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SOME TE-UNIVALENT FUNCTION SUBFAMILIES LINKED TO GENERALIZED BIVARIATE FIBONACCI POLYNOMIALS

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Abstract. Our present investigation is primarily motivated by the broad and impactful applications of special polynomials in geometric function theory. In particular, the generalized bivariate Fibonacci polynomials have recently attracted attention in the study of bi-univalent functions. In this article, we introduce and investigate a comprehensive subclass of Te-univalent functions, a recent development in geometric function theory, characterized by generalized bivariate Fibonacci polynomials. This polynomial sequence is chosen due to its flexibility, as numerous other polynomial families can be derived through appropriate specialization of its parameters. By applying the subordination technique, we derive bounds for the initial coefficients of functions belonging to this subfamily and investigate the associated Fekete–Szegö problem. In addition to presenting several new findings, we also explore meaningful connections with previously established results in the theory of bi-univalent and subordinate functions, thereby extending and unifying existing literature in a novel direction.

Keywords. Holomorphic functions, Te-univalent functions, Initial coefficients, Generalized bivariate Fibonacci polynomials
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1. Introduction and preliminaries

Geometric Function Theory (GFT) is a dynamic and influential area within complex analysis that focuses on the geometric properties of analytic functions. It explores how holomorphic functions map domains in the complex plane and investigates topics such as univalent (injective) functions, conformal mappings, and distortion theorems. GFT has a rich history, with foundational contributions from mathematicians like Riemann, Schwarz, and Koebe, and it continues to evolve with modern developments in quasiconformal mappings, Teichmüller theory, and applications in mathematical physics. The field's blend of geometric intuition and rigorous analysis makes it especially appealing, and its results

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have far-reaching implications across various branches of mathematics. Due to its depth and versatility, GFT has remained a fertile ground for research, consistently drawing the interest of mathematicians worldwide. One area of continued research interest within GFT is the study of Te-univalent functions. The study of Te-univalent functions not only enriches the theoretical foundations of GFT but also opens pathways to applications in areas such as fluid dynamics, engineering, and conformal mapping theory.

let $\mathfrak{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. The class of holomorphic functions ϕ in $\mathfrak{U}$ of the form

$$\phi(\varsigma) = \varsigma + d_2\varsigma^2 + d_3\varsigma^3 + \cdots = \varsigma + \sum_{j=2}^{\infty} d_j\varsigma^j, \quad \varsigma \in \mathfrak{U}, \tag{1.1}$$

is identified by $\mathcal{A}$. Let $\mathcal{S}$ be the subset of $\mathcal{A}$ defined by $\mathcal{S}$=$\{\phi \in \mathcal{A} : \phi$ is univalent in $\mathfrak{U}\}$. Bieberbach conjectured in [4] that for every function $\phi \in \mathcal{S}$, $|d_j| \leq j, j \geq 2$. In order to resolve the Bieberbach conjecture, many new subclasses of $\mathcal{S}$ were defined, and several results were established. Branges finally resolved this hypothesis for each $j \geq 2$ in [8] after years of research into its proof. For each function $\phi \in \mathcal{S}$ [13], Fekete-Szegö Functional (FSF) $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$ is another GFT problem. Researchers have published a large number of articles on FSF for functions that are members of subfamilies of $\mathcal{S}$. The bi-univalent function class σ is one of the most renowned subfamilies of $\mathcal{S}$. Levin presented the idea of σ in [19] and is defined as σ=$\{\phi \in \mathcal{A} : \phi$ and $\phi^{-1} = \psi$ are both univalent in $\mathfrak{U}\}$. The renowned theorem of Koebe (see[10]) says that every function $\phi \in \mathcal{S}$ of the form (1.1) has an inverse given by

$$\phi^{-1}(w) = w - d_2w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2d_3 + d_4)w^4 + \cdots = \psi(w) \tag{1.2}$$

such that $\varsigma = \psi(\phi(\varsigma)), \varsigma \in \mathfrak{U}$, and $w = \phi(\psi(w)), |w| < r_0(\phi)$, $1/4 \leq r_0(\phi), w \in \mathfrak{U}$. Since the functions $\frac{1}{2}\log\left(\frac{1+\varsigma}{1-\varsigma}\right)$, $\frac{\varsigma}{1-\varsigma}$, and $-\log(1-\varsigma)$ are members of the σ family, the class σ is not a null set. $\frac{2\varsigma-\varsigma^2}{2}$, $\frac{\varsigma}{1-\varsigma^2}$, and the Koebe function $\frac{\varsigma}{(1-\varsigma)^2}$, despite being in $\mathcal{S}$, are not elements of σ. For a brief analysis and see [6, 5, 20, 27] to find out more about some of the characteristics of the σ family. Research on the family of bi-univalent functions have recently gained momentum thanks to Srivastava and his co-authors for an article[24]. Since this article revived the topic, numerous researchers have looked into a number of fascinating special σ families; see [14, 28, 12, 11] as well as the citation provided in these articles.

Let the operator $\mathcal{T}\colon \mathcal{A} \to \mathcal{A}$ for the function $\phi \in \mathcal{A}$ be defined as

$$\mathcal{T}\phi(\varsigma) = \varsigma + t_2d_2\varsigma^2 + t_3d_3\varsigma^3 + \cdots = \varsigma + \sum_{j=2}^{\infty} t_jd_j\varsigma^j, \quad t_j \in \mathbb{C}, and\ \varsigma \in \mathfrak{U}. \tag{1.3}$$

Assume that the family of all univalent functions of the form (1.3) in $\mathfrak{U}$ is $\mathcal{T}_{\mathcal{S}}$. Every function $\mathcal{T}\phi \in \mathcal{T}_{\mathcal{S}}$ has an inverse $(\mathcal{T}\phi)^{-1}$, given by $(\mathcal{T}\phi)^{-1}(\mathcal{T}\phi(\varsigma)) = \varsigma$, $\varsigma \in \mathfrak{U}$ and $\mathcal{T}\phi((\mathcal{T}\phi)^{-1}(w)) = w$, $|w| < r_0(\mathcal{T}\phi)$; $1/4 \leq r_0(\mathcal{T}\phi), w \in \mathfrak{U}$, where

$$(\mathcal{T}\phi)^{-1}(w) = w - t_2d_2w^2 + (2t_2^2d_2^2 - t_3d_3)w^3 - (5t_2^3d_2^3 - 5t_2t_3d_2d_3 + t_4d_4)w^4 + \cdots. \tag{1.4}$$

The function ϕ provided by (1.1) is Te-univalent in $\mathfrak{U}$ associated with $\mathcal{T}$ if both $\mathcal{T}\phi$ and $(\mathcal{T}\phi)^{-1}$ are univalent in $\mathfrak{U}$. The set of all Te-univalent functions given by (1.1) in $\mathfrak{U}$ connected to $\mathcal{T}$ is represented by $\mathcal{T}_\sigma$. When $\mathcal{T}\phi = \phi$, we have $\mathcal{T}_\sigma = \sigma$ and if $t_j \neq 1$ for some j, we have $\mathcal{T}\phi(\mathcal{T}\psi(w)) = w + 2[t_3 - t_2^2]d_2^2 w^3 + ... \neq w$, where ψ is given by (1.2).

The convolution of the function ϕ given by (1.1) and $\Omega(\varsigma) = \varsigma + c_1\varsigma + c_2\varsigma^2 + \cdots = \varsigma + \sum_{j=2}^{\infty} c_j \varsigma^j$ is defined by $(\phi * \Omega)(\varsigma) = \varsigma + \sum_{j=2}^{\infty} d_j c_j \varsigma^j = (\Omega * \phi)(\varsigma)$. For complex parameters $\eta_1, \eta_2, \cdots, \eta_q$ and $\kappa_1, \kappa_2, \cdots, \kappa_s$, $(\kappa_n \neq 0, -1, -2, \cdots, n = 1, 2, 3, \ldots, s)$, the generalized hypergeometric function ${}_qF_s(\eta_1, \eta_2, \cdots, \eta_q; \kappa_1, \kappa_2, \cdots, \kappa_s, \varsigma)$= $\sum_{j=0}^{\infty} \frac{(\eta_1)_j, \cdots, (\eta_q)_j}{(\kappa_1)_j, \cdots, (\kappa_s)_j} \frac{\varsigma^j}{j!}$, where $q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $q \leq s + 1$, $\varsigma \in \mathfrak{U}$, and the shift factorial $(\delta)_j$ in terms of the Gamma function Γ is defined as

$$(\delta)_j = \frac{\Gamma(\delta + j)}{\Gamma(\delta)} = \begin{cases} 1 & ; j = 0 \\ \delta(\delta + 1) \cdots (\delta + j - 1) & ; j \in \mathbb{N}. \end{cases}$$

Let a function $\mathfrak{H}(\eta_1, \cdots, \eta_q; \kappa_1, \cdots, \kappa_s; \varsigma)$ be defined by

$$\mathfrak{H}(\eta_1, \cdots, \eta_q; \kappa_1, \cdots, \kappa_s; \varsigma) = \varsigma\, {}_qF_s(\eta_1, \eta_2, \cdots, \eta_q; \kappa_1, \kappa_2, \cdots, \kappa_s; \varsigma), \varsigma \in \mathfrak{U}.$$

The operator $\mathcal{H}(\eta_1, \cdots, \eta_q; \kappa_1, \cdots, \kappa_s) : \mathcal{A} \to \mathcal{A}$ was examined by Dziok and Srivastava [9]. It was defined by the following Hadamard product:

$$\mathcal{H}(\eta_1, \cdots, \eta_q; \kappa_1, \cdots, \kappa_s)\phi(\varsigma) = \mathfrak{H}(\eta_1, \cdots, \eta_q; \kappa_1, \cdots, \kappa_s; \varsigma) * \phi(\varsigma),$$

where $\phi \in \mathcal{A}$, $q, s \in \mathbb{N}_0, q \leq s + 1, \varsigma \in \mathfrak{U}$.

If $\phi \in \mathcal{A}$ is given by (1.1), then we have

$$\mathcal{H}(\eta_1, \cdots, \eta_q; \kappa_1, \cdots, \kappa_s)\phi(\varsigma) = \varsigma + \sum_{j=2}^{\infty} \Gamma_j[\eta_1; \kappa_1] d_j \varsigma^j,\ \varsigma \in \mathfrak{U}, \tag{1.5}$$

where

$$\Gamma_j[\eta_1; \kappa_1] = \frac{(\eta_1)_{j-1} \cdots (\eta_q)_{j-1}}{(\kappa_1)_{j-1} \cdots (\kappa_s)_{j-1}} \frac{1}{(j-1)!}. \tag{1.6}$$

To make the notation easier to understand, we write $\mathcal{H}(\eta_1, \cdots, \eta_q; \kappa_1, \cdots, \kappa_s)\phi = \mathcal{H}_{q,s}[\eta_1; \kappa_1]\phi$. Let $\mathcal{T}_{\mathcal{S}}^{q,s}[\eta_1; \kappa_1]$ represent the set of all univalent functions provided by (1.5) in $\mathfrak{U}$. Every function $\mathcal{H}_{q,s}[\eta_1; \kappa_1]\phi \in \mathcal{T}_{\mathcal{S}}^{q,s}[\eta_1; \kappa_1]$ has an inverse $(\mathcal{H}_{q,s}[\eta_1; \kappa_1]\phi)^{-1}$, defined by $\mathcal{G}(\mathcal{H}_{q,s}[\eta_1; \kappa_1]\phi(\varsigma)) = \varsigma$, and $\mathcal{H}_{q,s}[\eta_1; \kappa_1]\phi(\mathcal{G}(w)) = w$, $|w| < r_0(\mathcal{H}_{q,s}[\eta_1; \kappa_1]\phi)$; $1/4 \leq r_0(\mathcal{H}_{q,s}[\eta_1; \kappa_1]\phi)$, ς, $w \in \mathfrak{U}$, where

$$\mathcal{G}(w) = (\mathcal{H}_{q,s}[\eta_1; \kappa_1]\phi)^{-1}(w) = w - \Gamma_2[\eta_1; \kappa_1] d_2 w^2 + [2(\Gamma_2[\eta_1; \kappa_1])^2 d_2^2 - \Gamma_3[\eta_1; \kappa_1] d_3] w^3 - \cdots. \tag{1.7}$$

A function $\phi \in \mathcal{A}$ is said to be Te-univalent [1] in $\mathfrak{U}$ connected to the operator $\mathcal{H}_{q,s}[\eta_1; \kappa_1]$, if both $\mathcal{H}_{q,s}[\eta_1; \kappa_1]\phi$ and $(\mathcal{H}_{q,s}[\eta_1; \kappa_1]\phi)^{-1}$ are univalent in $\mathfrak{U}$ (see also [22, 2]). Let $\mathcal{T}_\sigma^{q,s}[\eta_1; \kappa_1]$ represent the class of all functions $\phi \in \mathcal{A}$, which are Te-univalent in $\mathfrak{U}$ associated with $\mathcal{H}_{q,s}[\eta_1; \kappa_1]$.

Remark 1.1. i). For $q = 2, s = 1, \eta_1 = \kappa_1 = c,$ and $\eta_2 = 1$, we have $\mathcal{T}_\sigma^{2,1}[c, 1; c] = \sigma$.

ii). If $\Gamma_j[\eta_1;\kappa_1]_j \neq 1$ for some j, we have $\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\psi(w)) = w + 2[\Gamma_3[\eta_1;\kappa_1] - \Gamma_2^2[\eta_1;\kappa_1]]d_2^2 w^3 + \cdots \neq w$, where ψ is given by (1.2).

Many polynomials such as Bernoulli, Fibonacci, Gegenbauer, Horadam, Lucas-Balancing, Lucas-Lehmer, and their extensions are essential in a wide range of disciplines, including combinatorics, computer science, engineering, number theory, numerical analysis, and physics. Because of their extensive use in science and engineering, generalizations of these polynomials have been defined in the literature. Over the past two decades, the emphasis has been on functions that are linked to special polynomials or known number sequences and that belong to a particular σ subfamily. For elements of σ subfamilies that are linked to number sequences or special polynomials, several researchers have discovered coefficient estmates and Fekete-Szegö functional $|d_3 - \xi d_2^2|$, $\xi \in \mathbb{R}$ (Refer to [23, 15, 25, 16, 26]). Recently, a particular polynomials known as Generalized Bivariate Fibonacci Polynomials (GBFP) has drawn the attention of researchers (see[17]).

The recursive relationship between the Fibonacci numbers is well known as $\mathcal{F}_j = \mathcal{F}_{j-1} + \mathcal{F}_{j-2}$, $(\mathcal{F}_0 = 0, \mathcal{F}_1 = 1, j \geq 2)$. Fibonacci polynomials with generalization Fibonacci numbers are described as (see [3]) $\mathcal{F}_j(\varkappa) = \varkappa\mathcal{F}_{j-1}(\varkappa) + \mathcal{F}_{j-2}(\varkappa)$, $(\mathcal{F}_0(\varkappa) = 0, \mathcal{F}_1(\varkappa) = 1, j \geq 2)$. A new generalization of $\mathcal{F}_j(\varkappa)$ is introduced in [18] and is expressed as below:

Assume that $k(\varkappa,\vartheta)$ and $l(\varkappa,\vartheta)$ are real-coefficient polynomials. The following recurrence relation defines the GBFP for $j \geq 2$:

$$\mathcal{F}_j(\varkappa,\vartheta) = k(\varkappa,\vartheta)\mathcal{F}_{j-1}(\varkappa,\vartheta) + l(\varkappa,y)\mathcal{F}_{j-2}(\varkappa,\vartheta), \tag{1.8}$$

where $\mathcal{F}_0(\varkappa,\vartheta) = 0$, $\mathcal{F}_1(\varkappa,\vartheta) = 1$ and $k^2(\varkappa,\vartheta) + 4l(\varkappa,\vartheta) > 0$. The generating function of GBFP is (see [18])

$$\mathcal{F}(\varkappa,\vartheta,\varsigma) = \sum_{j=0}^{\infty} \mathcal{F}_j(\varkappa,\vartheta)\varsigma^j = \frac{\varsigma}{1 - k(\varkappa,\vartheta)\varsigma - l(\varkappa,\vartheta)\varsigma^2}. \tag{1.9}$$

The article [7] and its references provide a brief history and a wealth of information for readers interested in GBFP. For the sake of conciseness, we state that $l(\varkappa,\vartheta)$=l and $k(\varkappa,\vartheta) = k$. Clearly

$$\mathcal{F}_2(\varkappa,\vartheta) = k, \mathcal{F}_3(\varkappa,\vartheta) = k^2 + l, \cdots \tag{1.10}$$

are clear from (1.8).

The main goal is to create a more comprehensive and potent analytical framework for Te-univalent function research, where GBFPs are useful, organized, and computationally manageable instruments. They are effective tools for creating and examining new subclasses of Te-univalent functions due to their parametric flexibility, recursive nature, and intrinsic generality. This allows for accurate coefficient estimates and unifies different previous results as special cases. Furthermore, GBFP are especially well-suited to recent advancements in geometric function theory since they support deeper geometric and combinatorial interpretations and make it easier to construct analytic operators.

Remark 1.2. Many polynomial sequences can be inferred from GBFP by specializing a and b (see [18]). They are i). If $a = \varkappa$ and $b = \vartheta$, then we obtain bivariate

Fibonacci polynmials, ii). If $a = \varkappa$ and $b = 1$, then we obtain Fibonacci polynmials, iii). If $a = 2\varkappa$ and $b = 1$, then we arrive at the Pell polynomials, iv). if $a = 1$ and $b = 2\varkappa$, then the Jacobsthal polynomials follow, v). If $a = 3\varkappa$ and $b = -2$, the Fermat polynomials are then obtained, and vi). If $a = 2\varkappa$ and $b = -1$, then we have the kind two Chebyshev polynomials

For $\varphi_1, \varphi_2 \in \mathcal{A}$ holomorphic in $\mathfrak{U}$, φ_1 is subordinate to φ_2, if there is a Schwarz function $\theta(\varsigma)$ that is holomorphic in $\mathfrak{U}$ with $\theta(0) = 0$ and $|\theta(\varsigma)| < 1$, such that $\varphi_1(\varsigma) = \varphi_2(\theta(\varsigma))$, $\varsigma \in \mathfrak{U}$. This is symbalized as $\varphi_1 \prec \varphi_2$ or $\varphi_1(\varsigma) \prec \varphi_2(\varsigma)$. Further, if $\varphi_2 \in \mathcal{S}$, then

$$\varphi_1(\varsigma) \prec \varphi_2(\varsigma) \;\Leftrightarrow\; \varphi_1(0) = \varphi_2(0) \quad and \quad \varphi_1(\mathfrak{U}) \subset \varphi_2(\mathfrak{U}).$$

This paper's goal is to introduce two subfamilies of $\mathcal{T}_\sigma^{q,s}[\eta_1;\kappa_1]$ that are subordinate to GBFP, namely $\mathfrak{T}_\sigma^{q,s}(\varrho,\tau,\nu,x)$ and $\mathfrak{Z}_\sigma^{q,s}(\varrho,\tau,\nu,x)$, and the bound for second and third coefficients of functions in these families are obtained. Also the Fekete-Szegö inequality[13] is addressed for these function families.

Throughout this paper, unless otherwise noted, we assume that $q, s \in \mathbb{N}_0$, $q \le s = 1$, $\varrho \in \mathbb{C}\backslash\{0\}$, and $\varsigma, w \in \mathfrak{U}$.

Definition 1.3. Let $0 \le \tau \le 1$, $\nu \ge 1$ and $F(\varsigma) = \frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma} = \frac{1}{1-k\varsigma-l\varsigma^2}, k \neq 0, k^2 + 4l > 0$. If $\phi \in \mathcal{A}$ satisfies

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{[(\varsigma(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')']^\nu}{(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))'}\right)+(1-\tau)\left(\frac{\varsigma((\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')^\nu}{\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma)}\right)-1\right) \prec F(\varsigma), \tag{1.11}$$

and

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{[(w\mathcal{G}'(w))']^\nu}{\mathcal{G}'(w)}\right)+(1-\tau)\left(\frac{w(\mathcal{G}'(w))^\nu}{\mathcal{G}(w)}\right)-1\right) \prec F(w), \tag{1.12}$$

then we say that $\phi \in \mathfrak{T}_\sigma^{q,s}(\varrho,\tau,\nu,F)$, where $\mathcal{G}(w) = (\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi)^{-1}(w)$ is given by (1.7).

The motivation for introducing different subclasses of the class of the class $\mathcal{T}_\sigma^{q,s}[\eta_1;\kappa_1]$ lies in the desire to generalize and unify various well-known classes of analytic and Te-univalent functions under a broader framework. These subclasses allow for a more flexible and comprehensive study of functions exhibiting distinct geometric properties such as starlikeness, convexity, or close-to-convexity. By incorporating parameters and special functions (such as generalized polynomials), these subclasses facilitate deeper analysis of coefficient bounds, Fekete-Szegő functionals, and distortion estimates. They also enable researchers to explore the influence of different operators and parameters on function behavior, recover known results as special cases, and provide a pathway for applications in complex analysis, mathematical modeling, and applied sciences.

The following subfamilies of $\mathcal{T}_\sigma^{q,s}[\eta_1;\kappa_1]$ are obtained for specific choices of the parameters τ and ν within the general class $\mathfrak{T}_\sigma^{q,s}(\varrho,\tau,\nu,F)$:

1. Let $\tau=0$. Then $\mathfrak{P}^{q,s}_{\sigma}(\varrho,\nu,F)\equiv\mathfrak{T}^{q,s}_{\sigma}(\varrho,0,\nu,F)$, $\nu\geq 1$ and $F(\varsigma)=\frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma}=\frac{1}{1-k\varsigma-l\varsigma^2}, k\neq 0, k^2+4l>0$ is a family of functions such that $\phi\in\mathcal{A}$ fulfills

$$1+\frac{1}{\varrho}\left(\left(\frac{\varsigma((\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')^{\nu}}{\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma)}\right)-1\right)\prec F(\varsigma),$$

and

$$1+\frac{1}{\varrho}\left(\left(\frac{w(\mathcal{G}'(w))^{\nu}}{\mathcal{G}(w)}\right)-1\right)\prec F(w),$$

where $\mathcal{G}(w)=(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi)^{-1}(w)$ is given by (1.7).

2. Let $\tau=1$. Then $\mathfrak{Q}^{q,s}_{\sigma}(\varrho,\nu,F)\equiv\mathfrak{T}^{q,s}_{\sigma}(\varrho,1,\nu,F)$, $\nu\geq 1$, and $F(\varsigma)=\frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma}=\frac{1}{1-k\varsigma-l\varsigma^2}, k\neq 0, k^2+4l>0$ represents a set of functions $\phi\in\mathcal{A}$ that meet

$$1+\frac{1}{\varrho}\left(\left(\frac{[(\varsigma(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')']^{\nu}}{(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))'}\right)-1\right)\prec F(\varsigma),$$

and

$$1+\frac{1}{\varrho}\left(\left(\frac{[(w\mathcal{G}'(w))']^{\nu}}{\mathcal{G}'(w)}\right)-1\right)\prec F(w),$$

where $\mathcal{G}(w)=(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi)^{-1}(w)$ is given by (1.7).

3. Let $\nu=1$. Then $\mathfrak{X}^{q,s}_{\sigma}(\varrho,\tau,F)\equiv\mathfrak{T}^{q,s}_{\sigma}(\varrho,\tau,1,F)$, $0\leq\tau\leq 1$, and $F(\varsigma)=\frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma}=\frac{1}{1-k\varsigma-l\varsigma^2}, k\neq 0, k^2+4l>0$ is a set of functions such that $\phi\in\mathcal{A}$ fulfills

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{[(\varsigma(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')']}{(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))'}\right)+(1-\tau)\left(\frac{\varsigma((\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')}{\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma)}\right)-1\right)\prec F(\varsigma),$$

and

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{[(w\mathcal{G}'(w))']}{\mathcal{G}'(w)}\right)+(1-\tau)\left(\frac{w(\mathcal{G}'(w))}{\mathcal{G}(w)}\right)-1\right)\prec F(w),$$

where $\mathcal{G}(w)=(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi)^{-1}(w)$ is as stated in (1.7).

Definition 1.4. Let $0\leq\tau\leq 1$, $\nu\geq 1$ and $F(\varsigma)=\frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma}=\frac{1}{1-k\varsigma-l\varsigma^2}, k\neq 0, k^2+4l>0$. If $\phi\in\mathcal{A}$ satisfies

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{[(\varsigma(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')']^{\nu}}{(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))'}\right)+(1-\tau)((\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')^{\nu}-1\right)\prec F(\varsigma) \tag{1.13}$$

and

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{[(w\mathcal{G}'(w))']^{\nu}}{\mathcal{G}'(w)}\right)+(1-\tau)(\mathcal{G}'(w))^{\nu}-1\right)\prec F(w), \tag{1.14}$$

then we say that $\phi\in\mathfrak{Z}^{q,s}_{\sigma}(\varrho,\tau,\nu,F)$, where $\mathcal{G}(w)=(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi)^{-1}(w)$ is given by (1.7).

For particular choices of τ and ν within the general class $\mathfrak{Z}^{q,s}_{\sigma}(\varrho,\tau,\nu,F)$, the following subclasses of σ are obtained:

1. Let $\tau=0$. Then $\mathfrak{V}^{q,s}_{\sigma}(\varrho,\nu,F)\equiv\mathfrak{Z}^{q,s}_{\sigma}(\varrho,0,\nu,F)$, $\nu\geq 1$, and $F(\varsigma)=\frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma}=\frac{1}{1-k\varsigma-l\varsigma^2}, k\neq 0, k^2+4l>0$ is the collection of functions $\phi\in\mathcal{A}$ that meet

$$1+\frac{1}{\varrho}((\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')^{\nu}-1)\prec F(\varsigma),$$

and

$$1+\frac{1}{\varrho}((\mathcal{G}'(w))^{\nu}-1)\prec F(w),$$

where $\mathcal{G}(w)=(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi)^{-1}(w)$ is given by (1.7).

2. Let $\nu=1$. Then $\mathfrak{H}^{q,s}_{\sigma}(\varrho,\tau,F)\equiv\mathfrak{Z}^{q,s}_{\sigma}(\varrho,\tau,1,F)$, $x\in\mathbb{R}$, $0\leq\tau\leq1$, and $F(\varsigma)=\frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma}=\frac{1}{1-k\varsigma-l\varsigma^2}, k\neq0, k^2+4l>0$ represents the set of functions $\phi\in\mathcal{A}$ that satisfies

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{(\varsigma(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')'}{(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))'}\right)+(1-\tau)(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))'-1\right)\prec F(\varsigma),$$

and

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{(w\mathcal{G}'(w))'}{\mathcal{G}'(w)}\right)+(1-\tau)\mathcal{G}'(w)-1\right)\prec F(w),$$

where $\mathcal{G}(w)=(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi)^{-1}(w)$ is given by (1.7).

Remark 1.5. $\mathfrak{Z}^{q,s}_{\sigma}(\varrho,1,\nu,F)\equiv\mathfrak{Q}^{q,s}_{\sigma}(\varrho,\nu,F)$.

The following is the structure of the article's content. For functions in the families $\mathfrak{T}^{q,s}_{\sigma}(\varrho,\tau,\nu,F)$, and $\mathfrak{Z}^{q,s}_{\sigma}(\varrho,\tau,\nu,F)$, the estimates for $|d_2|$, $|d_3|$, and $|d_3-\xi d_2^2|,\xi\in\mathbb{R}$ are found in Section 2. In Section 3, we highlight relevant instances of our primary findings, which were demonstrated in Section 2. In Section 4, we conclude the study with some observations.

2. Main results

The coefficient-related estimates are now provided for functions in the class $\mathfrak{T}^{q,s}_{\sigma}(\varrho,\tau,\nu,F)$.

Theorem 2.1. *Let $\xi\in\mathbb{R},\nu\geq1,0\leq\tau\leq1$, and $F(\varsigma)=\frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma}=\frac{1}{1-k\varsigma-l\varsigma^2}, k\neq0, k^2+4l>0$. If $\phi\in\mathcal{A}$ is an element of the family $\mathfrak{T}^{q,s}_{\sigma}(\varrho,\tau,\nu,F)$, then*

$$|d_2|\leq\frac{|\varrho k|}{\Gamma_2[\eta_1;\kappa_1]}\sqrt{\frac{|k|}{|\varrho(\mathsf{M}+\mathsf{N})k^2-\mathsf{L}^2(k^2+l)|}},\tag{2.1}$$

$$|d_3|\leq\frac{|\varrho|}{\Gamma_3[\eta_1;\kappa_1]}\left(\frac{|\varrho|k^2}{\mathsf{L}^2}+\frac{|k|}{\mathsf{M}}\right),\tag{2.2}$$

and

$$|d_3-\xi d_2^2|\leq\begin{cases}\frac{|\varrho k|}{\mathsf{M}\Gamma_3[\eta_1;\kappa_1]} & ;\ |1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi|\leq\daleth\\ \frac{|\varrho|^2|k|^3\,|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi|}{\Gamma_3[\eta_1;\kappa_1]|\varrho(\mathsf{M}+\mathsf{N})k^2-2\mathsf{L}^2(k^2+l)|} & ;\ |1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi|\geq\daleth,\end{cases}\tag{2.3}$$

where

$$\daleth=\left|\frac{\varrho(\mathsf{M}+\mathsf{N})k^2-\mathsf{L}^2(k^2+l)}{\varrho\mathsf{M}k^2}\right|,\tag{2.4}$$

$$\mathsf{L}=(2\nu-1)(\tau+1),\mathsf{M}=(3\nu-1)(2\tau+1),\textit{and }\mathsf{N}=(2\nu^2-4\nu+1)(3\tau+1).\tag{2.5}$$

Proof. Let $\phi \in \mathfrak{T}_{\sigma}^{q,s}(\varrho, \tau, \nu, F)$. Then, from (1.11) and (1.12), we can write

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{[(\varsigma(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')']^{\nu}}{(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))'}\right)+(1-\tau)\left(\frac{\varsigma((\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')^{\nu}}{\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma)}\right)-1\right)=F(\mathfrak{u}(\varsigma)), \tag{2.6}$$

and

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{[(w\mathcal{G}'(w))']^{\nu}}{\mathcal{G}'(w)}\right)+(1-\tau)\left(\frac{w(\mathcal{G}'(w))^{\nu}}{\mathcal{G}(w)}\right)-1\right)=F(\mathfrak{v}(w)), \tag{2.7}$$

where $\mathfrak{u}(\varsigma) = \sum_{j=1}^{\infty} \mathfrak{u}_j\varsigma^j$, *and* $\mathfrak{v}(w) = \sum_{j=1}^{\infty} \mathfrak{v}_j w^j$, $\varsigma, w \in \mathfrak{U}$ are Schwarz functions satisfying ([10])

$$|\mathfrak{u}_j| \le 1,\ \textit{and}\ |\mathfrak{v}_j| \le 1\,(j \in \mathbb{N}). \tag{2.8}$$

Using a few fundamental mathematical methods, equations (2.6) and (2.7) can be expressed as:

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{[(\varsigma(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')']^{\nu}}{(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))'}\right)+(1-\tau)\left(\frac{\varsigma((\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')^{\nu}}{\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma)}\right)-1\right)= \\ 1+\frac{1}{\varrho}\mathsf{L}\Gamma_2[\eta_1;\kappa_1]d_2\varsigma+\frac{1}{\varrho}\left(\{\mathsf{M}\Gamma_3[\eta_1;\kappa_1]d_3+\mathsf{N}(\Gamma_2[\eta_1;\kappa_1])^2d_2^2\}\varsigma^2+\cdots\right), \tag{2.9}$$

$$F(\mathfrak{u}(\varsigma)) = 1+\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{u}_1\varsigma+\left[\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{u}_2+\mathcal{F}_3(\varkappa,\vartheta)\mathfrak{u}_1^2\right]\varsigma^2+\cdots \tag{2.10}$$

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{[(w\mathcal{G}'(w))']^{\nu}}{\mathcal{G}'(w)}\right)+(1-\tau)\left(\frac{w(\mathcal{G}'(w))^{\nu}}{\mathcal{G}(w)}\right)-1\right)=1-\frac{1}{\varrho}\mathsf{L}\Gamma_2[\eta_1;\kappa_1]d_2\varsigma+ \\ \frac{1}{\varrho}\left(\{\mathsf{M}(2(\Gamma_2[\eta_1;\kappa_1])^2d_2^2-\Gamma_3[\eta_1;\kappa_1]d_3)+\mathsf{N}(\Gamma_2[\eta_1;\kappa_1])^2d_2^2\}\varsigma^2+\cdots\right), \tag{2.11}$$

and

$$F(\mathfrak{v}(w)) = 1+\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{v}_1w+\left[\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{v}_2+\mathcal{F}_3(\varkappa,\vartheta)\mathfrak{v}_1^2\right]w^2+\cdots, \tag{2.12}$$

where L, M, and N are as mentioned in (2.5). We compare the terms in (2.9) and (2.11) in order to draw following conclusions because of (2.6).

$$\mathsf{L}\Gamma_2[\eta_1;\kappa_1]d_2 = \varrho\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{u}_1, \tag{2.13}$$

and

$$\mathsf{M}\Gamma_3[\eta_1;\kappa_1]d_3+\mathsf{N}(\Gamma_2[\eta_1;\kappa_1])^2d_2^2 = \varrho[\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{u}_2+\mathcal{F}_3(\varkappa,\vartheta)\mathfrak{u}_1^2]. \tag{2.14}$$

The same is true for (2.7), which allows us to compare terms in (2.10) and (2.12) into

$$-\mathsf{L}\Gamma_2[\eta_1;\kappa_1]d_2 = \varrho\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{v}_1, \tag{2.15}$$

and

$$\mathsf{M}(2(\Gamma_2[\eta_1;\kappa_1])^2d_2^2-\Gamma_3[\eta_1;\kappa_1]d_3)+\mathsf{N}(\Gamma_2[\eta_1;\kappa_1])^2d_2^2 = \varrho[\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{v}_2+\mathcal{F}_3(\varkappa,\vartheta)\mathfrak{v}_1^2]. \tag{2.16}$$

From (2.13) and (2.15), we get

$$\mathfrak{u}_1 = -\mathfrak{v}_1, \tag{2.17}$$

and

$$2\mathsf{L}^2(\Gamma_2[\eta_1;\kappa_1])^2 d_2^2 = \varrho^2(\mathfrak{u}_1^2+\mathfrak{v}_1^2)\mathcal{F}_2^2(\varkappa,\vartheta). \tag{2.18}$$

Addition of (2.14) and (2.16) yields

$$2(\mathsf{M}+\mathsf{N})(\Gamma_2[\eta_1;\kappa_1])^2 d_2^2 = \varrho[\mathcal{F}_2(\varkappa,\vartheta)(\mathfrak{u}_2+\mathfrak{v}_2)+\mathcal{F}_3(\varkappa,\vartheta)(\mathfrak{u}_1^2+\mathfrak{v}_1^2)]. \tag{2.19}$$

Replacing $\mathfrak{u}_1^2+\mathfrak{v}_1^2$ from (2.18) in (2.19) we get

$$d_2^2 = \frac{\varrho^2\mathcal{F}_2^3(\varkappa,\vartheta)(\mathfrak{u}_2+\mathfrak{v}_2)}{2\left[(\mathsf{M}+\mathsf{N})(\Gamma_2[\eta_1;\kappa_1])^2\varrho\mathcal{F}_2^2(\varkappa,\vartheta)-\mathsf{L}^2(\Gamma_2[\eta_1;\kappa_1])^2\mathcal{F}_3(\varkappa,\vartheta)\right]}. \tag{2.20}$$

Applying (2.8) to $\mathfrak{u}_2$,$\mathfrak{v}_2$, and using (1.10) for $\mathcal{F}_2(\varkappa,\vartheta)$, $\mathcal{F}_3(\varkappa,\vartheta)$, yields (2.1).

The bound on $|d_3|$ is obtained by subtracting (2.16) from (2.14):

$$d_3 = \frac{(\Gamma_2[\eta_1;\kappa_1])^2}{\Gamma_3[\eta_1;\kappa_1]}d_2^2 + \frac{\varrho\mathcal{F}_2(\varkappa,\vartheta)(\mathfrak{u}_2-\mathfrak{v}_2)}{2\mathsf{M}\Gamma_3[\eta_1;\kappa_1]}. \tag{2.21}$$

If we replace d_2^2 from (2.18) in (2.21) we get

$$d_3 = \frac{\varrho}{2\Gamma_3[\eta_1;\kappa_1]}\left(\frac{\varrho\mathcal{F}_2^2(\varkappa,\vartheta)(\mathfrak{u}_1^2+\mathfrak{v}_1^2)}{\mathsf{L}^2}+\frac{\mathcal{F}_2(\varkappa,\vartheta)(\mathfrak{u}_2-\mathfrak{v}_2)}{\mathsf{M}}\right). \tag{2.22}$$

We deduce equation (2.2) from (2.22) by applying the relation given in (1.10) and (2.8). Finally, we compute the bound on the functional $|d_3-\xi d_2^2|$ based on the derivec coefficient estimates of d_2^2 and d_3 from (2.20) and (2.21), respectively. Consequently, we have

$$|d_3-\xi d_2^2| = \frac{|\varrho||\mathcal{F}_2(\varkappa,\vartheta)|}{2}| \left|\left(\frac{1}{\mathsf{M}\Gamma_3[\eta_1;\kappa_1]}+\mathcal{V}(\xi,\varkappa)\right)\mathfrak{u}_2-\left(\frac{1}{\mathsf{M}\Gamma_3[\eta_1;\kappa_1]}-\mathcal{V}(\xi,\varkappa)\right)\mathfrak{v}_2\right|$$

where

$$\mathcal{V}(\xi,\varkappa) = \frac{\left(\frac{(\Gamma_2[\eta_1;\kappa_1])^2}{\Gamma_3[\eta_1;\kappa_1]}-\xi\right)\varrho\mathcal{F}_2^2(\varkappa,\vartheta)}{(\Gamma_2[\eta_1;\kappa_1])^2\left[\varrho(\mathsf{M}+\mathsf{N})\mathcal{F}_2^2(\varkappa,\vartheta)-\mathsf{L}^2\mathcal{F}_3(\varkappa,\vartheta)\right]}.$$

Clearly

$$|d_3-\xi d_2^2| \le \begin{cases} \frac{|\varrho||\mathcal{F}_2(\varkappa,\vartheta)|}{\mathsf{M}\Gamma_3[\eta_1;\kappa_1]} & ; \ |\mathcal{V}(\xi,\varkappa)| \le \frac{1}{\mathsf{M}\Gamma_3[\eta_1;\kappa_1]} \\ |\varrho||\mathcal{F}_2(\varkappa,\vartheta)||\mathcal{V}(\xi,\varkappa)| & ; \ |\mathcal{V}(\xi,\varkappa)| \ge \frac{1}{\mathsf{M}\Gamma_3[\eta_1;\kappa_1]}. \end{cases} \tag{2.23}$$

From (2.23), we obtain (2.3), where $\daleth$ is as stated in (2.4). □

We obtain the following result by using $q=2$, $s=1$, $\eta_1=a$, $\eta_2=b$, and $\kappa_1=c$ in Theorem 2.1.

Corollary 2.2. *Let $\xi\in\mathbb{R}$, $\nu\ge 1$, $0\le\tau\le 1$, and $F(\varsigma)=\frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma}=\frac{1}{1-k\varsigma-l\varsigma^2}$, $k\neq 0, k^2+4l>0$. If $\phi\in\mathcal{A}$ is assigned to the family $\mathfrak{T}_\sigma^{2,1}(\varrho,\tau,\nu,F)$, then*

$$|d_2| \le \frac{|\varrho k|}{\Gamma_2[a,b;c]}\sqrt{\frac{2|k|}{|\varrho(\mathsf{M}+\mathsf{N})k^2-2\mathsf{L}^2(k^2+l)|}},$$

$$|d_3| \le \frac{|\varrho|}{\Gamma_2[a,b;c]}\left(\frac{|\varrho|k^2}{\mathsf{L}^2}+\frac{|k|}{\mathsf{M}}\right),$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\varrho k|}{\mathsf{M}\Gamma_3[a,b;c]} & ; |1 - \frac{\Gamma_3[a,b;c]}{(\Gamma_2[a,b;c])^2}\xi| \leq \daleth \\ \frac{|\varrho|^2|k|^3\,|1-\frac{\Gamma_3[a,b;c]}{(\Gamma_2[a,b;c])^2}\xi|}{\Gamma_3[a,b;c]|\varrho(\mathsf{M}+\mathsf{N})k^2-\mathsf{L}^2(k^2+l)|} & ; |1 - \frac{\Gamma_3[a,b;c]}{(\Gamma_2[a,b;c])^2}\xi| \geq \daleth, \end{cases}$$

where $\daleth$ *is as in* (2.4) *and* L, M, *and* N *are as stated in* (2.5) *and* $\Gamma_j[a,b;c] = \frac{(a)_{j-1}(b_{j-1})}{(c)_{j-1}}\frac{1}{(j-1)!}$ (*see*1.6).

The estimates related to the coefficients are found for any function $\phi \in \mathfrak{Z}_\sigma^{q,s}(\varrho,\tau,\nu,F)$.

Theorem 2.3. *Let* $\xi \in \mathbb{R}, \nu \geq 1, 0 \leq \tau \leq 1$ *and* $F(\varsigma) = \frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma} = \frac{1}{1-k\varsigma-l\varsigma^2}, k \neq 0, k^2+4l > 0$. *If* $\phi \in \mathcal{A}$ *is a member of the class* $\mathfrak{Z}_\sigma^{q,s}(\varrho,\tau,\nu,F)$, *then*

$$|d_2| \leq \frac{|\varrho k|}{\Gamma_2[\eta_1;\kappa_1]}\sqrt{\frac{|k|}{|\varrho(Q-R)k^2 - P^2(k^2+l)|}}, \tag{2.24}$$

$$|d_3| \leq \frac{|\varrho|}{\Gamma_3[\eta_1;\kappa_1]}\left(\frac{|\varrho|k^2}{P^2} + \frac{|k|}{Q}\right), \tag{2.25}$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\varrho k|}{Q\Gamma_3[\eta_1;\kappa_1]} & ; |1 - \frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi| \leq \mathcal{Z} \\ \frac{|\varrho|^2|k|^3\,|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi|}{\Gamma_3[\eta_1;\kappa_1]|\varrho(Q-R)k)^2-P^2(k^2+l)|} & ; |1 - \frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi| \geq \mathcal{Z}, \end{cases} \tag{2.26}$$

where

$$\mathcal{Z} = \left|\frac{\varrho(Q-R)k^2 - P^2(k^2+l)}{\varrho Q k^2}\right|, \tag{2.27}$$

$$P = 2(\tau(\nu-1)+\nu), Q = 3(\tau(2\nu-1)+\nu), \text{and } R = 2(2\tau(2\nu-1)-\nu(\nu-1)(3\tau+1)). \tag{2.28}$$

Proof. Let $\phi \in \mathfrak{Z}_\sigma^{q,s}(\varrho,\tau,\nu,F)$. Then, from (1.13) and (1.14), we get

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{[(\varsigma(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')']^\nu}{(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))'}\right) + (1-\tau)((\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')^\nu - 1\right) = F(\mathfrak{u}(\varsigma)), \tag{2.29}$$

and

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{[(w\mathcal{G}'(w))']^\nu}{\mathcal{G}'(w)}\right) + (1-\tau)(\mathcal{G}'(w))^\nu - 1\right) = F(\mathfrak{v}(w)), \tag{2.30}$$

where the Schwarz functions $\mathfrak{u}(\varsigma)$ and $\mathfrak{v}(w)$, $(\varsigma, w \in \mathfrak{U})$ have the property (2.8). We can write equations (2.29) and (2.30) as follows using a few simple mathematical techniques:

$$1+\frac{P}{\varrho}\Gamma_2[\eta_1;\kappa_1]d_2\varsigma + \frac{1}{\varrho}(Q\Gamma_3[\eta_1;\kappa_1]d_3 - R(\Gamma_2[\eta_1;\kappa_1])^2d_2^2)\varsigma^2 + \cdots = 1+\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{u}_1\varsigma + \left[\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{u}_2 + \mathcal{F}_3(\varkappa,\vartheta)\mathfrak{u}_1^2\right]\varsigma^2 + \cdots, \tag{2.31}$$

$$1-\frac{P}{\varrho}\Gamma_2[\eta_1;\kappa_1]d_2w + \frac{1}{\varrho}(Q(2(\Gamma_2[\eta_1;\kappa_1])^2d_2^2 - \Gamma_3[\eta_1;\kappa_1]d_3) - R(\Gamma_2[\eta_1;\kappa_1])^2d_2^2)w^2 + \cdots = 1+\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{v}_1w + \left[\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{v}_2 + \mathcal{F}_3(\varkappa,\vartheta)\mathfrak{v}_1^2\right]w^2 + \cdots, \tag{2.32}$$

where P, Q, and R are as stated in (2.28). As a result, by contrasting the matching coefficients in (2.31) and (2.32), one can get

$$P\Gamma_2[\eta_1;\kappa_1]d_2 = \varrho\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{u}_1, \tag{2.33}$$

$$Q\Gamma_3[\eta_1;\kappa_1]d_3 - R(\Gamma_2[\eta_1;\kappa_1])^2 d_2^2 = \varrho[\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{u}_2 + \mathcal{F}_3(\varkappa,\vartheta)\mathfrak{u}_1^2], \tag{2.34}$$

$$-P\Gamma_2[\eta_1;\kappa_1]d_2 = \varrho\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{v}_1, \tag{2.35}$$

and

$$Q(2(\Gamma_2[\eta_1;\kappa_1])^2 d_2^2 - \Gamma_3[\eta_1;\kappa_1]d_3) - R(\Gamma_2[\eta_1;\kappa_1])^2 d_2^2 = \varrho[\mathcal{F}_2(\varkappa,\vartheta)\mathfrak{v}_2 + \mathcal{F}_3(\varkappa,\vartheta)\mathfrak{v}_1^2]. \tag{2.36}$$

If we follow the same steps as in Theorem 2.1 for (2.13) – (2.16), the results (2.24) – (2.26) of this theorem follow from (2.33) – (2.36). □

The following result is obtained by taking $q = 2$, $s = 1$, $\eta_1 = a$, $\eta_2 = b$, and $\kappa_1 = c$ in Theorem 2.3.

Corollary 2.4. *Let $\xi \in \mathbb{R}, \nu \geq 1, 0 \leq \tau \leq 1$, and $F(\varsigma) = \frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma} = \frac{1}{1-k\varsigma-l\varsigma^2}, k \neq 0, k^2+4l > 0$. If $\phi \in \mathcal{A}$ is an element of $\mathfrak{Z}_\sigma^{2,1}(\varrho,\tau,\nu,F)$, then*

$$|d_2| \leq \frac{|\varrho k|}{\Gamma_2[a,b;c]}\sqrt{\frac{|k|}{|\varrho(Q-R)k^2 - P^2(k^2+l)|}},$$

$$|d_3| \leq \frac{|\varrho|}{\Gamma_3[a,b;c]}\left(\frac{|\varrho||k|^2}{P^2} + \frac{|k|}{Q}\right),$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\varrho k|}{Q\Gamma_3[a,b;c]} & ; |1 - \frac{\Gamma_3[a,b;c]}{(\Gamma_2[a,b;c])^2}\xi| \leq \mathcal{Z} \\ \frac{|\varrho|^2|k|^3\,|1-\frac{\Gamma_3[a,b;c]}{(\Gamma_2[a,b;c])^2}\xi|}{\Gamma_3[a,b;c]|\varrho(Q-R)k^2-P^2(k^2+l)|} & ; |1 - \frac{\Gamma_3[a,b;c]}{(\Gamma_2[a,b;c])^2}\xi| \geq \mathcal{Z}, \end{cases}$$

where $\mathcal{Z}$ is as in (2.27), P, Q, R are as stated in (2.28) and $\Gamma_j[a,b;c] = \frac{(a)_{j-1}(b_{j-1})}{(c)_{j-1}}\frac{1}{(j-1)!}$ (see 1.6).

3. Special cases

By specifying particular values for the parameters τ and ν one can derive various subfamilies of Te-univalent functions associated with GBFP. The corresponding results for these subfamilies are obtained as direct consequences of the general findings presented in this work. In this section, we highlight and analyze several of these notable cases.

The following result is obtained as a special case of Theorem 2.1 by setting $\tau = 0$:

Corollary 3.1. *Let $\xi \in \mathbb{R}$, $\nu \geq 1$, and $F(\varsigma) = \frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma} = \frac{1}{1-k\varsigma-l\varsigma^2}, k \neq 0, k^2+4l > 0$. If $\phi \in \mathcal{A}$ is an element class $\mathfrak{P}_\sigma^{q,s}(\varrho,\nu,F)$, then*

$$|d_2| \leq \frac{|\varrho k|}{\Gamma_2[\eta_1;\kappa_1]}\sqrt{\frac{|k|}{|\varrho\nu(2\nu-1)k^2 - (2\nu-1)^2(k^2+l)|}},$$

$$|d_3| \leq \frac{|\varrho|}{\Gamma_3[\eta_1;\kappa_1]}\left(\frac{|\varrho|k^2}{(2\nu-1)^2}+\frac{|k|}{3\nu-1}\right),$$

and

$$|d_3-\xi d_2^2| \leq \begin{cases} \frac{|\varrho k|}{(3\nu-1)\Gamma_3[\eta_1;\kappa_1]} & ; \left|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi\right| \leq \daleth_1 \\ \frac{|\varrho|^2|k|^3\left|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi\right|}{\Gamma_3[\eta_1;\kappa_1]|\varrho\nu(2\nu-1)k^2-(2\nu-1)^2(k^2+l)|} & ; \left|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi\right| \geq \daleth_1, \end{cases}$$

where

$$\daleth_1 = \left|\frac{\varrho\nu(2\nu-1)k^2-(2\nu-1)^2(k^2+l)}{\varrho(3\nu-1)(2x-1)^2}\right|.$$

In the case $\tau = 1$, Theorem 2.1 yields the following result:

Corollary 3.2. *Let $\xi \in \mathbb{R}$, $\nu \geq 1$, and $F(\varsigma) = \frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma} = \frac{1}{1-k\varsigma-l\varsigma^2}, k \neq 0, k^2+4l > 0$. If $\phi \in \mathcal{A}$ is an element of the class $\mathfrak{Q}_\sigma^{q,s}(\varrho,\nu,F)$, then*

$$|d_2| \leq \frac{|\varrho k|}{\Gamma_2[\eta_1;\kappa_1]}\sqrt{\frac{|k|}{|\varrho(8\nu^2-7\nu+1)k^2-4(2\nu-1)^2(k^2+l)|}},$$

$$|d_3| \leq \frac{|\varrho|}{\Gamma_3[\eta_1;\kappa_1]}\left(\frac{|\varrho|k^2}{4(2\nu-1)^2}+\frac{|k|}{3(3\nu-1)}\right),$$

and

$$|d_3-\xi d_2^2| \leq \begin{cases} \frac{|\varrho k|}{3(3\nu-1)\Gamma_3[\eta_1;\kappa_1]} & ; \left|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi\right| \leq \daleth_2 \\ \frac{|\varrho|^2|k|^3\left|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi\right|}{\Gamma_3[\eta_1;\kappa_1]|\varrho(8\nu^2-7\nu+1)k^2-4(2\nu-1)^2(k^2+l)|} & ; \left|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi\right| \geq \daleth_2, \end{cases}$$

where

$$\daleth_2 = \left|\frac{\varrho(8\nu^2-7\nu+1)k^2-4(2\nu-1)^2(k^2+l)}{3\varrho(3\nu-1)k^2}\right|.$$

For $\nu = 1$, the following result is obtained from Theorem 2.1:

Corollary 3.3. *Let $\xi \in \mathbb{R}$, $0 \leq \tau \leq 1$, and $F(\varsigma) = \frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma} = \frac{1}{1-k\varsigma-l\varsigma^2}, k \neq 0, k^2+4l > 0$. If $\phi \in \mathcal{A}$ is an element of $\mathfrak{X}_\sigma^{q,s}(\varrho,\tau,F)$, then*

$$|d_2| \leq \frac{|\varrho k|}{\Gamma_2[\eta_1;\kappa_1]}\sqrt{\frac{|k|}{|\varrho(\tau+1)k^2-(\tau+1)^2(k^2+l)|}},$$

$$|d_3| \leq \frac{|\varrho|}{\Gamma_3[\eta_1;\kappa_1]}\left(\frac{|\varrho|k^2}{(\tau+1)^2}+\frac{|k|}{2(2\tau+1)}\right),$$

and

$$|d_3-\xi d_2^2| \leq \begin{cases} \frac{|\varrho k|}{2(2\tau+1)\Gamma_3[\eta_1;\kappa_1]} & ; \left|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi\right| \leq \daleth_3 \\ \frac{|\varrho|^2|k|^3\left|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi\right|}{\Gamma_3[\eta_1;\kappa_1]|\varrho(\tau+1)k^2-2(\tau+1)^2(k^2+l)|} & ; \left|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi\right| \geq \daleth_3, \end{cases}$$

where

$$\daleth_3 = \left|\frac{\varrho(\tau+1)k^2-2(\tau+1)^2(k^2+l)}{2\varrho(2\tau+1)k^2}\right|.$$

The following are specific instances of special cases derived from the aforementioned corollary:

Example 3.1. Let $\tau = 1$. Then $S^{q,s}_{\sigma}(\varrho, F) \equiv \mathfrak{X}^{q,s}_{\sigma}(\varrho, 1, F)$ is the subclass of functions $\phi \in \mathcal{A}$ satisfying

$$1+\frac{1}{\varrho}\left(\left(\frac{[(\varsigma(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')']}{(\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))'}\right)-1\right)\prec F(\varsigma)$$

and

$$1+\frac{1}{\varrho}\left(\left(\frac{[(w\mathcal{G}'(w))']}{\mathcal{G}'(w)}\right)-1\right)\prec F(w),$$

where $\mathcal{G}(w) = (\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi)^{-1}(w)$ is as stated in (1.7), and $F(\varsigma) = \frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma} = \frac{1}{1-k\varsigma-l\varsigma^2}, k \neq 0, k^2 + 4l > 0$.

Corollary 3.4. *Let $\xi \in \mathbb{R}$. If $\phi \in \mathcal{A}$ is a member of the class $S^{q,s}_{\sigma}(\varrho, F)$, then*

$$|d_2| \leq \frac{|\varrho k|}{\Gamma_2[\eta_1;\kappa_1]}\sqrt{\frac{|k|}{2|\varrho k^2-2(k^2+l)|}}, \quad |d_3| \leq \frac{|\varrho|}{\Gamma_3[\eta_1;\kappa_1]}\left(\frac{|\varrho|k^2}{4}+\frac{|k|}{6}\right),$$

and

$$|d_3-\xi d_2^2| \leq \begin{cases} \frac{|\varrho k|}{6\Gamma_3[\eta_1;\kappa_1]} & ; |1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi| \leq \left|\dfrac{\varrho k^2-4(x^2-x+\frac{1}{6})}{3\varrho(2\tau+1)(2x-1)^2}\right| \\ \frac{|\varrho|^2|k|^3\,|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi|}{2\Gamma_3[\eta_1;\kappa_1]|\varrho k^2-2(k^2+l)|} & ; |1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi| \geq \left|\dfrac{\varrho k^2-4(x^2-x+\frac{1}{6})}{3\varrho(2\tau+1)(2x-1)^2}\right|. \end{cases}$$

Example 3.2. Let $\tau = 0$. Then $C^{q,s}_{\sigma}(\varrho, F) \equiv \mathfrak{X}^{q,s}_{\sigma}(\varrho, 0, F)$ is the collection of functions $\phi \in \mathcal{A}$ satisfying

$$1+\frac{1}{\varrho}\left(\left(\frac{\varsigma((\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma))')}{\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi(\varsigma)}\right)-1\right)\prec F(\varsigma),$$

and

$$1+\frac{1}{\varrho}\left(\left(\frac{w(\mathcal{G}'(w))}{\mathcal{G}(w)}\right)-1\right)\prec F(w),$$

where $\mathcal{G}(w) = (\mathcal{H}_{q,s}[\eta_1;\kappa_1]\phi)^{-1}(w)$ is as stated in (1.7), and $F(\varsigma) = \frac{\mathcal{F}(\varkappa,\vartheta,\varsigma)}{\varsigma} = \frac{1}{1-k\varsigma-l\varsigma^2}, k \neq 0, k^2 + 4l > 0$.

Corollary 3.5. *Let $\xi \in \mathbb{R}$. If $\phi \in \mathcal{A}$ is a member the class $C^{q,s}_{\sigma}(\varrho, F)$, then*

$$|d_2| \leq \frac{|\varrho k|}{\Gamma_2[\eta_1;\kappa_1]}\sqrt{\frac{|k|}{|\varrho k^2-(k^2+l)|}}, \quad |d_3| \leq \frac{|\varrho|}{\Gamma_3[\eta_1;\kappa_1]}\left(\frac{|\varrho|k^2}{1}+\frac{|k|}{2}\right),$$

and

$$|d_3-\xi d_2^2| \leq \begin{cases} \frac{|\varrho k|}{2\Gamma_3[\eta_1;\kappa_1]} & ; |1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi| \leq \left|\dfrac{\varrho k^2-2(k^2+l)}{2\varrho k^2}\right| \\ \frac{|\varrho|^2|k|^3\,|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi|}{\Gamma_3[\eta_1;\kappa_1]|\varrho k^2-2(k^2+l)|} & ; |1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi| \geq \left|\dfrac{\varrho k^2-2(k^2+l)}{2\varrho k^2}\right|. \end{cases}$$

In the case where $\tau = 0$, Theorem 2.3 yields the following result:

Corollary 3.6. *Let $\xi \in \mathbb{R}$, and $\nu \geq 1$. If $\phi \in \mathcal{A}$ is an element of $\mathfrak{V}_{\sigma}^{q,s}(\varrho,\nu,F)$, then*

$$|d_2| \leq \frac{|\varrho k|}{\Gamma_2[\eta_1;\kappa_1]}\sqrt{\frac{|k|}{|\varrho\nu(2\nu+1)k^2-4\nu^2(k^2+l)|}}, \quad |d_3| \leq \frac{|\varrho|}{\Gamma_3[\eta_1;\kappa_1]}\left(\frac{|\varrho|k^2}{4\nu^2}+\frac{|k|}{3\nu}\right),$$

and

$$|d_3-\xi d_2^2| \leq \begin{cases} \frac{|\varrho k|}{3\Gamma_3[\eta_1;\kappa_1]\nu} & ; |1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi| \leq \mathcal{Z}_1 \\ \frac{|\varrho|^2|k|^3\,|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi|}{\Gamma_3[\eta_1;\kappa_1]|2\varrho\nu(2\nu+1)k^2-4\nu^2(k^2+l)|} & ; |1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi| \geq \mathcal{Z}_1, \end{cases}$$

where

$$\mathcal{Z}_1 = \left|\frac{\varrho(2\nu+1)k^2-4\nu(k^2+l)}{3\varrho k^2}\right|.$$

In the case where $\nu = 1$, Theorem 2.3 yields the following result:

Corollary 3.7. *Let $\xi \in \mathbb{R}$, and $0 \leq \tau \leq 1$. If $\phi \in \mathcal{A}$ is an element of $\mathfrak{H}_{\sigma}^{q,s}(\varrho,\tau,F)$, then*

$$|d_2| \leq \frac{|\varrho k|}{\Gamma_2[\eta_1;\kappa_1]}\sqrt{\frac{|k|}{|\varrho(3-\tau)k^2-4(k^2+l)|}}, \quad |d_3| \leq \frac{|\varrho|}{\Gamma_3[\eta_1;\kappa_1]}\left(\frac{|\varrho|k^2}{4}+\frac{|k|}{3(\tau+1)}\right),$$

and

$$|d_3-\xi d_2^2| \leq \begin{cases} \frac{|\varrho k|}{3\Gamma_3[\eta_1;\kappa_1](\tau+1)} & ; |1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi| \leq \mathcal{Z}_2 \\ \frac{|\varrho|^2|k|^3\,|1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi|}{2\Gamma_3[\eta_1;\kappa_1]|\varrho(3-\tau)k^2-4(k^2+l)|} & ; |1-\frac{\Gamma_3[\eta_1;\kappa_1]}{(\Gamma_2[\eta_1;\kappa_1])^2}\xi| \geq \mathcal{Z}_2, \end{cases}$$

where

$$\mathcal{Z}_2 = \left|\frac{\varrho(3-\tau)k^2-4(k^2+l)}{3\varrho(\tau+1)k^2}\right|.$$

4. Conclusion

This study establishes upper bounds for $|d_2|$ *and* $|d_3|$ for functions belonging to certain σ subclasses associated with generalized bi-univalent function problems (GBFP). Additionally, we investigate an upper bound for the functional $|d_3-\xi d_2^2|, \xi \in \mathbb{R}$, within these subfamilies. As outlined in Section 1, specializing the parameters in our results yields novel and previously unexplored outcomes. However, this paper does not encompass all significant subclasses of σ that have been studied. For instance, the works in have examined various subclasses involving (p,q)-operators as introduced in (p,q)-calculus. Readers interested in these directions are encouraged to consult these papers and the references therein. Few scholars have investigated the application of initial coefficient bounds in domains such as special functions, signal processing, and image processing. Recent developments have highlighted their potential in these areas, leading to new and significant applications within our field. Motivated by these advancements, the present study explores possible extensions of these ideas within the framework of Te-univalent functions.

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