported in [1], [5] and [11]. Lastly, the assumption (F) stands as regularity condition, it permits us to evaluate the bias and the variance terms in order to obtain our results. This condition was used, for example, by Bosq [4] for density estimation based on the Parzen–Rosenblatt estimator.

Theorem 3.1. *Assume that assumptions (H.1), (H.2), (F), (K.1) and (K.2) hold. Then the sequence*

$$\left(\sqrt{nh_n} \left(f_n^\ell(t) - f(t) \right) \right)_n$$

converges in distribution as $n \rightarrow \infty$, to a centered normal law $\mathcal{N}(0, \sigma_\ell^2)$ with variance $\sigma_\ell^2 = \frac{\beta_{1-2\ell}}{\beta_{1-\ell}^2} \frac{f(t)}{G(t)} \int K^2(t) dt$.

Remark 3.2. • [8] and [12] obtained the asymptotic normality of recursive kernel estimators in the special cases $\ell = 1$ and $\ell = \frac{1}{2}$ under independent and α -mixing complete data. Further developments by [1] and [13] extended the estimator in (2.4) to α -mixing and η -dependent complete data settings.

- The estimator in (2.4) also encompasses several classical recursive kernel estimators as special cases. For example, setting $\ell = 0$ and $\ell = 1$ retrieves the estimators proposed by Deheuvels and Wolverton, respectively (see [6, 19]).
- [5] studied the asymptotic normality of the non-recursive kernel density estimator under censoring $f_n(t) = \frac{1}{nh_n} \sum_{i=1}^n K\left(\frac{t-Y_i}{h_n}\right) \frac{\delta_i}{G_n(Y_i)}$ and derived the asymptotic variance $\sigma^2 = \frac{f(t)}{G(t)} \int K^2(t) dt$. In contrast, Theorem 1 shows that the recursive estimator in (2.4) achieves a smaller asymptotic variance, i.e., $\sigma_\ell^2 < \sigma^2$. In fact, we have

$$\lim_{n \rightarrow \infty} \frac{\mathbb{V}(f_n^\ell(t))}{\mathbb{V}(f_n(t))} = \frac{\beta_{1-2\ell}}{\beta_{1-\ell}^2},$$

which, under the standard choice $h_n = n^{-p}$, $0 < p < 1$, reduces to

$$\frac{(1-p(1+\ell))^2}{1-p(1+2\ell)} < 1, \quad \forall \ell \in [0, 1],$$

due to the convergence of $\beta_{n,r}$ to $\beta_r = \frac{1}{1-rp}$.

(Here $\mathbb{V}(f_n^\ell(x))$ denotes the variance of the estimator $f_n^\ell(t)$.)

3.1. Application to density confidence interval. From theorem 3.1, we construct confidence interval that should contain the true density at a level of $1 - \alpha$ as follows

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(f_n^\ell(t) - q_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_\ell^2}{nh_n}} \leq f(t) \leq f_n^\ell(t) + q_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_\ell^2}{nh_n}} \right) = 1 - \alpha,$$

where $q_{1-\frac{\alpha}{2}}$ is the standard gaussian $1 - \frac{\alpha}{2}$ quantile.

Examining the asymptotic variance σ_ℓ^2 , which depends on the unknown functions f and G , we adopt a plug-in approach by substituting them with their estimators f_n^ℓ and G_n . This leads to the following corollary.

Corollary 3.3. *If assumptions of Theorem 3.1 are satisfied, we obtain*

$$\left(\sqrt{\frac{nh_n}{\sigma_{n,\ell}^2}} (f_n^\ell(t) - f(t)) \right)_n$$

converges in distribution as $n \rightarrow \infty$, to a centered normal law $\mathcal{N}(0, 1)$ with estimated variance given by $\sigma_{n,\ell}^2 = \frac{\beta_{n,1-2\ell}}{\beta_{n,1-\ell}^2} \frac{f_n^\ell(t)}{G_n(t)} \int K^2(t) dt$.

Accordingly, we obtain an asymptotically computable confidence interval

$$f(t) \in \left[f_n^\ell(t) - q_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_{n,\ell}^2}{nh_n}}, f_n^\ell(t) + q_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_{n,\ell}^2}{nh_n}} \right], \text{ with probability } 1 - \alpha.$$

4. SIMULATION STUDY

In the sequel, we focus on analyzing the behavior of the adaptive estimator for right-censored data with respect to the following two aspects

- Asymptotic normality.
- Mean square error.

For this reason, the values of censored and censoring data were generated independently from Weibull distribution $T \sim W(12, 14)$ and $C \sim W(10, 11)$. Note that these choices adopt the same simulation study example used in [5], where the authors addressed the case of twice censored data. However, in our case, we study right censored data. We use the optimal smoothing parameter $h_n = c.n^{-\frac{1}{5}}$, $c > 0$.

4.1. Asymptotic normality. To illustrate first the normality of the adaptive estimator in the right censored model. We take $k = 100$ realizations of the random variable Z_n given by

$$Z_n = \frac{f_n^\ell(t) - f(t)}{\sqrt{\mathbb{V}(f_n^\ell(t))}},$$

simulated at a fixed point t .

(Here $\mathbb{V}(f_n^\ell(t))$ is the variance of the estimator $f_n^\ell(t)$).

Based on the independent sample of censored data $(\min(T_i, C_i), \delta_i)_{i=1, \dots, n}$ for two

different values of the parameter $\ell = 0$ and $\ell = \frac{1}{2}$, we select Epachenekov kernel $K(u) = \frac{3}{4}(1-u)^2\mathbb{I}_{|u|\leq 1}$ and the bandwidth of the form $h_i = c.i^{-\frac{1}{5}}$, $c > 0$, $i = \overline{1, n}$ to define the recursive estimator $f_n^\ell(t)$ adapted to the censored model defined above with a censor ratio $CR = 25 - 30\%$.

To confirm the convergence of the distribution of Z_n to the standard Gaussian one, we plot both its histogram and the Q-Q plot and we calculate the kurtosis coefficients. The validity of the theoretical result is proven for k sample realizations of sizes $n = 100$, $n = 300$ and $n = 500$. We plot also in each case the asymptotic confidence interval containing the true density.

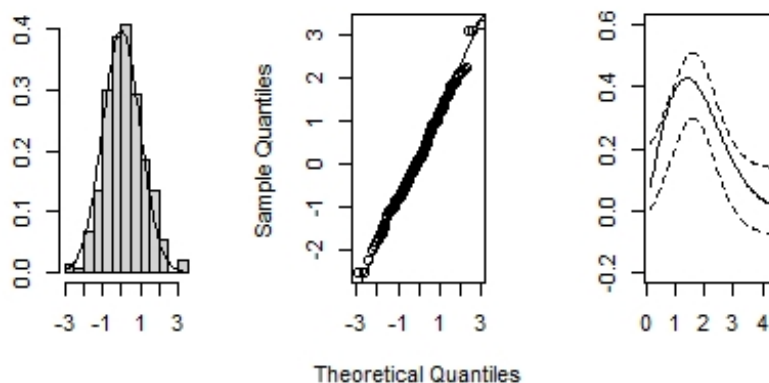


FIGURE 1. Asymptotic normality of adaptive kernel estimators with $l = 0.5, n = 100$.

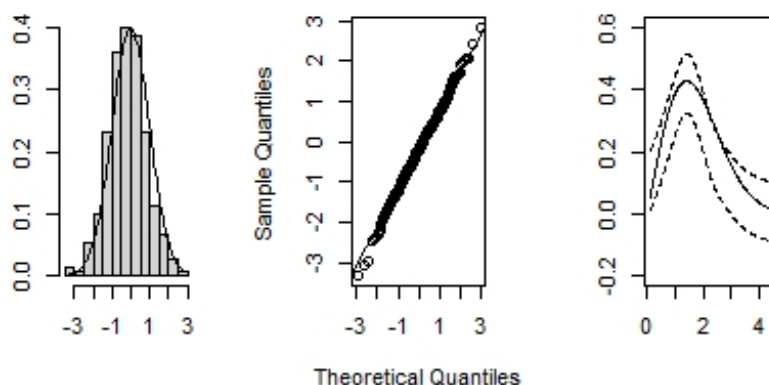


FIGURE 2. Asymptotic normality of adaptive kernel estimators with $l = 0.5, n = 300$.

TABLE 1. Kurtosis coefficient values according to sample sizes n .

	$n = 100$	$n = 300$	$n = 500$
$\ell = 0.5$	2.8862	2.8862	2.9467
$\ell = 0$	2.8876	2.9346	3.0437

From figures 1-6, we observe that the histogram is almost symmetric around the origin point and well shaped as standard normal distribution and the normal

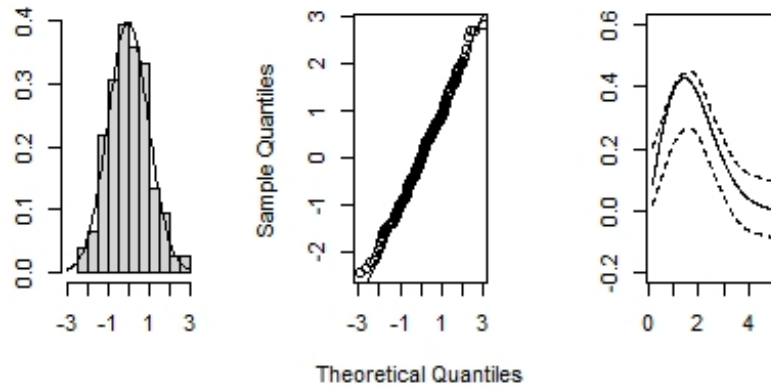


FIGURE 3. Asymptotic normality of adaptive kernel estimators with $l = 0.5, n = 500$.

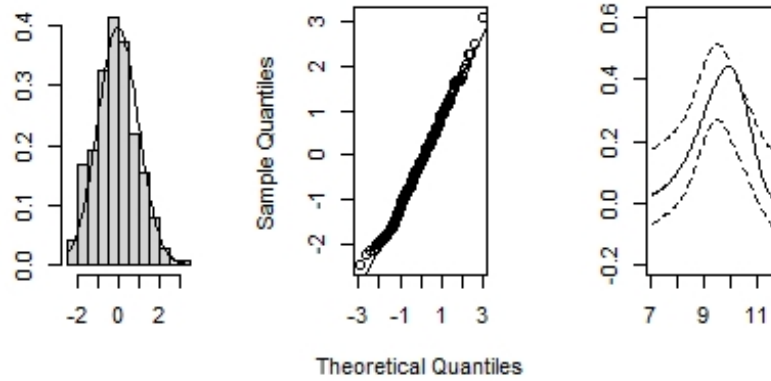


FIGURE 4. Asymptotic normality of adaptive kernel estimators with $l = 0, n = 100$.

Q-Q plot of Z_n is almost a line when the sample size n increases. Indeed, table 1 presents Kurtosis values of the empirical distribution of Z_n which are tended to be closer to the reference value 3 when the sample size n is increasing.

Consequently, these practical studies suggest that our estimator perform well through the convergence in law.

4.2. Mean square error. We generate $k = 200$ realizations of the classical kernel estimator (CKD) $f_n(t) = \frac{1}{nh_n} \sum_{i=1}^n K\left(\frac{t-T_i}{h_n}\right)$ and the adaptive kernel one (AKD) defined by (2.4) based on size sample $n_1 = 100$ and $n_2 = 300$, then we

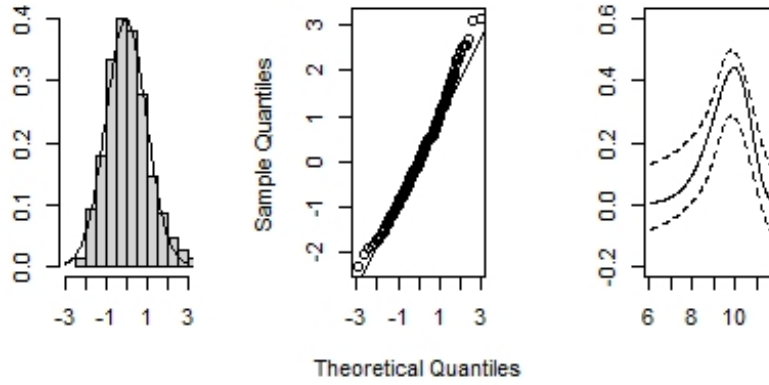


FIGURE 5. Asymptotic normality of adaptive kernel estimators with $l = 0$, $n = 300$.

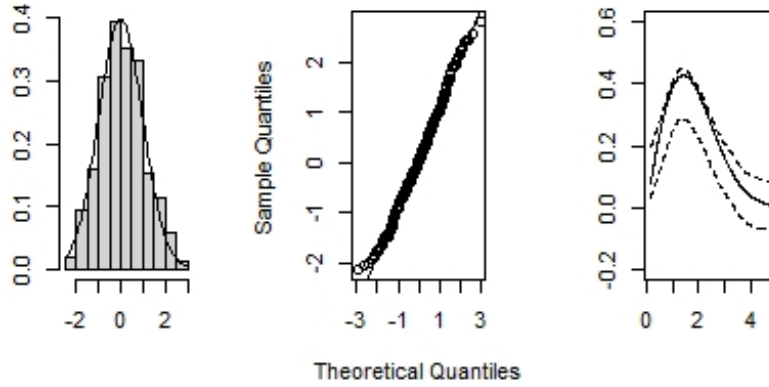


FIGURE 6. Asymptotic normality of adaptive kernel estimators with $l = 0$, $n = 500$.

analyze the empirical mean squared error (EMSE), defined as

$$EMSE = \frac{1}{k} \sum_{i=1}^k (f_{n_i}^\ell(t) - f(t))^2,$$

where $f(t)$ (resp. $f_{n_i}^\ell(t)$) is the real (resp. the estimated) value at a fixed point t . For the estimator variance we use an empirical variance through the k realisation. We display the distribution error of the two estimators for two distinct sample sizes, $n_1 = 100$ and $n_2 = 300$, using the box plot that follows. The obtained results are in table 2.

TABLE 2. Comparison of the empirical mean square error of the two classical kernel (CKD) and adaptive recursive kernel estimators (AKD)

n	AKD	CKD
100	0.00010	0.00024
300	$1.75 \cdot 10^{-5}$	$4.02 \cdot 10^{-5}$

Through the following box plot, we show the distribution error of the two estimators for two different size samples $n_1 = 100$ and $n_2 = 300$.

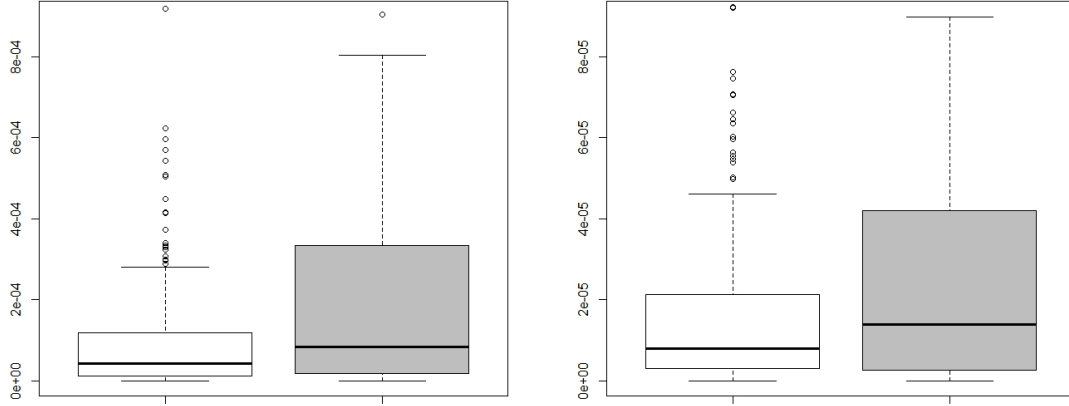


FIGURE 7. Box Plot of AKD (white) and CKD (grey) for $n = 100$, $n = 300$ respectively.

From table 2 and figure 7, by studying $EMSE$, we validate the advantageous behavior of the adaptive recursive kernel estimator (AKD) compared to the classical kernel estimator (CKD).

5. APPENDIX

In what follows, because G_n is discontinuous, we need to use the estimator (2.2). Our result will be obtained then from the following decomposition

$$\begin{aligned}
 \sqrt{nh_n}(f_n^\ell(t) - f(t)) &= \sqrt{nh_n}[(f_n^\ell(t) - \tilde{f}_n^\ell(t)) + (\tilde{f}_n^\ell(t) - \mathbb{E}\tilde{f}_n^\ell(t)) + (\mathbb{E}\tilde{f}_n^\ell(t) - f(t))] \\
 &:= I_1 + I_2 + I_3.
 \end{aligned}$$

For the first term

Set $T_Y = \sup\{t, F_Y(t) < 1\}$, for $\theta < T_Y$ we have

$$\begin{aligned} I_1 &\leq \sqrt{nh_n} \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n h_i^{-\ell} K\left(\frac{t-Y_i}{h_i}\right) \frac{|G_n(Y_i) - G(Y_i)|}{G_n(Y_i)G(Y_i)} \\ &\leq \sqrt{nh_n} \frac{\sup_{t \leq \theta} |G_n(t) - G(t)|}{G_n(\theta)G(\theta)} \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n h_i^{-\ell} K\left(\frac{t-Y_i}{h_i}\right). \end{aligned}$$

From [13], we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n h_i^{-\ell} K\left(\frac{t-Y_i}{h_i}\right) = f(t), \text{ in probability.}$$

Convergence of Kaplan-Meier estimator [3] gives

$$\sup_{t \leq \theta} |G_n(t) - G(t)| = O_{a.co} \left(\sqrt{\frac{\ln n}{n}} \right).$$

Thus, we obtain $\lim_{n \rightarrow \infty} I_1 = 0$, in probability.

For the second term,

$$I_2 = \sqrt{nh_n} (\tilde{f}_n^\ell(t) - \mathbb{E} \tilde{f}_n^\ell(t)) = \sum_{i=1}^n W_i,$$

$$\text{where } W_i = \sqrt{nh_n} \frac{h_i^{-\ell}}{\sum_{i=1}^n h_i^{1-\ell}} \left(K\left(\frac{t-Y_i}{h_i}\right) \frac{\delta_i}{G(Y_i)} - \mathbb{E} \left(K\left(\frac{t-Y_i}{h_i}\right) \frac{\delta_i}{G(Y_i)} \right) \right).$$

By assumption (K.1), we have

$$|W_i| \leq \frac{2\|K\|_\infty}{\sqrt{nh_n} \beta_{n,1-\ell} G(\theta)}.$$

Lindberg condition is satisfied, indeed,

$$\begin{aligned} \sum_{i=1}^n \frac{1}{\mathbb{V}(\tilde{f}_n^\ell(t))} \mathbb{E} \left(W_i^2 \mathbb{I}_{\{|W_i| > \epsilon \sqrt{\mathbb{V}(\tilde{f}_n^\ell(t))}\}} \right) &= \frac{n}{\mathbb{V}(\tilde{f}_n^\ell(t))} \mathbb{E} \left(W_i^2 \mathbb{I}_{\{|W_i| > \epsilon \sqrt{\mathbb{V}(\tilde{f}_n^\ell(t))}\}} \right) \\ &\leq \frac{4n\|K\|_\infty^2}{nh_n \mathbb{V}(\tilde{f}_n^\ell(t)) \beta_{n,1-\ell}^2 G^2(\theta)} \mathbb{E} \left(\mathbb{I}_{\{|W_i| > \epsilon \sqrt{\mathbb{V}(\tilde{f}_n^\ell(t))}\}} \right) \\ &\leq \frac{4n\|K\|_\infty^2}{nh_n \mathbb{V}(\tilde{f}_n^\ell(t)) \beta_{n,1-\ell}^2 G^2(\theta)} P \left(|W_1| > \epsilon \sqrt{\mathbb{V}(\tilde{f}_n^\ell(t))} \right), \end{aligned}$$

Tchebytchev inequality gives then

$$\begin{aligned} \sum_{i=1}^n \frac{1}{\mathbb{V}(\tilde{f}_n^\ell(t))} \mathbb{E} \left(W_i^2 \mathbb{I}_{\{|W_i| > \epsilon \sqrt{\mathbb{V}(\tilde{f}_n^\ell(t))}\}} \right) &\leq \frac{4\|K\|_\infty^2 n \mathbb{E}(W_1^2)}{nh_n \beta_{n,1-\ell}^2 G^2(\theta) \epsilon^2 (\mathbb{V}(\tilde{f}_n^\ell(t)))^2} \\ &\leq \frac{4\|K\|_\infty^2}{nh_n \epsilon^2 \mathbb{V}(\tilde{f}_n^\ell(t)) \beta_{n,1-\ell}^2 G^2(\theta)}. \end{aligned}$$

Now, we have

$$\begin{aligned}
\mathbb{V}(\tilde{f}_n^\ell(t)) &= \mathbb{V}\left(\frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n \frac{1}{h_i^\ell} K\left(\frac{t-Y_i}{h_i}\right) \frac{\delta_i}{G(Y_i)}\right) \\
&= \frac{1}{\left(\sum_{i=1}^n h_i^{1-\ell}\right)^2} \sum_{i=1}^n \mathbb{V}\left(\frac{1}{h_i^\ell} K\left(\frac{t-Y_i}{h_i}\right) \frac{\delta_i}{G(Y_i)}\right) \\
&= \frac{1}{n^2 h_n^{2-2\ell} \beta_{n,1-\ell}^2} \sum_{i=1}^n \mathbb{V}\left(\frac{1}{h_i^\ell} K\left(\frac{t-Y_i}{h_i}\right) \frac{\delta_i}{G(Y_i)}\right) \\
&\leq \frac{1}{n^2 h_n^{2-2\ell} \beta_{n,1-\ell}^2} \sum_{i=1}^n \mathbb{E}\left(h_i^{-2\ell} K^2\left(\frac{t-Y_i}{h_i}\right) \frac{\delta_i}{G^2(Y_i)}\right) \\
&\leq \frac{1}{n^2 h_n^{2-2\ell} \beta_{n,1-\ell}^2} \sum_{i=1}^n \mathbb{E}\left(h_i^{-2\ell} K^2\left(\frac{t-T_i}{h_i}\right) \frac{1}{G^2(T_i)} \mathbb{E}(\delta_i/T_i)\right) \\
&\leq \frac{1}{n^2 h_n^{2-2\ell} \beta_{n,1-\ell}^2} \sum_{i=1}^n \mathbb{E}\left(h_i^{-2\ell} K^2\left(\frac{t-T_i}{h_i}\right) \frac{1}{G(T_i)}\right).
\end{aligned}$$

The following result is derived through the change of variable $z = t - u$

$$\mathbb{V}(\tilde{f}_n^\ell(t)) \leq \frac{1}{nh_n \beta_{n,1-\ell}^2} \frac{1}{n} \sum_{i=1}^n \left(\frac{h_i}{h_n}\right)^{-2\ell+1} h_i^{-1} \int K^2\left(\frac{z}{h_i}\right) \frac{f(t-z)}{G(t-z)} dz < \infty.$$

Bochner theorem can be applied and provides

$$\lim_{n \rightarrow \infty} h_i^{-1} \int K^2\left(\frac{z}{h_i}\right) \frac{f(t-z)}{G(t-z)} dz = \frac{f(t)}{G(t)} \int K^2(z) dz.$$

Therefore, under (H.2), we can utilize Lemma 2.4.1 in ([1]), to get

$$\lim_{n \rightarrow \infty} nh_n \mathbb{V}(\tilde{f}_n^\ell(t)) = \sigma_\ell^2,$$

where $\sigma_\ell^2 = \frac{\beta_{1-2\ell}}{\beta_{1-\ell}^2} \cdot \frac{f(t)}{G(t)} \int K^2(z) dz < \infty$.

Consequently, under assumption (H.1), we obtain

$$\mathbb{V}(\tilde{f}_n^\ell(t)) = \frac{1}{nh_n} O(1) = o\left(\frac{1}{nh_n}\right).$$

Hence, we derive

$$\frac{1}{\mathbb{V}(\tilde{f}_n^\ell(t))} \sum_{i=1}^n \mathbb{E}\left(W_i^2 \mathbb{I}_{\{|W_i| > \epsilon \sqrt{\mathbb{V}(\tilde{f}_n^\ell(t))}\}}\right) \leq \frac{1}{nh_n} \times \frac{4\|K\|_\infty^2}{nh_n \epsilon^2 \beta_{n,1-\ell}^2 G^2(\theta)} \leq \frac{C}{(nh_n)^2}, \quad C < \infty.$$

So

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{\mathbb{V}(\tilde{f}_n^\ell(t))} \mathbb{E}\left(W_i^2 \mathbb{I}_{\{|W_i| > \epsilon \sqrt{\mathbb{V}(\tilde{f}_n^\ell(t))}\}}\right) = 0.$$

Thus central limit theorem gives

$$I_2 \sim N(0, \sigma_\ell^2), \quad \text{in Law, as } n \rightarrow \infty$$

Now for the third term I_3 , we have

$$\begin{aligned}\mathbb{E}\tilde{f}_n^\ell(t) &= \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n h_i^{-\ell} \mathbb{E}\left(K\left(\frac{t-Y_i}{h_i}\right) \frac{\delta_i}{G(Y_i)}\right) \\ &= \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n h_i^{-\ell} \mathbb{E}\left(K\left(\frac{t-T_i}{h_i}\right)\right). \\ &= \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n h_i^{-\ell} \int \left(\frac{t-u}{h_i}\right) f(u) du.\end{aligned}$$

By changing the variable $v = \frac{t-u}{h_i}$, we have

$$\mathbb{E}\tilde{f}_n^\ell(t) - f(t) = \frac{1}{\beta_{n,1-\ell}} \frac{1}{n} \sum_{i=1}^n \left(\frac{h_i}{h_n}\right)^{1-\ell} \int K(v)(f(t - vh_i) - f(t)) dv.$$

So, under assumptions (F) Taylor expansion provides

$$f(t - vh_i) = f(t) + vh_i f'(t) + \frac{v^2 h_i^2}{2!} f^{(2)}(t - \theta h_i v), \quad \eta \in]0, 1[.$$

Taken into consideration (K.2), we get

$$\mathbb{E}\tilde{f}_n^\ell(t) - f(t) = \frac{1}{\beta_{n,1-\ell}} \frac{1}{n} \sum_{i=1}^n \left(\frac{h_i}{h_n}\right)^{1-\ell} \int f^{(2)}(t - \eta h_i v) h_i^2 v^2 K(v) dv, \quad \theta \in]0, 1[.$$

By applying the dominated convergence theorem

$$\lim_{i \rightarrow \infty} \int v^2 K(t) f^{(2)}(t - \theta h_i v) dv = f^{(2)}(t) \int v^2 K(v) dv.$$

In conjunction with lemma 2.4.1 in ([1]) and under assumptions (H.2), (K.1) and (F), we obtain

$$\lim_{n \rightarrow \infty} h_n^{-2} \left(\mathbb{E}\tilde{f}_n^\ell(t) - f(t) \right) = \frac{\beta_{3-\ell}}{\beta_{1-\ell}} f^{(2)}(t) \int v^2 K(v) dv < \infty.$$

Thus, assumption (H.1) provides

$$I_3 = O\left(\sqrt{nh_n^5}\right) = o(1).$$

Finally, from terms I_1 , I_2 and I_3 , the claimed result follows by applying Slutsky theorem.

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