Gulf Journal of Mathematics ISSN: 2309-4966
Vol 20, Issue 2 (2025) 401-414 https://doi.org/10.56947/gjom.v20i2.3055

ON CERTAIN DOUBLE INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS AND LOGARITHMIC POWERS

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Abstract. In this paper, we evaluate certain novel double integrals that incorporate generalized hypergeometric functions and powers of the logarithmic functions, which are formulated using classical functions such as the Gamma function, Psi function, and Hurwitz zeta functions. These integrals are derived by utilizing advanced techniques such as series expansions and logarithmic differentiation. The double integrals are represented using the n-th derivative expressions of the gamma function ratios. A well-known double integral due to Edwards serves as a key foundation for deriving the results presented in this work.

Keywords. Gamma function, Psi function, Hurwitz zeta function, generalized hypergeometric function, Summation formulas
2020 Mathematics Subject Classification. 33C20, Secondary : 33B15, 33C05.

1. Introduction and preliminaries

We begin by adopting the standard notations $\mathbb{C}, \mathbb{R}, \mathbb{Z}$ to denote the sets of complex numbers, real numbers, and integers, respectively. We also write $\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} = \{0, -1, -2, -3, \dots\}$. For every complex number u, the Pochhammer symbol [4, 12, 14] denoted as $(u)_n$ is defined as,

$$(u)_n = \frac{\Gamma(u+n)}{\Gamma(u)} = \begin{cases} 1, & (n=0,\ u \text{ is a nonzero complex number}), \\ u(u+1)\cdots(u+n-1), & (n \in \mathbb{N}, u \in \mathbb{C}), \end{cases} \tag{1.1}$$

where $\Gamma(u)$, known as the Euler gamma function, valid for $\Re(u) > 0$, is defined by the integral

$$\Gamma(u) = \int_0^\infty t^{u-1} e^{-t}\, dt, \tag{1.2}$$

Date: Received: May 20, 2025; Accepted: Jun 15, 2025.
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and serves as an extension of the factorial concept to the complex domain. For a detailed explanation of the Pochhammer symbol, including special cases for negative integers and zero, see [14, 11].

Using the Pochhammer symbol, the generalized hypergeometric function, defined and expressed with p parameters in the numerator and q in the denominator as [2, 4, 12, 13, 14],

$$ {}_pF_q\left[\begin{array}{ccc} h_1, & \cdots . & h_p \\ w_1, & \cdots, & w_q \end{array};z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(h_i)_n}{\prod_{j=1}^{q}(w_j)_n}\frac{z^n}{n!}, \tag{1.3}$$

where $w_j \neq 0, -1, -2, \cdots$

We interpret an empty product as 1 by standard convention. The variable z along with the numerator parameters $h_1, \ldots, h_p$, and the denominator parameters $w_1, \ldots, w_q$ appearing in (1.3), are assumed to be complex numbers, with the condition that.

$$w_j \in \mathbb{C} \setminus \mathbb{Z}_0^-, \quad \text{for all } j = 1, \ldots, q.$$

The ratio test reveals that (see, [2, 3]), the series (1.3)

(i) converges universally across all values of $z \in \mathbb{C}$, if $p \leq q$;
(ii) converges for $|z| < 1$ and diverges for $|z| > 1$ if $p = q + 1$;
(iii) converges absolutely for $|z| = 1$, if $p = q + 1$ and $\Re(\delta) > 0$; where

$$\delta = \sum_{j=1}^{q} w_j - \sum_{j=1}^{p} h_j.$$

For a detailed discussion of the convergence of (1.3), including the divergent case $p > q + 1$ and the conditional behavior on the unit circle $|z| = 1$ when $p = q + 1$, governed by $\Re(\delta)$, refer to [14, 11].

The digamma function, commonly known as the Psi function [3], is the derivative of the natural logarithm of the Gamma function, given by

$$\psi(z) = \frac{d}{dz}\{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{1.4}$$

where $\Gamma'(z)$ represents its derivative of Gamma function (1.2) and $z \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

The polygamma functions [1, 3], which are the higher-order derivatives of the digamma function, are defined as,

$$\psi^{(n)}(z) = \frac{d^n(\psi(z))}{dz^n} = \frac{d^{n+1}\left(\log \Gamma(z)\right)}{dz^{n+1}}, \tag{1.5}$$

where $n \in \mathbb{N}_0$ and $z \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
Specifically,

$$\psi^{(0)}(z) = \psi(z),$$

is the digamma function, and

$$\psi^{(1)}(z) = \frac{d(\psi(z))}{dz} = \frac{d^2\left(\log \Gamma(z)\right)}{dz^2},$$

is the trigamma function.

The Hurwitz zeta function [1, 8], which generalizes the polygamma function, is denoted by $\zeta(s,h)$ and is defined as

$$\zeta(s,h)=\sum_{n=0}^{\infty}\frac{1}{(n+h)^{s}}, \tag{1.6}$$

where $s\in\mathbb{C}$ with $\Re(s)>1$, and h is a complex number such that $h\notin\{0,-1,-2,\dots\}$.

The poly gamma function [1] can be written as a series, expressed by

$$\psi^{(n)}(z)=(-1)^{n+1}\,n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{n+1}}, \tag{1.7}$$

The above representation is valid for all integers $n>0$ and for all z in the complex plane, excluding the non-positive integers. This can be rewritten using the Hurwitz zeta function [1, 8] as

$$\psi^{(n)}(z)=(-1)^{n+1}\,n!\,\zeta(n+1,z), \tag{1.8}$$

where $n\in\mathbb{N}$, and $z\in\mathbb{C}\backslash\mathbb{Z}_0^-$.
As evident from (1.8), it follows that

$$\frac{d^k}{dz^k}\left[\log\left(\prod_{j=1}^{m}\Gamma(h_j+z)\right)\right]=\sum_{j=1}^{m}\psi^{(k-1)}(h_j+z), \tag{1.9}$$

where $m,k\in\mathbb{N}$.
For $k\geq 2$, this can be further expressed as,

$$\frac{d^k}{dz^k}\left[\log\left(\prod_{j=1}^{m}\Gamma(h_j+z)\right)\right]=(-1)^k\,(k-1)!\sum_{j=1}^{m}\zeta(k,h_j+z), \tag{1.10}$$

A detailed explanation of these special functions can be found in [10].

The mathematical theory of special functions includes a wide array of summation formulas essential for evaluating various forms of integrals, both definite and indefinite, particularly those involving hypergeometric functions and their associated functions. In the present work, the following summation theorems for the generalized hypergeometric functions, ${}_4F_3$ and ${}_5F_4$ as established, for example, in the book authored by Slater [13], are employed.

$$\begin{aligned}&{}_4F_3\left[\begin{array}{llll}h, & 1+\frac{h}{2}, & w, & c\\ \frac{h}{2}, & h+1-w, & h+1-c\end{array};-1\right]\\ &=\frac{\Gamma(h+1-w)\Gamma(h+1-c)}{\Gamma(h+1)\Gamma(h+1-w-c)},\end{aligned} \tag{1.11}$$

provided $\Re(h-2w-2c)>-1$,
and

$$\begin{aligned}&{}_5F_4\left[\begin{array}{lllll}h, & 1+\frac{h}{2}, & w, & c, & d\\ \frac{h}{2}, & h+1-w, & h+1-c, & h+1-d\end{array};1\right]\\ &=\frac{\Gamma(h+1-w)\Gamma(h+1-c)\Gamma(h+1-d)\Gamma(h+1-w-c-d)}{\Gamma(h+1)\Gamma(h+1-w-c)\Gamma(h+1-w-d)\Gamma(h+1-c-d)},\end{aligned} \tag{1.12}$$

subject to the condition $\Re(h-w-c-d) > -1$.

This work also incorporates a Beta-type integral due to Edwards [7], formulated as

$$\int_0^1 \int_0^1 y^h(1-t)^{h-1}(1-y)^{w-1}(1-ty)^{1-h-w}dtdy = \frac{\Gamma(h)\Gamma(w)}{\Gamma(h+w)}, \tag{1.13}$$

which holds under the condition that $\Re(h) > 0$, and $\Re(w) > 0$.

In 2002, Brychkov [5] introduced two approaches to evaluate definite and indefinite integrals that include elementary functions, special functions, and the logarithmic function. In the past decade, specifically in 2015, Choi and Rathie [6] provided evaluations for several single integrals involving the Gauss hypergeometric function ${}_2F_1$ and n-th power of logarithmic functions. A year later, in 2016, Kim [9] published several double integrals with generalized hypergeometric functions and n-th power of logarithmic functions. The authors established their results by employing the Hurwitz zeta and Psi functions. Motivated by the contributions of Brychkov [5], Choi and Rathie [6], and Kim [9], this study builds upon Edward's well-known double integral, which serves as a key foundation for deriving the results presented in this work. To ensure the paper is self-contained and to improve the readability of the derivations, in section 2, we provide proofs for four double integrals previously presented by Kim [9] without derivation. In Section 3, we extend these results to include new findings involving powers of logarithmic functions.

Remark 1.1. In the following sections, we frequently encounter the arguments $\frac{y(1-t)}{(1-ty)}$, and $\frac{(1-y)}{(1-ty)}$ in the context of hypergeometric functions. For the sake of brevity, we will refer to these expressions as $X_1(t,y)$, $X_2(t,y)$, respectively, where

$$X_1(t,y) = \frac{y(1-t)}{(1-ty)}, \quad \text{and} \quad X_2(t,y) = \frac{(1-y)}{(1-ty)}.$$

These simplified notations will be used throughout the next two sections.

2. Double Integrals of Edward's Type with Generalized Hypergeometric Functions

Several results cited in the literature are stated without accompanying proofs. This section reviews the double integrals presented by Kim [9], for which detailed proofs were not provided in the original work. The explicit forms and corresponding proofs of these integrals are presented below, enabling readers to follow our work in the subsequent section.

Theorem 2.1 (Kim [9])**.** *For* $\Re(w) > 0$, $\Re(h-2w-2c) > -1$, *and* $\Re(h+1-2w) > 0$, *the following double integrals are valid.*

$$\int_0^1 \int_0^1 y^w (1-t)^{w-1} (1-y)^{h-2w} (1-ty)^{w-h} \times {}_3F_2 \left[\begin{matrix} h, & 1+\frac{h}{2}, & c \\ \frac{h}{2}, & h+1-c \end{matrix} ; -X_1(t,y) \right] dt\, dy = \frac{\Gamma(w)\Gamma(h+1-2w)\Gamma(h+1-c)}{\Gamma(h+1)\Gamma(h+1-w-c)}, \tag{2.1}$$

and

$$\int_0^1 \int_0^1 y^{h+1-2w} (1-t)^{h-2w} (1-y)^{w-1} (1-ty)^{w-h} \times {}_3F_2 \left[\begin{matrix} h, & 1+\frac{h}{2}, & c \\ \frac{h}{2}, & h+1-c \end{matrix} ; -X_2(t,y) \right] dt\, dy = \frac{\Gamma(w)\Gamma(h+1-2w)\Gamma(h+1-c)}{\Gamma(h+1)\Gamma(h+1-w-c)}. \tag{2.2}$$

We now proceed to provide the proofs of these double integrals, ensuring a comprehensive understanding of their derivations.

Proof. Our proof starts by considering the given double integral (2.1). To proceed with its evaluation, we denote the left-hand part of the equation (2.1) by I_1 and express the hypergeometric function ${}_3F_2$ in terms of its series representation using (1.3).

$${}_3F_2 \left[\begin{matrix} h, & 1+\frac{h}{2}, & c \\ \frac{h}{2}, & h+1-c \end{matrix} ; -X_1(t,y) \right] = \sum_{k=0}^{\infty} \frac{(h)_k \left(1+\frac{h}{2}\right)_k (c)_k}{\left(\frac{h}{2}\right)_k (h+1-c)_k\, k!} \left(-X_1(t,y)\right)^k,$$

where $(h)_k$ represents the Pochhammer symbol (1.1).
Substituting this series expansion into I_1, we obtain,

$$I_1 = \int_0^1 \int_0^1 y^w (1-t)^{w-1} (1-y)^{h-2w} (1-ty)^{w-h} \times \sum_{k=0}^{\infty} \frac{(h)_k \left(1+\frac{h}{2}\right)_k (c)_k}{\left(\frac{h}{2}\right)_k (h+1-c)_k\, k!} \left(-X_1(t,y)\right)^k dt\, dy.$$

By interchanging the summation and integration under appropriate convergence conditions, we obtain,

$$\begin{aligned} I_1 &= \sum_{k=0}^{\infty} \frac{(-1)^k (h)_k \left(1+\frac{h}{2}\right)_k (c)_k}{\left(\frac{h}{2}\right)_k (h+1-c)_k \, k!} \\ &\quad \times \int_0^1 \int_0^1 y^w (1-t)^{w-1} (1-y)^{h-2w} (1-ty)^{w-h} \left(\frac{y(1-t)}{(1-ty)}\right)^k dt\, dy \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (h)_k \left(1+\frac{h}{2}\right)_k (c)_k}{\left(\frac{h}{2}\right)_k (h+1-c)_k \, k!} \\ &\quad \times \int_0^1 \int_0^1 y^{w+k} (1-t)^{w+k-1} (1-y)^{h-2w} (1-ty)^{w-k-h} dt\, dy. \end{aligned}$$

Following the evaluation of the inner Beta-type integral using (1.13), we get

$$I_1 = \sum_{k=0}^{\infty} \frac{(-1)^k (h)_k \left(1+\frac{h}{2}\right)_k (c)_k}{\left(\frac{h}{2}\right)_k (h+1-c)_k \, k!} \frac{\Gamma(w+k)\Gamma(h+1-2w)}{\Gamma(h+1-w+k)}.$$

Using the identity (1.1), we can rewrite the ratio of Gamma functions as

$$\frac{\Gamma(w+k)}{\Gamma(h+1-w+k)} = \frac{\Gamma(w)}{\Gamma(h+1-w)} \frac{(w)_k}{(h+1-w)_k}.$$

Substituting this into the expression for I_1, we get

$$I_1 = \sum_{k=0}^{\infty} \frac{(-1)^k (h)_k \left(1+\frac{h}{2}\right)_k (c)_k}{\left(\frac{h}{2}\right)_k (h+1-c)_k \, k!} \frac{\Gamma(w)\Gamma(h+1-2w)}{\Gamma(h+1-w)} \frac{(w)_k}{(h+1-w)_k}.$$

Factoring out constants independent of k, the expression becomes

$$I_1 = \frac{\Gamma(w)\Gamma(h+1-2w)}{\Gamma(h+1-w)} \sum_{k=0}^{\infty} \frac{(-1)^k (h)_k \left(1+\frac{h}{2}\right)_k (w)_k (c)_k}{\left(\frac{h}{2}\right)_k (h+1-w)_k (h+1-c)_k \, k!}.$$

We now recognize the summation in this expression as the generalized hypergeometric series

$$\sum_{k=0}^{\infty} \frac{(-1)^k (h)_k \left(1+\frac{h}{2}\right)_k (w)_k (c)_k}{\left(\frac{h}{2}\right)_k (h+1-w)_k (h+1-c)_k \, k!} = {}_4F_3\left[\begin{matrix} h, & 1+\frac{h}{2}, & w, & c \\ \frac{h}{2}, & h+1-w, & h+1-c \end{matrix} ; -1\right].$$

Hence, the integral I_1 becomes

$$I_1 = \frac{\Gamma(w)\Gamma(h+1-2w)}{\Gamma(h+1-w)} {}_4F_3\left[\begin{matrix} h, & 1+\frac{h}{2}, & w, & c \\ \frac{h}{2}, & h+1-w, & h+1-c \end{matrix} ; -1\right]. \tag{2.3}$$

Using the result in (1.11), the generalized hypergeometric function ${}_4F_3$ in (2.3) can be evaluated, leading to the right-hand side expression in (2.1).
This concludes the derivation of (2.1). □

Similarly, we can prove the other integral given in (2.2).

Theorem 2.2 (Kim [9]). *For $\Re(w) > 0$, $\Re(h+1-2w) > 0$, and $\Re(h+1-w-c-d) > 0$, the following double integrals are valid.*

$$\begin{aligned}&\int_0^1\int_0^1 y^w(1-t)^{w-1}(1-y)^{h-2w}(1-ty)^{w-h}\\&\quad\times {}_4F_3\left[\begin{matrix} h, & 1+\frac{h}{2}, & c, & d \\ \frac{h}{2}, & h+1-c, & h+1-d\end{matrix};X_1(t,y)\right]dt\,dy\\&\quad=\frac{\Gamma(w)\Gamma(h+1-2w)\Gamma(h+1-c)\Gamma(h+1-d)\Gamma(h+1-w-c-d)}{\Gamma(h+1)\Gamma(h+1-w-c)\Gamma(h+1-w-d)\Gamma(h+1-c-d)},\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}&\int_0^1\int_0^1 y^{h+1-2w}(1-t)^{h-2w}(1-y)^{w-1}(1-ty)^{w-h}\\&\quad\times {}_4F_3\left[\begin{matrix} h, & 1+\frac{h}{2}, & c, & d \\ \frac{h}{2}, & h+1-c, & h+1-d\end{matrix};X_2(t,y)\right]dt\,dy\\&\quad=\frac{\Gamma(w)\Gamma(h+1-2w)\Gamma(h+1-c)\Gamma(h+1-d)\Gamma(h+1-w-c-d)}{\Gamma(h+1)\Gamma(h+1-w-c)\Gamma(h+1-w-d)\Gamma(h+1-c-d)}.\end{aligned} \tag{2.5}$$

Proof. To derive (2.4), we denote the left-hand part of the equation by I_2 and express the hypergeometric function ${}_4F_3$ in terms of its series representation,

$$\begin{aligned}&{}_4F_3\left[\begin{matrix} h, & 1+\frac{h}{2}, \ c, \ d \\ \frac{h}{2}, & h+1-c, & h+1-d\end{matrix};X_1(t,y)\right]\\&\quad=\sum_{k=0}^{\infty}\frac{(h)_k\left(1+\frac{h}{2}\right)_k(c)_k\,(d)_k}{\left(\frac{h}{2}\right)_k(h+1-c)_k\,(h+1-d)_k\,k!}\left(X_1(t,y)\right)^k.\end{aligned}$$

Substituting this series expansion into I_2, we obtain,

$$\begin{aligned}I_2&=\int_0^1\int_0^1 y^w(1-t)^{w-1}(1-y)^{h-2w}(1-ty)^{w-h}\\&\quad\times\sum_{k=0}^{\infty}\frac{(h)_k\left(1+\frac{h}{2}\right)_k(c)_k\,(d)_k}{\left(\frac{h}{2}\right)_k(h+1-c)_k\,(h+1-d)_k\,k!}\left(X_1(t,y)\right)^k dtdy.\end{aligned}$$

Now, interchanging the summation and the integration under appropriate conditions of convergence, we obtain,

$$\begin{aligned}I_2&=\sum_{k=0}^{\infty}\frac{(h)_k\left(1+\frac{h}{2}\right)_k(c)_k\,(d)_k}{\left(\frac{h}{2}\right)_k(h+1-c)_k\,(h+1-d)_k\,k!}\\&\quad\times\int_0^1\int_0^1 y^w(1-t)^{w-1}(1-y)^{h-2w}(1-ty)^{w-h}\left(\frac{y(1-t)}{(1-ty)}\right)^k dt\,dy.\end{aligned}$$

The resulting integral is identified as a Beta-type integral. Upon evaluating this integral using (1.13), we get

$$I_2=\sum_{k=0}^{\infty}\frac{(h)_k\left(1+\frac{h}{2}\right)_k(c)_k\,(d)_k}{\left(\frac{h}{2}\right)_k(h+1-c)_k\,(h+1-d)_k\,k!}\frac{\Gamma(w+k)\Gamma(h-2w+1)}{\Gamma(h-w+k+1)}.$$

Using the identity (1.1), by rewriting the ratio of Gamma functions and summing the resulting series, we obtain,

$$I_2 = \frac{\Gamma(w)\Gamma(h+1-2w)}{\Gamma(h+1-w)}\, {}_5F_4\left[\begin{matrix} h, & 1+\frac{h}{2}, & w, & c, & d \\ \frac{h}{2}, & h+1-w, & h+1-c, & h+1-d \end{matrix} ;1\right]. \tag{2.6}$$

The ${}_5F_4$ hypergeometric function can be evaluated using the result in (1.12), leading to the right-hand side of (2.4). Thus, we arrive at the desired result, completing the proof of (2.4). □

Similarly, we can prove the other integral (2.5).

3. Main Results

This section presents the derivation of four double integrals that involve generalized hypergeometric functions and the n-th powers of logarithmic functions by utilizing generalized hypergeometric functions with arguments $\pm X_1(t,y)$ and $\pm X_2(t,y)$, each with different sets of parameters. In deriving these integrals, we make use of a double integral attributed to Edwards [7], as given in (1.13). Our results are expressed in terms of the product of Gamma function ratios and the n-th derivatives of these ratios.

We now define the function A, which will appear in the statement of Theorem 3.1.

Let

$$A = \frac{\Gamma(w)\Gamma(h+1-2w)}{\Gamma(h+1-w-c)}. \tag{3.1}$$

Then, the n-th derivative of A with respect to w is given by

$$\begin{aligned} \frac{\partial^n A}{\partial w^n} = A \Bigg[\sum_{k_1=0}^{n-1} \Bigg\{ \binom{n-1}{k_1} \frac{\partial^{n-k_1-1} B}{\partial B^{n-k_1-1}} \sum_{k_2=0}^{k_1-1} \Bigg[\binom{k_1-1}{k_2} \frac{\partial^{k_1-k_2-1} w}{\partial w^{k_1-k_2-1}} \cdots \\ \sum_{k_n=0}^{k_{n-1}-1} \Bigg\{ \binom{k_{n-1}-1}{k_n} \frac{\partial^{k_{n-1}-k_n-1} w}{\partial w^{k_{n-1}-k_n-1}} \Bigg\} \Bigg] \Bigg\} \Bigg], \end{aligned} \tag{3.2}$$

where

$$B = \frac{\partial(\log A)}{\partial B} = \psi(w) - 2\psi(h+1-2w) + \psi(h+1-w-c), \tag{3.3}$$

and

$$\begin{aligned} \frac{\partial^{n-k-1} B}{\partial w^{n-k-1}} =& (n-k-1)! \left\{ (-1)^{n-k} \zeta(n-k,w) + 2^{n-k} \zeta\left(n-k, h+1-2w\right) \right. \\ & \left. - \zeta\left(n-k, h+1-w-c\right) \right\}. \end{aligned} \tag{3.4}$$

Theorem 3.1. *Let $n \in \mathbb{N}$, $\Re(w) > 0$, $\Re(h-2w-2c) > -1$ and $\Re(h+1-2w) > 0$. Then the following double integral holds.*

$$\begin{aligned} &\int_0^1 \int_0^1 y^w (1-t)^{w-1} (1-y)^{h-2w} (1-ty)^{w-h} \log^n \left(\frac{y(1-t)(1-ty)}{(1-y)^2} \right) \\ &\quad \times {}_3F_2 \left[\begin{matrix} h, & 1+\frac{h}{2}, & c \\ \frac{h}{2}, & h+1-c \end{matrix} ; -X_1(t,y) \right] dt\, dy \\ &\quad = \frac{\Gamma(h+1-c)}{\Gamma(h+1)} \frac{\partial^n A}{\partial w^n}, \end{aligned} \tag{3.5}$$

where the term A is the ratio involving the product of Gamma functions as given by (3.1).

Proof. Our approach begins by differentiating each side of (2.1) n times with respect to w, which leads to,

$$\begin{aligned} &\int_0^1 \int_0^1 y^w (1-t)^{w-1} (1-y)^{h-2w} (1-ty)^{w-h} \log^n \left(\frac{y(1-t)(1-ty)}{(1-y)^2} \right) \\ &\quad \times {}_3F_2 \left[\begin{matrix} h, & 1+\frac{h}{2}, & c \\ \frac{h}{2}, & h+1-c \end{matrix} ; -X_1(t,y) \right] dt\, dy \\ &\quad = \frac{\Gamma(h+1-c)}{\Gamma(h+1)} \frac{\partial^n A}{\partial w^n}, \end{aligned} \tag{3.6}$$

where the expression for A is provided in (3.1).
Through the logarithmic diferentiation of A with respect to w, we obtain ,

$$\begin{aligned} \frac{\partial A}{\partial w} =& A \frac{\partial (log A)}{\partial w} = AB \\ =& \frac{\Gamma(w)\Gamma(h+1-2w)}{\Gamma(h+1-w-c)} \left[\psi(w) - 2\psi(h+1-2w) + \psi(h+1-w-c) \right], \end{aligned} \tag{3.7}$$

where, B is defined as in (3.3).
From (3.7) we have,

$$\frac{\partial^n A}{\partial w^n} = \frac{\partial^{n-1}}{\partial w^{n-1}} \left(\frac{\partial A}{\partial w} \right) = \frac{\partial^{n-1}}{\partial w^{n-1}} (AB),$$

which is expanded using Leibniz's theorem to obtain,

$$\frac{\partial^n A}{\partial w^n} = \sum_{k=0}^{n-1} \left(\begin{matrix} n-1 \\ k \end{matrix} \right) \frac{\partial^k A}{\partial w^k} \frac{\partial^{n-k-1} B}{\partial w^{n-k-1}}, \tag{3.8}$$

where, $\dfrac{\partial^{n-k-1} B}{\partial w^{n-k-1}}$ is as given in (3.4).
By substituting the expression for $\frac{\partial^n A}{\partial w^n}$ from (3.8) into (3.6), we obtain the desired result in (3.5). $\square$

Next, we define a function P, which will appear in the statement of Theorem 3.2.

Let

$$P=\frac{\Gamma(w)\Gamma(h+1-2w)}{\Gamma(h+1-w-c)}. \tag{3.9}$$

Then the n-th derivative of P with respect to w takes the form

$$\begin{aligned}\frac{\partial^n P}{\partial w^n}=P\Bigg[\sum_{k_1=0}^{n-1}\Bigg\{\left(\begin{array}{c}n-1\\k_1\end{array}\right)\frac{\partial^{n-k_1-1}Q}{\partial w^{n-k_1-1}}\sum_{k_2=0}^{k_1-1}\Bigg[\left(\begin{array}{c}k_1-1\\k_2\end{array}\right)\frac{\partial^{k_1-k_2-1}Q}{\partial w^{k_1-k_2-1}}\cdots\\ \sum_{k_n=0}^{k_{n-1}-1}\left\{\left(\begin{array}{c}k_{n-1}-1\\k_n\end{array}\right)\frac{\partial^{k_{n-1}-k_n-1}Q}{\partial w^{k_{n-1}-k_n-1}}\right\}\Bigg]\Bigg\}\Bigg],\end{aligned} \tag{3.10}$$

where

$$Q=\frac{\partial(\log P)}{\partial w}=\psi(w)-2\psi(h+1-2w)+\psi(h+1-w-c), \tag{3.11}$$

and

$$\begin{aligned}\frac{\partial^{n-k-1}Q}{\partial w^{n-k-1}}&=(n-k-1)!\left\{(-1)^{n-k}\zeta(n-k,w)+2^{n-k}\zeta\left(n-k,h+1-2w\right)\right.\\&\left.\quad-\zeta\left(n-k,h+1-w-c\right)\right\}.\end{aligned} \tag{3.12}$$

Theorem 3.2. *For $n\in\mathbb{N}$, $\Re(w)>0$, $\Re(h-2w-2c)>-1$ and $\Re(h+1-2w)>0$, the following double integral holds.*

$$\begin{aligned}&\int_0^1\int_0^1 y^{h+1-2w}(1-t)^{h-2w}(1-y)^{w-1}(1-ty)^{w-h}\log^n\left(\frac{(1-y)(1-ty)}{y^2(1-t)^2}\right)\\&\quad\times{}_3F_2\left[\begin{array}{ccc}h, & 1+\frac{h}{2}, & c\\ \frac{h}{2}, & h+1-c & \end{array};-X_2(t,y)\right]dt\,dy\\&\quad=\frac{\Gamma(h+1-c)}{\Gamma(h+1)}\frac{\partial^n P}{\partial w^n},\end{aligned} \tag{3.13}$$

where the term P is the ratio involving the product of Gamma functions as given in (3.9).

Using a similar approach as in the proof of Theorem 3.1, by differentiating both sides of (2.2) n times concerning w, and following analogous steps to those in the preceding proof, we can establish Theorem 3.2.

We proceed to define the term R, which appears in the statement of Theorem 3.3.

Let

$$R=\frac{\Gamma(w)\Gamma(h+1-2w)\Gamma(h+1-w-c-d)}{\Gamma(h+1-w-c)\Gamma(h+1-w-d)}. \tag{3.14}$$

Then the n-th derivative of R with respect to w takes the form

$$\begin{aligned}\frac{\partial^n R}{\partial w^n}=R\Bigg[\sum_{k_1=0}^{n-1}\Bigg\{\left(\begin{array}{c}n-1\\k_1\end{array}\right)\frac{\partial^{n-k_1-1}S}{\partial w^{n-k_1-1}}\sum_{k_2=0}^{k_1-1}\Bigg[\left(\begin{array}{c}k_1-1\\k_2\end{array}\right)\frac{\partial^{k_1-k_2-1}\,S}{\partial w^{k_1-k_2-1}}\cdots\\ \sum_{k_n=0}^{k_{n-1}-1}\left\{\left(\begin{array}{c}k_{n-1}-1\\k_n\end{array}\right)\frac{\partial^{k_{n-1}-k_n-1}S}{\partial w^{k_{n-1}-k_n-1}}\right\}\Bigg]\Bigg\}\Bigg],\end{aligned} \tag{3.15}$$

where

$$S = \frac{\partial(\log R)}{\partial w} = \psi(w) - \psi(h+1-w-c-d) - 2\psi(h+1-2w) + \psi(h+1-w-c) + \psi(h+1-w-d), \tag{3.16}$$

and

$$\begin{aligned}\frac{\partial^{n-k-1}S}{\partial w^{n-k-1}} = (n-k-1)!\,\{&(-1)^{n-k}\zeta(n-k,w) + 2^{n-k}\zeta(n-k,h+1-2w) \\ &- \zeta(n-k,h+1-w-c) + \zeta(n-k,h+1-w-c-d) \\ &- \zeta(n-k,h+1-w-d)\}.\end{aligned} \tag{3.17}$$

Theorem 3.3. *For $n \in \mathbb{N}$, $\Re(w) > 0$, $\Re(h+1-2w) > 0$ and $\Re(h+1-w-c-d) > 0$, the following double integral holds.*

$$\begin{aligned}&\int_0^1\int_0^1 y^w(1-t)^{w-1}(1-y)^{h-2w}(1-ty)^{w-h}\log^n\left(\frac{y(1-t)(1-ty)}{(1-y)^2}\right) \\ &\quad\times {}_4F_3\left[\begin{matrix} h, & 1+\frac{h}{2}, & c, & d \\ \frac{h}{2}, & h+1-c, & h+1-d \end{matrix}; X_1(t,y)\right] dt\,dy \\ &\quad= \frac{\Gamma(h+1-c)\Gamma(h+1-d)}{\Gamma(h+1)\Gamma(h+1-c-d)}\frac{\partial^n R}{\partial w^n},\end{aligned} \tag{3.18}$$

where the term R is the expression as given in (3.14).

Proof. To prove the result, we begin by differentiating both sides of (2.4) n times, concerning w, which yields,

$$\begin{aligned}&\int_0^1\int_0^1 y^w(1-t)^{w-1}(1-y)^{h-2w}(1-ty)^{w-h}\log^n\left(\frac{y(1-t)(1-ty)}{(1-y)^2}\right) \\ &\quad\times {}_4F_3\left[\begin{matrix} h, & 1+\frac{h}{2}, & c, & d \\ \frac{h}{2}, & h+1-c, & h+1-d \end{matrix}; X_1(t,y)\right] dt\,dy \\ &\quad= \frac{\Gamma(h+1-c)\Gamma(h+1-d)}{\Gamma(h+1)\Gamma(h+1-c-d)}\frac{\partial^n R}{\partial w^n},\end{aligned} \tag{3.19}$$

where the expression for R is provided in (3.14).
Differentiating R logarithmically with respect to w, we have,

$$\begin{aligned}\frac{\partial R}{\partial w} &= R\frac{\partial(\log R)}{\partial w} = RS \\ &= \frac{\Gamma(w)\Gamma(h+1-w-c-d)\Gamma(h+1-2w)}{\Gamma(h+1-w-c)\Gamma(h+1-w-d)} \\ &\quad\times[\psi(w) - \psi(h+1-w-c-d) - 2\psi(h+1-2w) \\ &\qquad+\psi(h+1-w-c) + \psi(h+1-w-d)],\end{aligned} \tag{3.20}$$

where S is as provided in (3.16).
Referring to (3.20), it follows that,

$$\frac{\partial^n R}{\partial w^n} = \frac{\partial^{n-1}}{\partial w^{n-1}}\left(\frac{\partial R}{\partial w}\right) = \frac{\partial^{n-1}}{\partial w^{n-1}}(RS),$$

which is expanded using Leibniz's theorem as follows,

$$\frac{\partial^n R}{\partial w^n} = \sum_{k=0}^{n-1} \begin{pmatrix} n-1 \\ k \end{pmatrix} \frac{\partial^k R}{\partial w^k} \frac{\partial^{n-k-1} S}{\partial w^{n-k-1}}, \tag{3.21}$$

Here, the term $\frac{\partial^{n-k-1} S}{\partial w^{n-k-1}}$ is identical to the expression provided in (3.17). Substituting the expression for $\frac{\partial^n R}{\partial w^n}$ from (3.21) into (3.19), we arrive at the desired result (3.18). □

We proceed to define the term E, which appears in the statement of Theorem 3.4.
Let

$$E = \frac{\Gamma(w)\Gamma(h+1-w-c-d)\Gamma(h+1-2w)}{\Gamma(h+1-w-c)\Gamma(h+1-w-d)}. \tag{3.22}$$

The n-th derivative of E is given by

$$\begin{aligned} \frac{\partial^n E}{\partial w^n} = E \Bigg[\sum_{k_1=0}^{n-1} \Bigg\{ \begin{pmatrix} n-1 \\ k_1 \end{pmatrix} \frac{\partial^{n-k_1-1} F}{\partial w^{n-k_1-1}} \sum_{k_2=0}^{k_1-1} \Bigg[\begin{pmatrix} k_1-1 \\ k_2 \end{pmatrix} \frac{\partial^{k_1-k_2-1} F}{\partial w^{k_1-k_2-1}} \cdots \\ \sum_{k_n=0}^{k_{n-1}-1} \Bigg\{ \begin{pmatrix} k_{n-1}-1 \\ k_n \end{pmatrix} \frac{\partial^{k_{n-1}-k_n-1} F}{\partial w^{k_{n-1}-k_n-1}} \Bigg\} \Bigg] \Bigg\} \Bigg], \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} F = \frac{\partial(\log E)}{\partial w} =& \psi(w) - 2\psi(h+1-2w) + \psi(h+1-w-c) \\ &+ \psi(h+1-w-d) - \psi(h+1-w-c-d), \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \frac{\partial^{n-k-1} F}{\partial w^{n-k-1}} =& (n-k-1)! \left\{ (-1)^{n-k} \zeta(n-k,w) + 2^{n-k} \zeta(n-k,h+1-2w) \right. \\ & - \zeta(n-k,h+1-w-c) + \zeta(n-k,h+1-w-c-d) \\ & \left. - \zeta(n-k,h+1-w-d) \right\}. \end{aligned} \tag{3.25}$$

Theorem 3.4. *For $n \in \mathbb{N}$, $\Re(w) > 0$, $\Re(h+1-2w) > 0$ and $\Re(h+1-w-c-d) > 0$, the following double integral holds.*

$$\begin{aligned} &\int_0^1 \int_0^1 y^{h+1-2w}(1-t)^{h-2w}(1-y)^{w-1}(1-ty)^{w-h} \log^n \left(\frac{(1-y)(1-ty)}{y^2(1-t)^2} \right) \\ &\quad \times {}_4F_3 \left[\begin{matrix} h, & 1+\frac{h}{2}, & c, & d \\ \frac{h}{2}, & h+1-c, & h+1-d \end{matrix} ; X_2(t,y) \right] dt\, dy \\ &\quad = \frac{\Gamma(h+1-c)\Gamma(h+1-d)}{\Gamma(h+1)\Gamma(h+1-c-d)} \frac{\partial^n E}{\partial w^n}, \end{aligned} \tag{3.26}$$

where E is the term as given in (3.22).

Using a similar approach as in the proof of Theorem 3.3, upon taking the n-th derivative of both sides of (2.2) concerning w, and following analogous steps to those in the preceding proof, we can establish Theorem 3.4.

4. Conclusion

In this work, we derived four distinct finite double integrals incorporating the generalized hypergeometric function and the n-th powers of logarithmic functions. By employing advanced techniques such as series expansions, logarithmic differentiation, and Leibniz's theorem, we successfully expressed the results in terms of Gamma functions and their higher derivatives. The obtained results demonstrate a connection between hypergeometric functions and logarithmic derivatives, which may be of interest in future theoretical developments.

Acknowledgement. The authors thank the editor and anonymous reviewers for their constructive feedback and insightful suggestions, which greatly contributed to improving the manuscript after revision.

References

[1] Abramowitz, M., & Stegun, I. A. (Eds.). (1972). *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Dover Publications.

[2] Andrews, G. E., Askey, R., & Roy, R. (1999). *Special functions* (Vol. 71). Cambridge University Press. https://doi.org/10.1017/CBO9781107325937

[3] Arfken, G. W., Weber, H. J., & Harris, F. E. (2012). *Mathematical methods for physicists* (7th ed.). Academic Press. https://doi.org/10.1016/C2009-0-30629-7

[4] Bailey, W. N. (1964). *Generalized hypergeometric series* (Cambridge tracts in mathematics and mathematical physics, No. 32). Cambridge University Press. (Reprinted by Stechert-Hafner Service Agency)

[5] Brychkov, Y. A. (2002). Evaluation of some classes of definite and indefinite integrals. *Integral transforms and special functions, 13*(2), 163–167. https://doi.org/10.1080/10652460212896

[6] Choi, J., & Rathie, A. K. (2015). Evaluation of a certain new class of definite integrals. *Integral transforms and special functions, 26*(4), 282–294. https://doi.org/10.1080/10652469.2014.1001385

[7] Edwards, J. (1954). *A treatise on the integral calculus with applications, examples, and problems* (Vol. II). Chelsea Publishing Company.

[8] Hurwitz, A. (1882). Einige Eigenschaften der Dirichlet'schen Funktionen $F(s) = \sum \left(\frac{D}{n}\right) \cdot \frac{1}{n^s}$. *Zeitschrift für Mathematik und Physik, 27*, 86–101. https://doi.org/10.1007/978-3-0348-4161-0_3

[9] Kim, Y. (2016). New class of integrals involving generalized hypergeometric function and logarithmic function. *Communications of the Korean Mathematical Society, 31*(2), 329–342. https://doi.org/10.4134/CKMS.2016.31.2.329

[10] Prudnikov, A. P., Brychkov, Y. A., & Marichev, O. I. (1990). *Integrals and series, volume 3: More special functions*. Gordon and Breach Science Publishers.

[11] Qureshi, M. I., & Malik, S. H. (2022). Certain cubic reduction formulas involving hypergeometric functions. *Gulf Journal of Mathematics, 13*(2). https://doi.org/10.56947/gjom.v13i2.788

[12] Rainville, E. D. (1960). *Special functions*. Macmillan Company. (Reprinted by Chelsea Publishing Company, 1971)

[13] Slater, L. J. (1966). *Generalized hypergeometric functions*. Cambridge University Press.

[14] Srivastava, H. M., & Choi, J. (2012). *Zeta and q-zeta functions and associated series and integrals*. Elsevier. https://doi.org/10.1016/C2010-0-67023-4

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