

INVESTIGATION OF A MATHEMATICAL MODEL FROM THE THEORY OF VISCOUS FLUIDS

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ABSTRACT. This study is devoted to a nonlinear third-order ordinary differential equation featuring movable singularities. Two principal research tasks are addressed. The first concerns establishing a theorem on the existence and uniqueness of the solution to the Cauchy problem in the vicinity of a movable singular point. To tackle this, an approximate analytical solution is constructed, and corresponding error estimates are derived. The theoretical findings are supported by numerical simulations. The second task focuses on deriving both pointwise and interval-based criteria that ensure the presence of a movable singularity. These results form the foundation for constructing an algorithm to detect the location of such singular points.

Keywords. Approximate analytical solution, Cauchy problem, nonlinear ODE, majorant method, movable singular point

2020 Mathematics Subject Classification. Primary 34A25; Secondary 34C15

1. INTRODUCTION AND PRELIMINARIES

Most natural phenomena and processes can be described using differential equations, making them one of the primary mathematical modeling tools today. Specific applications of differential equations can be found, for example, in fluid and nanofluid flow problems [4, 31]; in thermal engineering [14, 6, 18]; in cosmology [19], [7]; and in many other areas. Often, due to the complexity of the processes under study, the mathematical models exhibit nonlinear structures. One aspect of this nonlinearity is the presence of movable singular points.

The initial classification of singularities into movable and fixed types was introduced by L. Fuchs. According to the definitions presented in Golubev's monograph [13].

Singular points of the integrals of differential equations, whose position does not depend on the initial data that determine these integrals, are called *fixed singular points*, while those whose position depends on the initial data, are called *movable singular points*.

Date: Received: Jun 23, 2025; Accepted: Jul 27, 2025.

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The class of *algebraic movable singularities* comprises simple and multiple poles, as well as branch points of finite order. In the neighborhood of such points, a local expansion in the form of a Puiseux series is typically assumed.

Therefore, each class of equations requires an individual approach. Currently, the most commonly used methods include:

- (1) *Finding exact solutions.* Exact solutions are most often understood as particular solutions, that is, solutions of a specific form, or those obtained under certain assumptions. One of the most popular methods for finding exact solutions is group analysis methods [33, 32, 1]. These methods are most commonly applied when understanding the laws and physical processes, thus anticipating the behavior of the solution in advance.
- (2) *Approximate methods.* This method is based on simplifying the model by discretizing derivatives followed by the application of numerical methods [2, 20], or linearizing the model [3]. For the application of this method, it is important to understand that there are no singular points over the entire region where the problem is being solved, as otherwise, linearization or discretization could lead to an incorrect solution.
- (3) *Qualitative theory and asymptotic methods.* These methods help to understand the nature of solutions and its behavior near singular points or at infinity [9, 17], which may be sufficient, but if it is necessary to find a solution with a certain degree of accuracy, then such a method is not applicable.

It is also worth noting the actively developing direction based on the *Geometric approach* to studying differential equations in the complex domain [5].

Considering the advantages and disadvantages of the aforementioned methods, the *analytical approximation method* is applied. This method is based on dividing the solution search area into two: the region of analyticity and the neighborhood of the movable singular point. In the region of analyticity, the solution is sought in the form of a power series, and in the neighborhood of the movable singular point, it is sought in the form of a Puiseux series.

The tasks formulated in this study are as follows:

- (1) The theorem of existence and uniqueness of a solution in the analyticity domain.
- (2) Continuation of the solution along the chain of curves.
- (3) Theorem of existence and uniqueness of the solution in the neighborhood of a movable singular point.
- (4) The influence of the solution's application area, considering the approximate value of the movable singular point.
- (5) Necessary as well as necessary and sufficient conditions for the existence of a movable singular point.

This method is actively used for the study of various nonlinear differential equations, both ordinary [27, 28, 29, 10, 11, 12] and partial differential equations [16], as well as equations with fractional derivatives [30].

The author has conducted research on a nonlinear third-order differential equation with a polynomial right-hand side of second and seventh degree, respectively

[21, 22, 23, 24, 25, 26], and demonstrated how the theory can be used to construct an algorithm for locating a movable singular point with any pre-specified accuracy [26].

In reference [15], the following equation was mentioned:

$$y''' + yy'' - (y')^2 + 1 = 0,$$

in the third chapter (Section 7), where its application in the theory of viscous fluids is discussed, with a reference to the original source [8]. A boundary value problem was considered, and the solution was sought using Taylor series expansions.

In the present work, the Cauchy problem is studied for a more general mathematical model of the form:

$$y''' + yy'' = y'^2 + f(x) \quad (1.1)$$

$$\begin{cases} y(x_1) = y_1 \\ y'(x_1) = y_2 \\ y''(x_1) = y_3 \end{cases} \quad (1.2)$$

For this problem, a theorem on the existence and uniqueness of the solution in the neighborhood of a movable singular point is proved. An approximate analytical solution is constructed, along with error estimates. The validity of the theoretical results is confirmed by a numerical experiment. Furthermore, pointwise and interval-based criteria for the existence of a movable singular point are presented.

2. THEOREM ON THE EXISTENCE AND UNIQUENESS OF THE SOLUTION

Let us consider the Cauchy problem (1.1)–(1.2) and state a theorem on the existence and uniqueness of its solution.

Theorem 2.1. *Let the following conditions hold:*

1. $f(x)$ is a real analytic function.
2. x^* is a movable singular point of the Cauchy problem (1.1)–(1.2).

Then the solution to the Cauchy problem (1.1)–(1.2) has the form:

$$y(x) = (x^* - x)^{-1} \sum_{n \geq 0} A_n (x^* - x)^n, \quad (2.1)$$

in the domain of convergence of the regular part of the series

$$|x^* - x| < \frac{1}{\sqrt{M+1}},$$

where $M = \max \left\{ \sup_n (|B_n|) \right\}$, $B_n = \frac{f^{(n)}(x^*)}{n!}$, $n \in \mathbb{N} \cup \{0\}$.

Proof. Given that the function $f(x)$ is analytic by the conditions of Theorem 2.1, it can be represented as a power series:

$$f(x) = \sum_{n \geq 0} B_n (x^* - x)^n. \quad (2.2)$$

Next, we seek the solution of the Cauchy problem (1.1)–(1.2) in the form of a generalized power series:

$$y(x) = (x^* - x)^\alpha \sum_{n \geq 0} A_n (x^* - x)^n. \quad (2.3)$$

Substitute the expressions (2.2) and (2.3) into equation (1.1):

$$\begin{aligned} & - \sum_{n \geq 0} A_n (n + \alpha) (n + \alpha - 1) (n + \alpha - 2) (x^* - x)^{n+\alpha-3} \\ & + \sum_{n \geq 0} A_n (x^* - x)^{n+\alpha} \sum_{n \geq 0} A_n (n + \alpha) (n + \alpha - 1) (x^* - x)^{n+\alpha-2} \\ & = \left(\sum_{n \geq 0} A_n (n + \alpha) (x^* - x)^{n+\alpha-1} \right)^2 + \sum_{n \geq 0} B_n (x^* - x)^n. \end{aligned}$$

After the transformation, we obtain:

$$\begin{aligned} & - (x^* - x)^{\alpha-3} \sum_{n \geq 0} A_{n,1} (x^* - x)^n + (x^* - x)^{2\alpha-2} \sum_{n \geq 0} A_{n,2} (x^* - x)^n \\ & = (x^* - x)^{2\alpha-2} \sum_{n \geq 0} A_{n,3} (x^* - x)^n + \sum_{n \geq 0} B_n (x^* - x)^n \quad (2.4) \end{aligned}$$

where

$$\begin{cases} A_{n,1} = -A_n (n + \alpha) (n + \alpha - 1) (n + \alpha - 2) \\ A_{n,2} = \sum_{i=0}^n (A_i A_{n-i} (n - i + \alpha) (n - i + \alpha - 1)) \\ A_{n,3} = \sum_{i=0}^n (A_i (i + \alpha) A_{n-i} (n - i + \alpha)) \end{cases}$$

From equation (2.4), we obtain the following relations:

$$\begin{cases} n + \alpha - 3 = n + 2\alpha - 2 \\ A_{n,1} + A_{n,2} = A_{n,3}, \text{ if } n = 0, 1, 2, 3 \\ A_{n,1} + A_{n,2} = A_{n,3} + B_n, \text{ if } n > 3 \end{cases}$$

From the first equation, we obtain the value of the parameter $\alpha = -1$, which indicates the presence of a movable singular point x^* . The remaining equations of the system form a recurrence relation for finding the coefficients of the solution to the Cauchy problem (1.1)–(1.2). The coefficients that cannot be determined using the recurrence relations are defined by the initial conditions. Note that due to the uniqueness of the coefficients, the solution is unique.

To determine the region of convergence of the regular part of the series (2.1), we estimate the coefficients of the expansion by first calculating the first few terms of the series:

$$A_0 = -6, A_1 = A_2 = A_3 = 0, A_4 = -\frac{B_0}{90}, \dots$$

Let us write the recurrence relation:

$$A_{n,1} + A_{n,2} = A_{n,3} + B_n.$$

Substituting the expressions for the terms, we then obtain:

$$\begin{aligned} -(n-1)(n-2)(n-3)A_n &= \sum_{i=0}^n (A_i(i-1)A_{n-i}(n-i-1)) \\ &\quad - \sum_{i=0}^n (A_iA_{n-i}(n-i-1)(n-i-2)) + B_n \quad (2.5) \end{aligned}$$

Let's simplify the right-hand side:

$$\begin{aligned} &\sum_{i=0}^n (A_i(i-1)A_{n-i}(n-i-1)) - \sum_{i=0}^n (A_iA_{n-i}(n-i-1)(n-i-2)) \\ &= \sum_{i=0}^n (A_iA_{n-i}(n-i-1)(n-i-2-n-2i+1)) - \sum_{i=0}^n (A_iA_{n-i}(n-i-1)(n-i-2)) \\ &= - \sum_{i=0}^n (A_iA_{n-i}(n-i-1)(n+2i-1)). \end{aligned}$$

Substitute the obtained expression into (2.5):

$$-(n-1)(n-2)(n-3)A_n = - \sum_{i=0}^n (A_iA_{n-i}(n-i-1)(n+2i-1)) + B_n. \quad (2.6)$$

From equation (2.6), we solve for A_n :

$$A_n = \frac{- \sum_{i=1}^{n-1} (A_iA_{n-i}(n-i-1)(n+2i-1)) + B_n}{(n-1)(n-2)(n-3) + n^2 - 5n + 2}.$$

where $M = \max \left\{ \sup_n (|B_n|) \right\}$, $B_n = \frac{f^{(n)}(x^*)}{n!}$.

Let us prove by induction that for sufficiently large n , the following inequality holds:

$$|A_n| \leq \frac{(M+1)^{\lceil \frac{n}{2} \rceil}}{|n^3 - 5n^2 + 6n - 4|} \quad (2.7)$$

Indeed,

$$\begin{aligned} |A_n| &= \left| \frac{- \sum_{i=1}^{n-1} (A_iA_{n-i}(n-i-1)(n+2i-1)) + B_n}{(n-1)(n-2)(n-3) + n^2 - 5n + 2} \right| \\ &\leq \frac{|\sum_{i=1}^{n-1} (A_iA_{n-i}(n-i-1)(n+2i-1))| + |B_n|}{|n^3 - 5n^2 + 6n - 4|}. \end{aligned}$$

Let us estimate the expression $|\sum_{i=1}^{n-1} (A_iA_{n-i}(n-i-1)(n+2i-1))|$:

$$\begin{aligned}
 & \left| \sum_{i=1}^{n-1} (A_i A_{n-i} (n-i-1)(n+2i-1)) \right| \leq \sum_{i=1}^{n-1} |A_i| |A_{n-i}| \cdot |(n-i-1)(n+2i-1)| \\
 & \leq \sum_{i=1}^{n-1} \frac{(M+1)^{\lfloor \frac{i}{2} \rfloor}}{|i^3 - 5i^2 + 6i - 4|} \cdot \frac{(M+1)^{\lfloor \frac{n-i}{2} \rfloor}}{|(n-i)^3 - 5(n-i)^2 + 6(n-i) - 4|} \cdot |(n-i-1)(n+2i-1)| \\
 & \leq \sum_{i=1}^{n-1} \frac{(M+1)^{\lfloor \frac{i+n-i}{2} \rfloor}}{|i^3 - 5i^2 + 6i - 4|} \cdot \frac{|(n-i-1)(n+2i-1)|}{|(n-i)^3 - 5(n-i)^2 + 6(n-i) - 4|} \\
 & = (M+1)^{\lfloor \frac{n}{2} \rfloor} \sum_{i=1}^{n-1} \frac{|(n-i-1)(n+2i-1)|}{|i^3 - 5i^2 + 6i - 4| \cdot |(n-i)^3 - 5(n-i)^2 + 6(n-i) - 4|}
 \end{aligned}$$

Note that for sufficiently large n , the sum

$$\sum_{i=1}^{n-1} \frac{|(n-i-1)(n+2i-1)|}{|i^3 - 5i^2 + 6i - 4| \cdot |(n-i)^3 - 5(n-i)^2 + 6(n-i) - 4|} \ll 1,$$

and therefore,

$$\left| \sum_{i=1}^{n-1} (A_i A_{n-i} (n-i-1)(n+2i-1)) \right| \leq \frac{(M+1)^{\lfloor \frac{n}{2} \rfloor}}{2}.$$

Based on the obtained estimate for $\left| \sum_{i=1}^{n-1} (A_i A_{n-i} (n-i-1)(n+2i-1)) \right|$, we obtain the following bound for $|A_n|$:

$$|A_n| \leq \frac{\frac{(M+1)^{\lfloor \frac{n}{2} \rfloor}}{2} + M}{|n^3 - 5n^2 + 6n - 4|} \leq \frac{(M+1)^{\lfloor \frac{n}{2} \rfloor}}{|n^3 - 5n^2 + 6n - 4|}.$$

Taking into account the obtained error estimates, we obtain the region of convergence of the regular part of the series:

$$|x^* - x| < \frac{1}{\sqrt{M+1}}.$$

□

Next, we construct an approximate analytical solution to the Cauchy problem (1.1) – (1.2):

$$y_N(x) = (x^* - x)^{-1} \sum_{n=0}^N A_n (x^* - x)^n. \quad (2.8)$$

Theorem 2.2. *Under the conditions of Theorem 2.1, the error estimate for the approximate analytical solution (2.8) is given by:*

$$\Delta y_N \leq (M+1)^{\lfloor \frac{N+1}{2} \rfloor} |x^* - x|^N \left(\frac{1 + |x^* - x|}{1 - (M+1) |x^* - x|^2} \right),$$

in the area

$$|x^* - x| < \frac{1}{\sqrt{M+1}}.$$

Proof. We use the triangle inequality to estimate the error of the series remainder:

$$\begin{aligned}
 \Delta y_N = |y(x) - y_N(x)| &= \left| (x^* - x)^{-1} \sum_{n \geq 0} A_n (x^* - x)^n - (x^* - x)^{-1} \sum_{n=0}^N A_n (x^* - x)^n \right| \\
 &= \left| \sum_{n=N+1}^{\infty} A_n (x^* - x)^{n-1} \right| \leq \sum_{n=N+1}^{\infty} |A_n| |x^* - x|^{n-1} \\
 &\leq \sum_{n=N+1}^{\infty} (M+1)^{\left[\frac{n}{2}\right]} |x^* - x|^{n-1} \\
 &\leq (M+1)^{\left[\frac{N}{2}\right]} |x^* - x|^N \left(\frac{1 + |x^* - x|}{1 - (M+1)|x^* - x|^2} \right).
 \end{aligned}$$

□

2.1. Numerical Experiment. Consider the Cauchy problem (1.1)–(1.2) for the case $f(x) = -1$, as in [33], with initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = -0.5$. The approximate value of the movable singular point in this case is $x^* \approx 2.4352$. For this problem, we determine the radius of convergence $\rho = \frac{1}{\sqrt{2}} \approx 0.707$.

TABLE 1. Numerical characteristics of the approximate analytical solution.

x_1	$y_{10}(x_1)$	Δ_1	Δ_2
2	-4.111	0.005	0.0038

The parameters in Table 1 are defined as follows: $y_{10}(x_1)$ is the analytically approximated solution (2.8); Δ_1 is the a priori estimate; Δ_2 is the a posteriori estimate.

For $\Delta_1 = 0.005$, according to Theorem 2.2, we determine $N = 10$. Indeed,

$$\Delta y_N \leq 1.5^{\left[\frac{N+1}{2}\right]} \cdot 0.4352^N \cdot 2.004 \quad (2.9)$$

For $N = 10$, we obtain $\Delta y_N \leq 0.0038$. Thus, our initial estimate is slightly overestimated and can be optimized using an a posteriori estimate $\Delta_2 = 0.0038$. It is obvious that the expression (2.9) will asymptotically decrease since

$$\begin{aligned}
 1.5^{\left[\frac{N}{2} + \frac{1}{2}\right]} \cdot 0.4352^N &< 1.5^{\frac{N+1}{2}} \cdot 0.4352^N \\
 &< 1.23^{N+1} \cdot 0.4352^N < 1.23 \cdot 0.534^N \rightarrow 0
 \end{aligned}$$

as $N \rightarrow +\infty$. Therefore, various software packages can be used to determine the smallest N for which the inequality $\Delta y_N \leq 0.005$ holds. Below is the program implementation, as well as a plot of the solution in the vicinity of the movable singular point (Figure 1).

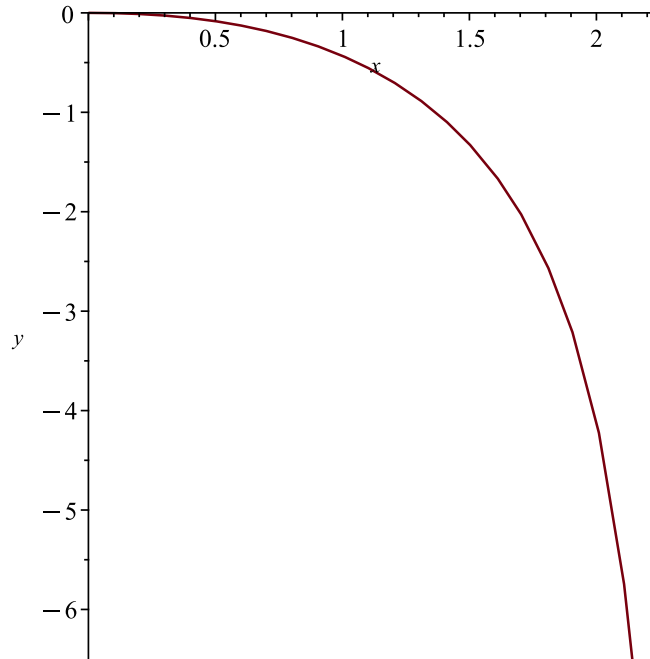
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E := (N, M, x, x_star) -> (M + 1)^floor(1/2*N + 1/2)*abs(
  x_star - x)^N*(1 + abs(x_star - x))/(1 - (M + 1)*abs(
  x_star - x)^2);

x := 2;
x_star := 2.4352;
M := 1/2;

for N from 1 to 50 do
  val := evalf(E(N, M, x, x_star));
  if val < 0.005 then
    printf("Minimal N: %d, E(N): %.6f\n", N, val);
    break;
  end if;
end do;

```

LISTING 1. Computation of the minimal N such that $\Delta y_N < 0.005$ FIGURE 1. The analytically approximated solution of the Cauchy problem (1.1)–(1.2) for the case $f(x) = -1$, with initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = -0.5$.

3. THE NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF MOVABLE SINGULAR POINTS.

In this section, we examine pointwise and interval-based criteria that characterize the existence of movable singularities. The pointwise criterion provides a

condition confirming the presence of a movable singular point, while the interval-based criterion serves to localize its position. The theorems formulated below form the basis for developing an algorithm to identify a movable singularity in a given mathematical model with prescribed accuracy.

Before proceeding to the theorems, we transform the original equation into its reciprocal form by introducing the substitution $y(x) = \frac{1}{u(x)}$. This yields the following equation:

$$\begin{aligned} y' &= -u^{-2}u', \\ y'' &= 2u^{-3}(u')^2 - u^{-2}u'', \\ y''' &= -6(u')^3u^{-4} + 6u'u''u^{-3} - u'''u^{-2}. \end{aligned}$$

Substituting the obtained expressions into equation (1.1), we obtain the inverse equation for the Cauchy problem (1.1)–(1.2):

$$\begin{aligned} & -6(u')^3u^{-4} + 6u'u''u^{-3} - u'''u^{-2} + \frac{2u^{-3}(u')^2 - u^{-2}u''}{u} \\ & \quad = \left(2u^{-3}(u')^2 - u^{-2}u''\right)^2 + f(x), \\ & -6(u')^3u^{-4} + 6u'u''u^{-3} - u'''u^{-2} + \frac{2u^{-3}(u')^2 - u^{-2}u''}{u} \\ & \quad = 4u^{-6}(u')^4 - 4u^{-5}(u')^2u'' + u^{-4}(u'')^2 + f(x), \\ & -u'''u^4 - 6(u')^3u^2 + 6u'u''u^3 + 2u^2(u')^2 - u^3u'' \\ & \quad = 4(u')^4 - 4u(u')^2u'' + u^2(u'')^2 + f(x)u^6. \quad (3.1) \end{aligned}$$

Initial conditions for the inverse Cauchy problem:

$$\begin{cases} u(x_0) = u_0 \\ u'(x_0) = u'_0 \\ u''(x_0) = u''_0 \end{cases} \quad (3.2)$$

It is evident that the movable singular point of the solution to the Cauchy problem (1.1)–(1.2) is regular for the solution of the inverse Cauchy problem (3.1)–(3.2), with $u(x^*) = 0$.

Let us formulate the theorems that provide the necessary and sufficient conditions for the existence of a movable singular point.

Theorem 3.1. *Let x^* be a movable singularity of the Cauchy problem (1.1)–(1.2), and suppose the function $y(x)$ is defined on the half-open interval $[x_0, x^*)$. Then there exists a point $\xi \in [x_0, x^*)$ such that on the interval $[\xi, x^*)$, the following inequalities hold:*

$$y(x) < 0, \quad y'(x) < 0, \quad y''(x) < 0.$$

Proof. Consider the formula (2.1) obtained in Theorem 2.1. According to the existence theorem, there exists ξ such that $\xi \in [x_0, x^*)$ and the regular part (2.1) converges in the region $[\xi, x^*)$. Then, we obtain:

$$y(x) = -6(x^* - x)^{-1} - \frac{B_0}{90}(x^* - x)^3 + \dots$$

Since $-6(x^* - x)^{-1} \xrightarrow{x \rightarrow x^*-0} -\infty$ and $-\left(\frac{B_0}{90}(x^* - x)^3 + \dots\right) \xrightarrow{x \rightarrow x^*-0} 0$, we obtain the required condition $y(x) < 0$.

Similarly, for the first and second derivatives:

$$y'(x) = -6(x^* - x)^{-2} + \frac{B_0}{30}(x^* - x)^2 + \dots$$

Since $-6(x^* - x)^{-2} \xrightarrow{x \rightarrow x^*-0} -\infty$ and $\left(\frac{B_0}{30}(x^* - x)^2 + \dots\right) \xrightarrow{x \rightarrow x^*-0} 0$, we obtain the required condition $y'(x) < 0$.

$$y''(x) = -12(x^* - x)^{-3} - \frac{B_0}{15}(x^* - x) + \dots$$

Since $-12(x^* - x)^{-3} \xrightarrow{x \rightarrow x^*-0} -\infty$ and $\left(-\frac{B_0}{15}(x^* - x) + \dots\right) \xrightarrow{x \rightarrow x^*-0} 0$, we obtain the required condition $y''(x) < 0$.

□

Theorem 3.2. *The point x^* is a movable singular point of the function $y(x)$ if and only if the function $x(u)$, which is the inverse function of the inverse Cauchy problem (3.1)–(3.2), satisfies the following conditions:*

$$x(0) = x^*, \quad x'(0) = \frac{1}{6}, \quad x''(0) = 0.$$

Proof. Necessity. First, we establish the existence of the inverse function $u(x)$. Since $u(x) = \frac{1}{y(x)}$, and by Theorem 3.2, on the half-interval $[\xi, x^*)$, we obtain that the function $u(x)$ is continuous and satisfies the condition

$$u(x) < 0, \quad u'(x) = -\frac{y'(x)}{y^2(x)} > 0$$

on the half-interval $[\xi, x^*)$. For a continuous and monotonic function, the inverse function exists. Next, we proceed with the proof of the theorem.

Let x^* be a movable singular point of the function $y(x)$. Then, taking into account that $u(x^*) = 0$ and the relationship $y(x) = \frac{1}{u(x)}$, we obtain:

$$\begin{aligned} u(x) &= \frac{1}{y(x)} = \frac{1}{(x^* - x)^{-1} \sum_{n \geq 0} A_n (x^* - x)^n} \\ &= \frac{(x^* - x)}{\sum_{n \geq 0} A_n (x^* - x)^n} = \sum_{n \geq 0} D_n (x^* - x)^{n+1}, \end{aligned}$$

$$u(x) = D_0(x^* - x) + D_1(x^* - x)^2 + \dots$$

Based on Lagrange's theorem [13] on the inversion of series, we obtain that:

$$x^* - x(u) = \sum_{n \geq 0} K_n u^{n+1}, \quad (3.3)$$

where $K_0 = \frac{1}{A_0}$, $K_2 = 0$, $K_3 \neq 0$.

From equation (3.3), we obtain that $x(0) = x^*$.

Next, by differentiating the equality (3.3), we obtain:

$$-x'(u) = (K_0 u + K_2 u^3 + \dots)' = K_0 + 3K_2 u^2 + \dots \quad (3.4)$$

From equation (3.4), we obtain that $-x'(0) = K_0 = \frac{1}{A_0} = -\frac{1}{6}$, which implies that $x'(0) = \frac{1}{6}$.

Next, by differentiating the equality (3.4), we obtain:

$$-x''(u) = (K_0 + 3K_2 u^2 + \dots)' = 6K_2 u + \dots \quad (3.5)$$

From equation (3.5), we obtain that $x''(0) = 0$. This completes the proof.

Sufficiency. Let $x(u)$ be the inverse function of the inverse Cauchy problem (3.1) – (3.2), and the following conditions are satisfied:

$$x(0) = x^*, \quad x'(0) = \frac{1}{6}, \quad x''(0) = 0.$$

From Theorem 3.2, it follows that the function $x(u)$ can be expanded into a regular series:

$$x(u) = \sum_{n \geq 0} B_n u^n. \quad (3.6)$$

Using the initial conditions, we obtain that $B_0 = x^*$.

Next, by differentiating (3.6) and substituting the condition $x'(0) = \frac{1}{6}$, we obtain:

$$x'(u) = B_1 + 2B_2 u + 3B_3 u^2 + \dots \Rightarrow B_1 = \frac{1}{6}.$$

Similarly, by differentiating equation (3.6) twice and using the condition $x''(0) = 0$, we obtain:

$$x''(u) = 2B_2 + 6B_3 u + \dots \Rightarrow B_2 = 0.$$

Considering the obtained values of the coefficients of the expansion of the function $x(u)$, we have the following expansion:

$$\begin{aligned} x(u) &= x^* + 2u + B_3 u^3 + \dots, \\ x^* - x(u) &= -2u - B_3 u^3 + \dots \end{aligned}$$

Applying Lagrange's theorem once more, we arrive at the following identity:

$$u(x) = \tilde{D}_0 (x^* - x) + \tilde{D}_1 (x^* - x)^2 + \dots,$$

where $\tilde{D}_0 = \frac{1}{A_0} = -\frac{1}{6}$. Using the relationship $y(x) = \frac{1}{u(x)}$, we show that x^* is a movable singular point of algebraic type for the Cauchy problem (1.1) – (1.2):

$$y(x) = \frac{1}{u(x)} = \frac{1}{\sum_{n \geq 0} \tilde{D}_n (x^* - x)^{n+1}} = (x^* - x)^{-1} \left(\frac{1}{\tilde{D}_0} + \tilde{A}_1 (x^* - x) + \dots \right).$$

□

Theorem 3.3. *A point x^* is a movable singularity of the function $y(x)$ if and only if there exists a neighborhood around x^* where the function $u(x)$ is continuous and takes values of opposite signs at the endpoints of this neighborhood.*

Proof. Necessity. Since $u(x)$ is defined as the reciprocal of $y(x)$, it follows that $u(x^*) = 0$. Moreover, $u(x)$ is continuous in a neighborhood of x^* . Considering the expansion

$$u(z) = -\frac{1}{6}(x^* - x) + o(x^* - x),$$

it is clear that $u(x)$ changes sign when crossing x^* . Hence, there exists an interval around x^* where $u(x)$ assumes values of different signs at its endpoints.

Sufficiency. Given that $u(x)$ is continuous and takes opposite signs at the ends of some interval, the Bolzano-Cauchy theorem guarantees the existence of a point ξ within this interval such that $u(\xi) = 0$. Since $y(x) = \frac{1}{u(x)}$, it follows that $y(x)$ has a movable singularity at ξ for the solution of the Cauchy problem (1.1)–(1.2).

□

4. CONCLUSION

In this paper, we investigated an equation arising in the theory of viscous fluids. Two problems related to nonlinear differential equations were addressed. The first problem is the formulation and proof of the existence and uniqueness theorem for solutions in the vicinity of a movable singular point, which resulted in a solution expressed as a generalized power series. The parameter $\alpha = -1$ was found, and a recurrence relation was derived, which allows the determination of the coefficients for the power series expansion of the solution to the Cauchy problem (1.1) – (1.2). Estimates for the coefficients were obtained to determine the region of convergence for the regular part of the series. An analytical approximate solution was constructed, and its error estimates were derived. The obtained theoretical results were verified through numerical experiments. The second problem involved deriving pointwise and interval criteria for the existence of a movable singular point. Theorem 3.2 establishes the mere existence of a movable singular point and is referred to as the pointwise criterion. Theorem 3.3 shows the interval in which the movable singular point should be sought and is referred to as the interval criterion. These theorems allow the construction of an algorithm for finding movable singular points, as demonstrated in the work [26].

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