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ON THE STRUCTURE OF S-2-PRIME IDEALS IN NONCOMMUTATIVE RINGS

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Abstract. This paper investigates right S-2-prime ideals in noncommutative rings, extending the concept of 2-prime and right S-prime ideals and their related structures. A proper ideal $Y \subseteq H$ such that $Y \cap S = \emptyset$ where S is an m-system, is called S-2-prime if for all elements $x, y \in H$ satisfying $xHy \subseteq Y$, there is $s \in S$ such that either $x^2\langle s\rangle \subseteq Y$ or $y^2\langle s\rangle \subseteq Y$. We present various characterizations of these ideals, particularly in domains and prime rings. One of the key results includes a characterization for prime rings, along with conditions under which right S-2-prime ideals exhibit specific properties, such as the IFP property in certain domains. This study broadens the framework of 2-prime and S-prime ideals and provides deeper insights into their structure across different ring settings.

Keywords. S-prime ideals, 2-prime ideals, m-system, noncommutative rings, S-2-prime ideals
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1. Introduction and Preliminaries

Prime ideals and prime rings serve as cornerstones in both algebraic geometry and topology due to their capacity to encapsulate the structure and behavior of algebraic systems. Their foundational nature has inspired a wide range of investigations, resulting in significant progress across various theoretical and practical domains. For more on these developments, we refer the reader to [7, 4, 13], which provide detailed discussions of various contributions expanding upon these foundational notions.

The concepts of 2-prime ideals and S-prime ideals were first introduced in [5] and [10], respectively. These notions extend the classical definition of prime ideals and provide a more general framework for examining the structure of ideals within rings. Specifically, a proper ideal Y in a commutative ring $\mathcal{H}$ is termed a 2-prime ideal if, for any $a, b \in \mathcal{H}$ satisfying $ab \in Y$, it follows that either $a^2 \in Y$ or $b^2 \in Y$. This modified condition offers an alternative perspective on multiplicative behavior by incorporating a squared element criterion.

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Similarly, a proper ideal Y of a commutative ring $\mathcal{H}$ is called an S-prime ideal if there is an element $s \in S$, where S denotes a multiplicatively closed subset of $\mathcal{H}$, such that whenever $ab \in Y$ for some $a, b \in \mathcal{H}$, it follows that either $as \in Y$ or $bs \in Y$. The inclusion of the element s introduces a refined structure to the classical notion of prime ideals while maintaining key algebraic characteristics.

In [5], Beddani et al. provided a characterization of valuation rings through the notion of 2-prime ideals, emphasizing their role in analyzing the structure of certain commutative rings. This approach has led to numerous follow-up investigations focusing on further properties of both S-prime and 2-prime ideals. For commutative settings, several aspects of these ideals have been explored in [19, 16, 14, 12, 18, 17]. In the context of noncommutative rings, extended versions of these definitions and additional results are discussed in [11, 1, 9, 2, 3], demonstrating their applicability to a broader range of noncommutative algebraic structures.

In the realm of noncommutative ring theory, the concept of S-prime ideals has been further generalized through the use of m-systems, as introduced by Abouhalaka in [1]. More precisely, in a noncommutative ring $\mathcal{H}$, a proper ideal J is said to satisfy this condition if, whenever two ideals K and L of $\mathcal{H}$ satisfy $KL \subseteq J$, there is an element $s \in S$ such that either $K\langle s\rangle \subseteq J$ or $L\langle s\rangle \subseteq J$, where $\langle s\rangle$ represents the two-sided ideal generated by s. This broader formulation proves valuable in analyzing ideal structures in noncommutative settings.

The notion of 2-prime ideals in the setting of noncommutative rings was first formulated in [11]. In this context, an ideal Y of a noncommutative ring $\mathcal{H}$ is defined to be 2-prime if, for all $a, b \in \mathcal{H}$, the condition $a\mathcal{H}b \subseteq Y$ implies that either $a^2 \in Y$ or $b^2 \in Y$. This definition parallels the commutative case while accommodating the added structural intricacies present in noncommutative rings.

The *pseudo radical* of an ideal K in a ring $\mathcal{H}$, as introduced by Birkenmeier in [8], is characterized as the sum of all ideals J of $\mathcal{H}$ for which there is some positive integer $n \in \mathbb{Z}^+$ satisfying $J^n \subseteq K$. Formally, it is given by

$$\sqrt{K} = \sum \left\{ J \subseteq \mathcal{H} \,\middle|\, J \text{ is an ideal of } \mathcal{H} \text{ and } J^n \subseteq K \text{ for some } n \in \mathbb{Z}^+ \right\}.$$

This construction yields an ideal $\sqrt{K}$ of the ring $\mathcal{H}$.

The concept of S-2-prime ideals was initially formulated by Yavuz et al. in [19] in the context of commutative ring theory. This notion offers a unified approach that extends both S-prime and 2-prime ideals. In a commutative ring $\mathcal{H}$ with a multiplicatively closed subset S, a proper ideal Y is called S-2-prime if there is an element $s \in S$ such that whenever $ab \in Y$ for some $a, b \in \mathcal{H}$, it follows that either $a^2 s \in Y$ or $b^2 s \in Y$.

In this work, we broaden the concept of S-2-prime ideals to the setting of noncommutative rings by treating S as an m-system. This generalization offers a more comprehensive theoretical framework appropriate for noncommutative algebra. To illustrate the importance and correctness of this approach, we provide counterexamples in Example 2.4 and Example 2.5.

Moreover, we investigate extensions of domain concepts—including S-domains, square domains, S-square domains, and prime rings—under the theory of S-2-prime ideals. This study uncovers relationships between the ideal-theoretic characteristics and the underlying structural aspects of rings. Additionally, we introduce the concept of pseudo semiprimary ideals to analyze the interaction between total orderings and 2-primeness in noncommutative rings.

By utilizing Proposition 2.8, we derive necessary and sufficient criteria under which the concepts of 2-prime ideals and S-2-prime ideals are equal, providing deeper understanding and guidance for subsequent research.

Throughout this paper, all rings are assumed to be noncommutative and are denoted by $\mathcal{H}$, unless otherwise stated. We let $Z_r(\mathcal{H})$ stand for the set of right zero divisors of $\mathcal{H}$, which is fundamental to our study. Moreover, the term "ideal" will refer to a two-sided ideal, and $\langle x\rangle$ denotes the two-sided ideal generated by the element x.

2. Main results

An S-2-prime ideal is defined as a generalization of both S-prime ideals and 2-prime ideals as follows:

Definition 2.1. Let A be a proper ideal of $\mathcal{H}$ and let S be an m-system in $\mathcal{H}$ such that $A \cap S = \emptyset$. The ideal A is called a right S-2-prime ideal if for any $x, y \in \mathcal{H}$ satisfying $x\mathcal{H}y \subseteq A$, there is an element $s \in S$ with the property that either $x^2\langle s\rangle \subseteq A$ or $y^2\langle s\rangle \subseteq A$.

Definition 2.2. Let A be a proper ideal of $\mathcal{H}$ and let S be an m-system in $\mathcal{H}$ such that $A \cap S = \emptyset$. The ideal A is called a right fixed S-2-prime ideal if for any $x, y \in \mathcal{H}$ satisfying $x\mathcal{H}y \subseteq A$, for a fixed element $s \in S$ with the property that either $x^2\langle s\rangle \subseteq A$ or $y^2\langle s\rangle \subseteq A$.

Remark 2.3. It is clear that every right fixed S-2-prime ideal is a right S-2-prime ideal. Definition 2.2 is necessary to establish the connection with s in the context of colon ideals.

A right S-2-prime ideal constitutes a broader class than a right S-prime ideal. To demonstrate this, we provide the following example.

Example 2.4. Let $\mathcal{H} = M_2(\mathbb{Z}[X])$ and $Y = M_2(\langle 4X^2\rangle)$. The set $S = \{s, s^2, s^4, \dots\}$ is an m-system where $s = 2(e_{11} + e_{22})$. It is clear that $Y \cap S = \emptyset$. The ideal Y is not a right S-prime since $M_2(\langle 4X\rangle)M_2(\langle X\rangle) \subseteq Y$, but neither $M_2(\langle 4X\rangle)\langle s\rangle \subseteq Y$ nor $M_2(\langle X\rangle)\langle s\rangle \subseteq Y$. Let $\mathcal{M}_1\mathcal{H}\mathcal{M}_2 \subseteq Y$ for some matrices $\mathcal{M}_1, \mathcal{M}_2$ of $\mathcal{H}$. Since $\mathcal{M}_1\mathcal{H}\mathcal{M}_2 \subseteq Y = M_2(\langle 4X^2\rangle) \subseteq M_2(\langle X\rangle) = L$ and L is prime, then $\mathcal{M}_1 \in L$ or $\mathcal{M}_2 \in L$. Thus there exist matrices B_1,B_2 and an element $X \in \mathbb{Z}[X]$ such that $\mathcal{M}_1 = XB_1$ or $\mathcal{M}_2 = XB_2$. Hence $\mathcal{M}_1^2\langle s^2\rangle \subseteq Y$ or $\mathcal{M}_2^2\langle s^2\rangle \subseteq Y$ for $s^2 \in S$. Therefore Y is a right S-2-prime ideal.

A right S-2-prime ideal also generalizes the concept of a 2-prime ideal. To clarify this, we present the following example.

Example 2.5. Let $\mathcal{H} = M_2(\mathbb{Z}[X])$ and $Y = M_2(\langle 5X \rangle)$. The set $S = \{s, s^2, s^4, \ldots\}$ is an m-system where $s = 5(e_{11} + e_{22})$. Clearly, $Y \cap S = \emptyset$. Y is not a 2-prime ideal since

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} M_2(\mathbb{Z}[X]) \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \subseteq Y,$$

but neither

$$\left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right)^2 \in Y$$

nor

$$\left(\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \right)^2 \in Y.$$

Let $\mathcal{M}_1 \mathcal{H} \mathcal{M}_2 \subseteq Y$ for some matrices $\mathcal{M}_1, \mathcal{M}_2$ of $\mathcal{H}$. Then, as in the previous example, we observe that there exist matrices A_1 and A_2 and an element $X \in \mathbb{Z}[X]$ such that $\mathcal{M}_1 = XA_1$ or $\mathcal{M}_2 = XA_2$. Thus $\mathcal{M}_1^2 \langle s \rangle \subseteq Y$ or $\mathcal{M}_2^2 \langle s \rangle \subseteq Y$ for $s \in S$. Hence Y is a right S-2-prime ideal.

The notion of a colon ideal in noncommutative rings varies somewhat from the classical definition in commutative rings. We now recall this definition as presented by Lam in [15].

Definition 2.6. In the setting of noncommutative rings, colon ideals can be defined in two distinct ways:

Let I, J be proper ideals of $\mathcal{H}$.

(1) The right colon ideal is defined as $(I : J) = \{r \in \mathcal{H} \mid Jr \subseteq I\}$.
(2) The left colon ideal is defined as $(I : J)^* = \{r \in \mathcal{H} \mid rJ \subseteq I\}$.

For convenience, throughout the remainder of the paper, the term *S-2-prime ideal* will be abbreviated as *S-2-PI*.

Theorem 2.7. *Let $Y \trianglelefteq \mathcal{H}$ be an ideal such that $Y \cap S = \emptyset$.*

(1) *Y is a right fixed S-2-PI related to $s \in S$.*
(2) *$(Y : \langle s \rangle)^*$ is a 2-prime ideal.*

Proof. (**1**) $\Rightarrow$ (**2**) Assume that Y is a right fixed S-2-PI and let $a, b \in \mathcal{H}$ satisfy $a\mathcal{H}b \subseteq (Y : \langle s \rangle)^*$. Then we have

$$a\mathcal{H}b\langle s \rangle \subseteq Y.$$

Therefore, either

$$a^2\langle s \rangle \subseteq Y \quad \text{or} \quad b^2\langle s \rangle^2 \langle s \rangle \subseteq Y.$$

If $a^2\langle s \rangle \subseteq Y$, then $a^2 \in (Y : \langle s \rangle)^*$ and there is nothing further to prove. Suppose instead that $a^2\langle s \rangle \nsubseteq Y$. In this case, since Y is a right S-2-PI, from

$$b^2\langle s \rangle^2 \langle s \rangle \subseteq Y.$$

Note that

$$b^2\mathcal{H}s\mathcal{H}s\mathcal{H}s\mathcal{H} = b^2\langle s \rangle^2 \langle s \rangle \subseteq Y$$

As S forms an m-system, we may choose $r_1 \in \mathcal{H}$ such that $s' = sr_1s \in S$, and therefore

$$b^2\mathcal{H}s'\mathcal{H}s\mathcal{H}s\mathcal{H} \subseteq Y.$$

Similarly, there exists $r_2 \in \mathcal{H}$ such that $s_1 = s'r_2s \in S$ with

$$b^2\mathcal{H}s_1\mathcal{H}s\mathcal{H} \subseteq Y.$$

Since S is an m-system and $s \in S$, there exists an element $r_k \in \mathcal{H}$ such that $s = s_1r_ks \in S$. Thus

$$b^2\mathcal{H}s\mathcal{H} = b^2\langle s\rangle \subseteq Y.$$

Therefore, we obtain $b^2 \in (Y : \langle s\rangle)^*$, so $(Y : \langle s\rangle)^*$ is a 2-prime ideal.

(2) $\Rightarrow$ **(1)** Assume that $(Y : \langle s\rangle)^*$ is a 2-prime ideal and $a, b \in \mathcal{H}$ satisfy $a\mathcal{H}b \subseteq Y$. Since $Y \subseteq (Y : \langle s\rangle)^*$, we have

$$a\mathcal{H}b \subseteq (Y : \langle s\rangle)^*.$$

Because $(Y : \langle s\rangle)^*$ is 2-prime, it follows that

$$a^2 \in (Y : \langle s\rangle)^* \quad \text{or} \quad b^2 \in (Y : \langle s\rangle)^*,$$

which means

$$a^2\langle s\rangle \subseteq Y \quad \text{or} \quad b^2\langle s\rangle \subseteq Y.$$

Thus, Y is a right fixed S-2-PI related to $s \in S$. □

Recall that the set $\overline{S} = \{s + Y : s \in S\}$ is an m-system of $\mathcal{H}/Y$ where S is an m-system of $\mathcal{H}$. It is easy to see that $(I/Y) \cap \overline{S} = \emptyset$ if and only if $I \cap S = \emptyset$.

Proposition 2.8. *Let $\mathcal{H}$ has an identity and Y an ideal of $\mathcal{H}$ such that $Y \cap S = \emptyset$, where S is an m-system in $\mathcal{H}$. Assume further that the set of right zero divisors of the quotient ring $\mathcal{H}/Y$ such that $Z_r(\mathcal{H}/Y) \cap \overline{S} = \emptyset$. Under these conditions, Y is a right S-2-PI if and only if it is a 2-prime ideal.*

Proof. Assume that Y is a right S-2-PI. Let $x \in \mathcal{H}$ and $s \in S$ be such that

$$x^2 \in (Y : \langle s\rangle)^*,$$

which implies

$$x^2\langle s\rangle \subseteq Y.$$

Passing to the quotient $\mathcal{H}/Y$, we have

$$\overline{x}^2\overline{s} \in \overline{x}^2\langle\overline{s}\rangle \subseteq \langle\overline{0}\rangle,$$

so $\overline{x}^2\overline{s} = \overline{0}$. Since $Z_r(\mathcal{H}/Y) \cap \overline{S} = \emptyset$, it follows that $\overline{x}^2 = \overline{0}$, which means $x^2 \in Y$. Therefore,

$$(Y : \langle s\rangle)^* \subseteq Y \subseteq (Y : \langle s\rangle)^*.$$

By Theorem 2.7, it follows that $Y = (Y : \langle s\rangle)^*$ is a 2-prime ideal. □

Proposition 2.9. *Every 2-prime ideal that shares no element with S is also an S-2-PI.*

Proof. Assume that Y is a 2-prime ideal of $\mathcal{H}$. Thus the set $S = \mathcal{H} \setminus Y$ forms an m-system with $S \cap Y = \emptyset$. For any $x, y \in \mathcal{H}$ satisfying $x\mathcal{H}y \subseteq Y$, it follows that either $x^2 \in Y$ or $y^2 \in Y$. Consequently, we have

$$x^2\langle s\rangle \subseteq Y \quad \text{or} \quad y^2\langle s\rangle \subseteq Y,$$

which completes the proof. □

Proposition 2.10. *Let $\mathcal{H}$ has an identity, S an m-system of $\mathcal{H}$, and Y a proper ideal of $\mathcal{H}$ such that $Y \cap S = \emptyset$. Then Y is a right fixed S-2-PI related to $s \in S$ if and only if there exists a fixed $s \in S$ for which $(Y : \langle s\rangle)^*$ is also a right fixed S-2-PI.*

Proof. Assume that Y is a right fixed S-2-PI. By Theorem 2.7, $(Y : \langle s\rangle)^*$ is a 2-prime ideal. If there existed an element $s' \in S \cap (Y : \langle s\rangle)^*$, then

$$s'\langle s\rangle \subseteq Y,$$

implying

$$s'\mathcal{H}s \subseteq Y,$$

which contradicts the assumption $S \cap Y = \emptyset$. Hence,

$$S \cap (Y : \langle s\rangle)^* = \emptyset.$$

By Proposition 2.9, it can be concluded that $(Y : \langle s\rangle)^*$ is a right (fixed) S-2-PI.

Conversely, assume that $(Y : \langle s\rangle)^*$ is a right (fixed) S-2-PI and let $x, y \in \mathcal{H}$ satisfy

$$x\mathcal{H}y \subseteq Y.$$

By Theorem 2.7, there is $s' \in S$ such that

$$((Y : \langle s\rangle)^* : \langle s'\rangle)^*$$

is a 2-prime ideal. Since

$$Y \subseteq ((Y : \langle s\rangle)^* : \langle s'\rangle)^*,$$

we have

$$x\mathcal{H}y \subseteq ((Y : \langle s\rangle)^* : \langle s'\rangle)^*.$$

Because the ideal is 2-prime, either

$$x^2 \in ((Y : \langle s\rangle)^* : \langle s'\rangle)^* \quad \text{or} \quad y^2 \in ((Y : \langle s\rangle)^* : \langle s'\rangle)^*,$$

which implies

$$x^2\langle s'\rangle \subseteq (Y : \langle s\rangle)^* \quad \text{or} \quad y^2\langle s'\rangle \subseteq (Y : \langle s\rangle)^*.$$

There exists $r_1 \in \mathcal{H}$ such that

$$s'r_1s = s_2 \in S,$$

hence

$$x^2\langle s_2\rangle \subseteq Y \quad \text{or} \quad y^2\langle s_2\rangle \subseteq Y.$$

Therefore, Y is a right (fixed) S-2-PI. □

Corollary 2.11. *Let $\mathcal{H}$ be a unital ring and $Y \lhd \mathcal{H}$ such that $Y \cap S = \emptyset$. Then $(Y : \langle s\rangle)^*$ is a 2-prime ideal if and only if $(Y : \langle s\rangle)^*$ is a right S-2-PI.*

Proof. The conclusion follows directly from Theorem 2.7 and Proposition 2.10. □

Corollary 2.12. *Let Y be a right S-2-PI of $\mathcal{H}$ for an $s \in S$. Let $B \lhd \mathcal{H}$ such that $B \cap S \neq \emptyset$. Then the following holds:*

(1) $Y \cap B$ is a right S-2-PI of $\mathcal{H}$.
(2) YB is a right S-2-PI of $\mathcal{H}$.

Proof. (1) Assume that $x, y \in \mathcal{H}$ such that $x\mathcal{H}y \subseteq Y \cap B$. Thus $x\mathcal{H}y \subseteq Y$. Hence since Y is a right S-2-PI, $x^2\langle s\rangle \subseteq Y$ or $y^2\langle s\rangle \subseteq Y$. For an element $c \in B \cap S$, we get $x^2\langle s\rangle\langle c\rangle \subseteq YB \subseteq Y \cap B$ or $y^2\langle s\rangle\langle c\rangle \subseteq YB \subseteq Y \cap B$. Since $c \in S$ and S is an m-system of $\mathcal{H}$, there exists an element $r_1 \in \mathcal{H}$ such that $s' = sr_1c$. Thus we get $x^2\langle s'\rangle \subseteq Y \cap B$ or $y^2\langle s'\rangle \subseteq Y \cap B$. Therefore $Y \cap B$ is a right S-2-PI of $\mathcal{H}$.

(2) The proof is similar to (1). □

Remark 2.13. If the pseudo-radical of an ideal is prime, the ideal is called semiprimary. In Proposition 2.8 (1) of [17], Sharifi and Hadjirezaei proved that every 2-prime ideal is semiprimary in commutative rings. As will be shown below, this result holds in the noncommutative case.

Proposition 2.14. *Let $\mathcal{H}$ has a nonzero identity and I be a proper ideal of $\mathcal{H}$. If I is a 2-prime ideal, then I is a pseudo semiprimary ideal.*

Proof. Let $x\mathcal{H}y \subseteq \sqrt{I}$ for $x, y \in \mathcal{H}$. Then $\langle x\rangle\langle y\rangle \subseteq \sqrt{I}$. Since $\langle x\rangle$ and $\langle y\rangle$ are finitely generated ideals, we have $\langle x\rangle^n\langle y\rangle^n \subseteq I$ for some $n \in \mathbb{Z}$. Thus $x^n\mathcal{H}y^n = 1x1x1...1x\mathcal{H}1y1y1...1y1 \subseteq \langle x\rangle^n\langle y\rangle^n \subseteq I$, which implies either $x^{2n} \in I$ or $y^{2n} \in I$. Hence $\langle x\rangle^{2n} \subseteq I$ or $\langle y\rangle^{2n} \subseteq I$, so $\langle x\rangle \subseteq \sqrt{I}$ or $\langle y\rangle \subseteq \sqrt{I}$. Thus we obtain either $x \in \sqrt{I}$ or $y \in \sqrt{I}$. Therefore $\sqrt{I}$ is a prime ideal, hence I is a pseudo semiprimary ideal. □

Remark 2.15. In a noncommutative ring, the *prime radical* of an ideal I, represented by $\mathcal{P}^*(I)$, is defined as the intersection of all prime ideals contained within I. Although the prime radical and the pseudo-radical coincide in the commutative setting, in the noncommutative context, the prime radical of an ideal always includes its pseudo-radical. Moreover, it is evident that an ideal whose prime radical is itself a prime ideal is referred to as *semiprimary.*

Remark 2.16. A ring is called totally ordered under inclusion if, for every pair of ideals A and B of $\mathcal{H}$, either $A \subseteq B$ or $B \subseteq A$.

Proposition 2.17. *Let $Spec(\mathcal{H})$ be the set of all prime ideals of $\mathcal{H}$. $Spec(\mathcal{H})$ is totally ordered under inclusion if and only if each ideal of $\mathcal{H}$ is semiprimary.*

Proof. Let $Spec(\mathcal{H})$ be totally ordered under inclusion and $B \lhd \mathcal{H}$. Then $\mathcal{P}^*(B) \in Spec(\mathcal{H})$. Thus B is semiprimary. Conversely, let each ideal of $\mathcal{H}$ be semiprimary and let $A, B \in Spec(\mathcal{H})$. Then $A \cap B$ is a semiprimary ideal. Assume that $A \nsubseteq B$. Then there exists an element $a \in A - B$. Suppose that $b \in B$. Thus $a\mathcal{H}b \subseteq A \cap B$. Since $A \cap B$ is semiprimary, we get $a \in \mathcal{P}^*(A \cap B)$ or $b \in \mathcal{P}^*(A \cap B)$. If $a \in \mathcal{P}^*(A \cap B)$, then $a \in B$, a contradiction. Hence $b \in \mathcal{P}^*(A \cap B) \subseteq A$. Thus $B \subseteq A$. □

Corollary 2.18. *If all the ideals of the unital ring $\mathcal{H}$ are 2-prime, then $Spec(\mathcal{H})$ is totally ordered by inclusion.*

Proof. From Proposition 2.14 and Remark 2.15, we get each 2-prime ideal is semiprimary. Hence by Proposition 2.17, we get the result. □

2.1. **S-2-PIs in Various Ring Structures.** In [1], Lemma 2.1. (1), Abouhalaka proves that $\varphi(S)$ is an m-system in the $\mathcal{H}_2$, where $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ is a ring epimorphism.

The following theorem asserts that being a right S-2-PI is preserved under a ring epimorphism φ when passing to the image $\mathcal{H}_2$.

Theorem 2.19. *Let $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ be a ring epimorphism and S be an m-system of $\mathcal{H}_1$. Let $A \lhd \mathcal{H}_1$ such that $Ker(\varphi) \subseteq A$. A is a right S-2-PI of $\mathcal{H}_1$ for an $s \in S$ if and only if $\varphi(A)$ is a right $\varphi(S)$-2-PI of $\mathcal{H}_2$ for a $\varphi(s)$.*

Proof. Let A be a right S-2-PI of $\mathcal{H}_1$. Suppose that $x\mathcal{H}_2 y \subseteq \varphi(A)$ for some $x, y \in \mathcal{H}_2$. Then there exist $a, b \in \mathcal{H}_1$ such that $a = \varphi^{-1}(x), b = \varphi^{-1}(y)$. Hence $\varphi(a) = x, \varphi(b) = y$ as φ is an epimorphism. So $\varphi(a\mathcal{H}_1 b) = x\mathcal{H}_2 y \subseteq \varphi(A)$. Hence $a\mathcal{H}_1 b \subseteq A$. Therefore $a^2\langle s\rangle \subseteq A$ or $b^2\langle s\rangle \subseteq A$. Thus $x^2\langle\varphi(s)\rangle \subseteq \varphi(A)$ or $y^2\langle\varphi(s)\rangle \subseteq \varphi(A)$. Now suppose that $\varphi(A)\cap\varphi(S) \neq \emptyset$. So there exists $k \in \varphi(A)\cap\varphi(S)$, such that $k = \varphi(a) = \varphi(s_1)$ for $a \in A, s_1 \in S$. Thus $a - s_1 \in Ker(\varphi) \subseteq A$ which implies $s_1 \in A$, a contradiction with $A\cap S = \emptyset$. Hence $\varphi(A)\cap\varphi(S) = \emptyset$. So $\varphi(A)$ is a right $\varphi(S)$-2-PI of $\mathcal{H}_2$. Conversely, suppose that $\varphi(A)$ is a right $\varphi(S)$-2-PI of $\mathcal{H}_2$ and $x, y \in \mathcal{H}_1$ such that $x\mathcal{H}_1 y \subseteq A$. Hence $\varphi(x)\mathcal{H}_2\varphi(y) \subseteq \varphi(A)$, that implies either $\varphi(x^2)\langle\varphi(s_1)\rangle \subseteq \varphi(A)$ or $\varphi(y^2)\langle\varphi(s_1)\rangle \subseteq \varphi(A)$ for a $\varphi(s_1) \in \varphi(S)$. Thus either

$$x^2\langle s_1\rangle \subseteq A$$

or

$$y^2\langle s_1\rangle \subseteq A.$$

Now assume $A \cap S \neq \emptyset$. Then there exists $a \in A \cap S$ that implies $\varphi(a) \in \varphi(A \cap S) \subseteq \varphi(A) \cap \varphi(S) = \emptyset$, a contradiction. So A is a right S-2-PI of $\mathcal{H}_1$. □

In the following, we will analyze the characteristics of prime ideals and their generalizations in the context of prime rings.

The $\mathcal{H}$ is called a prime ring if the zero ideal of $\mathcal{H}$ is prime. For more please see [15].

According to the definition in Bell [6], a right or left ideal I is said to have the IFP property (intersection of factors property) if it satisfies the condition that for $a, b \in \mathcal{H}$, whenever $ab \in I$, it follows that $a\mathcal{H}b \subseteq I$.

Let us now revisit a well-established and fundamental property of paramount importance in the theory of commutative rings: Let $\mathcal{H}$ be a commutative ring and Y be a proper ideal of $\mathcal{H}$. The ideal Y is prime if and only if $\mathcal{H}/Y$ is an integral domain. In [11], Kim demonstrated that this property can be similarly defined for noncommutative rings. According to this property in his paper, given a noncommutative ring $\mathcal{H}$ and an ideal Y possessing the IFP property, Y is prime if and only if $\mathcal{H}/Y$ is a domain. Now, we will show that a similar relationship can be established between the primality of the ideal Y and the primality of the quotient ring $\mathcal{H}/Y$ in a noncommutative ring, assuming that Y is a prime ideal.

Proposition 2.20. *The proper ideal Y of $\mathcal{H}$ is prime if and only if $\mathcal{H}/Y$ is a prime ring.*

Proof. Assume that Y is a prime ideal and let $\overline{x}(\mathcal{H}/Y)\overline{y} \subseteq \langle\overline{0}\rangle$ for some $\overline{x}, \overline{y} \in \mathcal{H}/Y$. Then $x\mathcal{H}y \subseteq Y$, so either $x \in Y$ or $y \in Y$. Hence we get either $\overline{x} \in \langle\overline{0}\rangle$ or $\overline{y} \in \langle\overline{0}\rangle$. Thus $\langle\overline{0}\rangle$ is prime. So $\mathcal{H}/Y$ is prime. Conversely, suppose that $\mathcal{H}/Y$ is a prime ring. Then the zero ideal $\langle\overline{0}\rangle$ is prime. Assume that $a\mathcal{H}b \subseteq Y$ for some $a, b \in \mathcal{H}$. Then $\overline{a}(\mathcal{H}/Y)\overline{b} \subseteq \langle\overline{0}\rangle$. Since $\langle\overline{0}\rangle$ is prime, then $\overline{a} \in \langle\overline{0}\rangle$ or $\overline{b} \in \langle\overline{0}\rangle$, so either $a \in Y$ or $b \in Y$. Hence Y is a prime ideal. □

Now, we will show that a similar property can also be obtained for S-prime ideals in noncommutative rings under certain additional special conditions.

Proposition 2.21. *Let $\overline{S} \cap Z_r(\mathcal{H}/Y) = \emptyset$ for the m-system S of $\mathcal{H}$. If the proper ideal Y of $\mathcal{H}$ is a right S-prime, then $\mathcal{H}/Y$ is a prime ring.*

Proof. Assume that Y is a right S-prime ring and $\overline{x}(\mathcal{H}/Y)\overline{y} \subseteq \langle\overline{0}\rangle$ for $\overline{x}, \overline{y} \in \mathcal{H}/Y$. Then $x\mathcal{H}y \subseteq Y$. Since Y is a right S-prime ideal, then either $x\langle s\rangle \subseteq Y$ or $y\langle s\rangle \subseteq Y$ for some $s \in S$. Hence either $\overline{x}\langle\overline{s}\rangle \subseteq \langle\overline{0}\rangle$ or $\overline{y}\langle\overline{s}\rangle \subseteq \langle\overline{0}\rangle$. Thus $\overline{x} \in \langle\overline{0}\rangle$ or $\overline{y} \in \langle\overline{0}\rangle$, since $\overline{S} \cap Z_r(\mathcal{H}/Y) = \emptyset$. Therefore $\langle\overline{0}\rangle$ is prime, so $\mathcal{H}/Y$ is prime. □

Remark 2.22. Since all prime ideals are S-prime, the converse of Proposition 2.21 also holds.

In [11], Kim defined a 2-prime ring as follows: A ring is called 2-prime if its zero ideal is 2-prime, meaning that for any elements a and b in the ring $\mathcal{H}$, if $a\mathcal{H}b \subseteq \langle 0\rangle$ implies either $a^2 \in \langle 0\rangle$ or $b^2 \in \langle 0\rangle$, then the ring is 2-prime.

Lemma 2.23. *Let Y be a proper ideal of $\mathcal{H}$ and S be an m-system of $\mathcal{H}$. Let $\overline{S} \cap Z_r(\mathcal{H}/Y) = \emptyset$. If Y is a right S-2-PI of $\mathcal{H}$, then $\mathcal{H}/Y$ is 2-prime.*

Proof. Suppose that Y is a right S-2-PI and $\overline{x}(\mathcal{H}/Y)\overline{y} \subseteq \langle\overline{0}\rangle$ for $\overline{x}, \overline{y} \in \mathcal{H}/Y$. Then $x\mathcal{H}y \subseteq Y$. Since Y is a right S-2-PI, then either $x^2\langle s\rangle \subseteq Y$ or $y^2\langle s\rangle \subseteq Y$ for some $s \in S$. Hence either $\overline{x}^2\langle\overline{s}\rangle \subseteq \langle\overline{0}\rangle$ or $\overline{y}^2\langle\overline{s}\rangle \subseteq \langle\overline{0}\rangle$. Thus $\overline{x}^2 \in \langle\overline{0}\rangle$ or $\overline{y}^2 \in \langle\overline{0}\rangle$, since $\overline{S} \cap Z_r(\mathcal{H}/Y) = \emptyset$. Therefore $\langle\overline{0}\rangle$ is 2-prime, so $\mathcal{H}/Y$ is 2-prime. □

In [11], Kim defined the concept of a square domain as follows: Let $\mathcal{H}$ be a ring and let $a, b \in \mathcal{H}$. If $ab = 0$ implies that either $a^2 = 0$ or $b^2 = 0$, then $\mathcal{H}$ is called a square domain. Now, let us examine a property between right S-2-PIs and square domains under certain special conditions.

Lemma 2.24. *Let Y be an IFP ideal of $\mathcal{H}$ and $\overline{S} \cap Z_r(\mathcal{H}/Y) = \emptyset$ for the m-system S of $\mathcal{H}$. If Y is a right S-2-PI of $\mathcal{H}$, then $\mathcal{H}/Y$ is a square domain.*

Proof. Assume that Y is a right S-2-PI and $\overline{xy} = \overline{0}$ for $\overline{x}, \overline{y} \in \mathcal{H}/Y$. Then $xy \in Y$. Since Y is an IFP ideal, we get $x\mathcal{H}y \subseteq Y$. Thus either $x^2\langle s\rangle \subseteq Y$ or $y^2\langle s\rangle \subseteq Y$ for an $s \in S$. Therefore $\overline{x}^2\langle\overline{s}\rangle \subseteq \langle\overline{0}\rangle$ or $\overline{y}^2\langle\overline{s}\rangle \subseteq \langle\overline{0}\rangle$, so $\overline{x}^2\overline{s} = \overline{0}$, or $\overline{y}^2\overline{s} = \overline{0}$. Since $\overline{S} \cap Z_r(\mathcal{H}/Y) = \emptyset$ this implies $\overline{x}^2 = \overline{0}$ or $\overline{y}^2 = \overline{0}$. Thus $\mathcal{H}/Y$ is a square domain. □

Corollary 2.25. *[Lemma 1.5 in [11]] Let Y be an IFP ideal of a noncommutative ring $\mathcal{H}$. Y is a 2-prime ideal of $\mathcal{H}$ if and only if $\mathcal{H}/Y$ is a square domain.*

Proof. By taking $S = \{1\}$ in Lemma 2.24, we get the result. □

Remark 2.26. Since each 2-prime ideal is a right S-2-PI, then the converse of Lemma 2.24 also holds.

Definition 2.27. The $\mathcal{H}$ is called a right S-square domain if, for any $x, y \in \mathcal{H}$ and an $s \in S$, the condition $xy = 0$ implies that either $x^2 s = 0$ or $y^2 s = 0$.

Definition 2.28. If for any $a, b \in \mathcal{H}$, the condition $ab = 0$ implies that for some $s \in S$, either $as = 0$ or $bs = 0$, then $\mathcal{H}$ is called a right S-domain.

It is obvious that each domain is also a right S-domain. But the converse doesn't hold in general. Now, let us justify our claim with the following example.

Example 2.29. Let $\mathcal{H} = M_2(\mathbb{Z}) \times \mathbb{Z}_p$ where p is a prime integer and $S = \{(A, s) : det(A) \neq 0, s \in \mathbb{Z}_p \setminus \{0\}\}$. Clearly, S is an m-system of $\mathcal{H}$. $\mathcal{H}$ is not a domain since

$$(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, 0)(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, 0) = (\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, 0).$$

Let $(\begin{bmatrix}a & b\\c & d\end{bmatrix}, m)(\begin{bmatrix}k & l\\j & y\end{bmatrix}, n) = (\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, 0)$ for $a, b, c, d, k, l, j, y \in \mathbb{Z}, m, n \in \mathbb{Z}_p$. Then since $\mathbb{Z}_p$ is an integral domain with $mn = 0$, either $m = 0$ or $n = 0$. Thus for $s = (\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, 1)$, we obtain either $(\begin{bmatrix}a & b\\c & d\end{bmatrix}, m)s = (\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, 0)$ or $(\begin{bmatrix}k & l\\j & y\end{bmatrix}, n)s = (\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, 0)$. Hence $\mathcal{H}$ is a right S-domain.

It is a well-known property in commutative rings that if K is a prime ideal, then the quotient ring $\mathcal{H}/K$ is an integral domain.

We will show, through the following two propositions, that the version of prime ideals defined via m-systems in noncommutative rings possesses structural properties similar to those of domains.

Proposition 2.30. *Let $Y \lhd \mathcal{H}$ be an IFP ideal. Then Y is a right S-prime ideal of $\mathcal{H}$ if and only if $\mathcal{H}/Y$ is a right $\overline{S}$-domain.*

Proof. Assume that Y is a right S-prime ideal of $\mathcal{H}$ and $\overline{x}\,\overline{y} = \overline{0}$. Since Y is an IFP ideal and $xy \in Y$, we obtain $x\mathcal{H}y \subseteq Y$. This implies either $x\langle s\rangle \subseteq Y$ or $y\langle s\rangle \subseteq Y$. Hence either $\overline{x}\,\overline{s} = \overline{0}$ or $\overline{y}\,\overline{s} = \overline{0}$. Thus $\mathcal{H}/Y$ is a $\overline{S}$-domain. Conversely, assume that $\mathcal{H}/Y$ is a right $\overline{S}$-domain and $x\mathcal{H}y \subseteq Y$. Then $xy \in Y$ which implies $\overline{x}\,\overline{y} = \overline{0}$. Since $\mathcal{H}/Y$ is a right $\overline{S}$-domain, then $\overline{x}\,\overline{s} = \overline{0}$ or $\overline{y}\,\overline{s} = \overline{0}$. Therefore either $x\langle s\rangle \subseteq Y$ or $y\langle s\rangle \subseteq Y$. Hence Y is a right S-prime ideal. □

Proposition 2.31. *Let $Y \lhd \mathcal{H}$ be an IFP ideal. Then Y is a right S-2-PI of $\mathcal{H}$ if and only if $\mathcal{H}/Y$ is a right S-square domain.*

Proof. The proof is similar to Proposition 2.30. □

In [1], Abouhalaka defined fully right S-prime rings. A ring is called a fully right S-prime ring if all of its ideals are right S-prime. Similarly, we define a fully right S-2-prime ring as a ring in which every ideal is right S-2-prime.

Definition 2.32. If each ideal of the $\mathcal{H}$ is right S-2-prime, then the ring is said to be a fully right S-2-prime ring.

Corollary 2.33. *Let $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ be a ring epimorphism, and S be an m-system in $\mathcal{H}_1$. Then $\mathcal{H}_1$ is a fully right S-2-prime ring if and only if $\mathcal{H}_2$ is a fully right $\varphi(S)$-2-prime ring.*

Proof. Assume that $\mathcal{H}_1$ is a fully right S-2-prime ring, and let $Y \lhd \mathcal{H}_2$. Then $\varphi^{-1}(Y)$ is a right S-2-PI of $\mathcal{H}_1$ and contains $\ker\varphi$. Hence by Theorem 2.19, we get $\varphi(\varphi^{-1}(Y)) = Y$ is a right $\varphi(S)$-2-PI of $\mathcal{H}_2$. Conversely, let Y be an ideal of $\mathcal{H}_1$. Then $\varphi(Y)$ is a right $\varphi(S)$-2-PI of $\mathcal{H}_2$ since $\mathcal{H}_2$ is a fully right $\varphi(S)$-2-prime ring. Thus, by Theorem 2.19, we obtain Y is a right S-2-PI of $\mathcal{H}_1$. □

Conflict of interest. The authors declare no conflicts of interest.

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