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STUDY OF RECURSIVE RELATIVE REGRESSION PREDICTOR

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ABSTRACT. In this work, we develop first the recursive relative regression estimator using independent identically real random variables. The classical regression estimators may give unreliable prediction results with the existence of aberrant values. The recursive relative regression estimator works better in this case, further it allows the estimation to be updated whenever new observations are included in the estimation at any time. We provide next its almost complete convergence by indicating its rate. In a simulation study, we employ the recursive relative estimator as a predictor and assess its performance against various nonparametric estimators to substantiate our results.

1. INTRODUCTION

Nonparametric regression approach is a statistical tool aimed at exploring the relationship between a dependent variable and one or more independent variables without making assumptions about the functional form of this association. It is commonly used for forecasting and is a powerful tool to make decisions.

The field of regression has been explored in several papers under various contexts, we note, among others, [8],[12], [11],[4] and [8]. We highlight in this paper the relationship of two variables (T, Y) considering the influence of outliers. These data points situated generally far from the least squares line may skew the result of regression analysis, and give irrelevant forecasting values. It can distort the interpretation of the studied relationship between the variables.

Let us consider $\{(T_i, Y_i)\}_{i=1}^n$ a sequence of independent and identically distributed (i.i.d) real random variables as (T, Y). Let $r(t) = E(Y/T = t)$ represent the unknown regression function of Y conditioned on T supposed to be continuous. Moreover, we represent the marginal density of T using the notation f.

We subsequently present the following predictive model

$$Y_i = r(T_i) + \epsilon_i,\ i = \overline{1, n}, \tag{1.1}$$

with ϵ_i are random variables that are independent and identically distributed, satisfying $E(\epsilon_i|X_i) = 0$, almost surely.

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The regression function r is calculated as a solution of the following minimization problem

$$\min_r E\left[(Y - r(t))^2 \middle| T = t \right] , \tag{1.2}$$

which is clearly expressed by

$$r(t) = E(Y| T = t) \, ..$$

It is estimated first by Nadaraya-Watson (1965) [12] using the following expression

$$\widehat{r}_n(t) = \frac{\sum_{i=1}^{n} Y_i K\left(\frac{t-T_i}{h_n}\right)}{\sum_{i=1}^{n} K\left(\frac{t-T_i}{h_n}\right)}. \tag{1.3}$$

where

- K is a kernel function.
- h_n is the smoothing parameter, it is a given positive sequence decreasing to zero.

A recursive form of the earlier estimator introduced by Amiri (2010) is defined by

$$r_n^l(t) = \frac{g_n^l(t)}{f_n^l(t)}, \quad t \in \mathbb{R} \tag{1.4}$$

where

$$f_n^l(t) = \frac{1}{\sum_{i=1}^{n} h_i^{1-l}} \sum_{i=1}^{n} \frac{1}{h_i^l} K\Big(\frac{t - T_i}{h_i}\Big), \quad t \in \mathbb{R} \tag{1.5}$$

and

$$g_n^l(t) = \frac{1}{\sum_{i=1}^{n} h_i^{1-l}} \sum_{i=1}^{n} \frac{1}{h_i^l} Y_i K\Big(\frac{t - T_i}{h_i}\Big), \quad t \in \mathbb{R}$$

with $l \in [0, 1]$ is another smoothing parameter.
In [1] the authors investigated the consistency of the recursive regression estimator for the fixed value $l = 0$. In [5] the consistency in L^1 is examined for two different values of the parameter l. [2] provides the almost sure convergence of the generalized sequential estimators (1.4) and (1.5) for $l \in [0, 1]$ under independence. In the same spirit, Mezhoud (2020)[11] investigated the estimator (1.4) in the case of weak dependent data.
The optimization problem (1.2) treats all variables as having the same weight, in that case data are not suitably fit by the model (1.1) using (1.3) or (1.4), we need so to define another regression estimate because the existence of outliers may lead to biased estimation.
In this setting, an alternative optimization problem based on the quadratic relative error can be expressed by

$$\min_r E\left[\left(\frac{Y - r(t)}{Y}\right)^2 \middle| T = t \right] ,$$

Park and Stefanski (1998)[13] proposed its following solution

$$r(t) = \frac{E\left(Y^{-1} \middle| T = t\right)}{E\left(Y^{-2} \middle| T = t\right)} := \frac{g_1(t)}{g_2(t)},$$

where $g_j(t) = r_j(t)f(t)$, with $r_j(t) = E\left(Y^{-j} \middle| T = t\right),\ j = 1, 2$.
Subsequently, Jones and al. (2008)[10] introduced the relative kernel estimator of $r(t)$ if the two first inverse conditional moments exist and are finite. It is given by

$$\begin{aligned} \overline{r}_n(t) &= \frac{\sum_{i=1}^{n} Y_i^{-1} K\left(\frac{t-T_i}{h_n}\right)}{\sum_{i=1}^{n} Y_i^{-2} K\left(\frac{t-T_i}{h_n}\right)} \\ &:= \frac{\overline{g}_1(t)}{\overline{g}_2(t)}, \end{aligned} \tag{1.6}$$

where

$$\overline{g}_j(t) = \overline{r}_j(t) f(t),$$

and

$$\overline{r}_j(t) = \frac{1}{nh_n} \sum_{i=1}^{n} Y_i^{-j} K\left(\frac{t-T_i}{h_n}\right), \qquad j = 1, 2.$$

For more details about relative regression, we point the reader's attention for example to the work of Jones and al. (2008), where the kernel estimation by relative error regression is investigated, Demongeot and al.(2016)[6] introduced and studied the functional relative kernel regression estimator, Bouhadjera and al.(2023)[3], studied also asymptotic properties of the relative kernel regression estimator for censored data, and Bouhedjra and al. (2022)[4] worked on almost sure convergence of the local linear relative regression estimator for censored data. In [7] the nonparametric relative error regression estimator is investigated for $\eta-$weak dependent data.
Combining the two regression estimates (1.4) and (1.6), we propose here the recursive relative regression estimator

$$\begin{aligned} \widehat{r}_n^l(t) &= \frac{\sum_{i=1}^{n} \frac{Y_i^{-1}}{h_i^l} K\left(\frac{t-T_i}{h_i}\right)}{\sum_{i=1}^{n} \frac{Y_i^{-2}}{h_i^l} K\left(\frac{t-T_i}{h_i}\right)} \\ &:= \frac{\widehat{g}_1^l(t)}{\widehat{g}_2^l(t)}, \end{aligned} \tag{1.7}$$

such as

$$\widehat{g}_j(t) = \widehat{r}_{j,n}^l(t) \widehat{f}_n^l(t),$$

and

$$\widehat{r}_{j,n}^l(t) = \frac{1}{\sum_{i=1}^{n} h_i^{1-l}} \sum_{i=1}^{n} \frac{Y_i^{-j}}{h_i^l} K\left(\frac{t-T_i}{h_i}\right), \qquad j = 1, 2.$$

It may be noted that the estimator (1.7) satisfies the recursive form

$$\widehat{r}^{l}_{n+1}(t) = \frac{(\sum_{i=1}^{n} h_i^{1-l})\widehat{r}^{l}_{1,n}(t) + (\sum_{i=1}^{n+1} h_i^{1-l})W_{n+1}\left(\frac{t-T_{n+1}}{h_{n+1}}\right)Y_{n+1}^{-1}}{(\sum_{i=1}^{n} h_i^{1-l})\widehat{r}^{l}_{2,n}(t) + (\sum_{i=1}^{n+1} h_i^{1-l})W_{n+1}\left(\frac{t-T_{n+1}}{h_{n+1}}\right)Y_{n+1}^{-2}} \tag{1.8}$$

where $W_{n+1}(.) := \frac{h_{n+1}^{-l}}{\sum_{i=1}^{n+1} h_i^{1-l}} K\left(\frac{t-T_{n+1}}{h_{n+1}}\right)$.

As a result, the recursive procedure can continuously update their estimates as new data becomes available, enabling a significant reduction in computational time and memory, additionally it performs well with real time applications where data arrives sequentially such as in control problems.

Overall, the objective of this investigation is applying the recursive method to the relative regression estimate, in order to benefit from the advantages of both approaches, and then studying its strong consistency. We state firstly basic assumptions in section two, followed by the almost complete convergence of the relative recursive estimator. Section three is eventually accorded to a numerical study aimed at showing performance of our predictor. Section four provides proofs.

2. Main results

The obtained results are based essentially on the following assumptions.

[$A1$]: $E\left(|Y^{-m}|/T\right) < \infty$, $m \geq 1$.
[$A2.1$]: f and r_j, $j = 1, 2$ are continuous in the neighborhood of t.
[$A2.2$]: f and r_j, $j = 1, 2$ are twice continuously derivable at t.
[$A3.1$]: K is a density function that is bounded and has compact support.
[$A3.2$]: $\int |t|K(t)dt = 0$.
[$A4$]: $h_n \to 0$, $nh_n^3 \to \infty$ and $\frac{\log n}{nh_n} \to 0$, as $n \to \infty$.

[$A5$]: $\beta_{n,r} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{h_i}{h_n}\right)^r \to \beta_r < \infty$, as $n \to \infty, r \leq 3$.

Remark 2.1.

- Assumption [$A.1$] is classic in regression estimation study, and is generally used when the random variable Y is unbounded.
- Notice that even the random variables are identically distributed, this is not the case with the transformed sequence $\left(K\left(\frac{t-T_i}{h_i}\right)\right)_{i=1,\ldots,n}$. The particular recursion assumption [$A5$] is designed to smooth such complexity of calculations associated with recursive kernels. It is previously mentioned in [18] for $r = -1$, and [17] for $r = \frac{1}{2}$.
- The condition $nh_n^3 \to \infty$ is classic in the recursive estimation, it allows to deal with the bias terms, whereas the choice $h_n = n^{-\gamma}$ with $\frac{1}{5} < \gamma < \frac{1}{3}$, fulfills the assumption [$A.4$].
- Epanechnikov kernel $K(u) = \frac{3}{4}(1-u^2)\mathbb{I}_{\{|u|\leq 1\}}$ and quadratic kernel $K(u) = \frac{15}{16}(1-u^2)^2\mathbb{I}_{\{|u|\leq 1\}}$ are kernels that fulfill assumptions [$A3.1$] and [$A3.2$].

- The assumption $[A2.1]$ allows to obtain the convergence of the bias term to 0 meanwhile assumption $[A3.1]$ derives the rate of deceasing to 0.

We subsequently provide consistency using the almost complete (a.co.) convergence of the recursive relative predictive estimator defined in (1.7), pointing out that it is stronger than the almost sure one. We specify additionally the rate of this convergence.

Theorem 2.2. *If the assumptions* $[A1]$, $[A2.1]$, $[A3.1]$, $[A4]$ *and* $[A5]$ *hold, then*

$$\lim_{n\to+\infty} \widehat{r}_n^l(t) = r(t), \qquad a.co.$$

Proof. This result is obtained by showing the next two lemmas which demonstrate the convergence of the different terms of the forthcoming expression

$$\begin{aligned}\widehat{r}_n^l(t) - r(t) = \frac{1}{\widehat{r}_{2,n}^l(t)}\Big(&\left[\widehat{r}_{1,n}^l(t) - E\left(\widehat{r}_{1,n}^l(t)\right) + E\left(\widehat{r}_{1,n}^l(t)\right) - g_1(t)\right] \\ &+ r(t)\left[g_2(t) - E\left(\widehat{r}_{2,n}^l(t)\right) + E\left(\widehat{r}_{2,n}^l(t)\right) - \widehat{r}_{2,n}^l(t)\right]\Big).\end{aligned} \tag{2.1}$$

Lemma 2.3. *Under* $[A1]-[A4]$ *we have*

$$\widehat{r}_{j,n}^l(t) - E\left(\widehat{r}_{j,n}^l(t)\right) = O_{a.co.}\left(\sqrt{\frac{\log n}{nh_n}}\right).$$

Proof. To prove this lemma, we first apply Bernstein's inequality([9], proposition A.7) on the variables Z_i defined by

$$Z_i = \frac{n}{\sum_{i=1}^{n} h_i^{1-l}}\left[\frac{Y_i^{-j}}{h_i^l}K\left(\frac{t-T_i}{h_i}\right) - E\left(\frac{Y_i^{-j}}{h_i^l}K\left(\frac{t-T_i}{h_i}\right)\right)\right],$$

where

$$\widehat{r}_{j,n}^l(t) - E\left(\widehat{r}_{j,n}^l(t)\right) = \frac{1}{n}\sum_{i=1}^{n} Z_i.$$

Now, by conditioning on T_i, thanks to assumption [A1] we obtain

$$\begin{aligned}|EZ_i^m| \leq& \frac{1}{h_n^{m(1-l)}}\sum_{k=0}^{m} C_m^k E\Big(\frac{|Y_i^{-j}|}{h_i^l}K\left(\frac{t-T_i}{h_i}\right)\Big)^k\left(E\frac{|Y_i^{-j}|}{h_i^l}K\left(\frac{t-T_i}{h_i}\right)\right)^{m-k} \\ \leq& \frac{C}{h_n^{m(1-l)}}\sum_{k=0}^{m}\frac{1}{h_i^{ml}}E\Big(K^k(\frac{t-T_i}{h_i})\Big)\Big(EK(\frac{t-T_i}{h_i})\Big)^{m-k}.\end{aligned}$$

Firstly,

$$\begin{aligned}\frac{1}{h_i}E\left[K^k\left(\frac{t-T_i}{h_i}\right)\right] &= \int \frac{1}{h_i}K^k\left(\frac{t-u}{h_i}\right)f(u)\,du. \\ &= \int \frac{1}{h_i}K^k\left(\frac{z}{h_i}\right)f(t-z)\,dz,\end{aligned}$$

by setting $z = t - u$.
Now under assumptions $[A2.1]$ and $[A3.1]$, Bochner's theorem ([16]) implies that

$$E\left[K^k\left(\frac{t-T_i}{h_i}\right)\right] = O\left(h_i\right),$$

then

$$\left[EK\left(\frac{t-T_i}{h_i}\right)\right]^{m-k} = O\left(h_i^{m-k}\right).$$

Consequently, we obtain

$$\begin{aligned} \left|EZ_i^m\right| &\leq \frac{C}{h_n^m}\sum_{k=0}^{m} h_i^{1+m-k} \\ &\leq \frac{Ch_i^m}{h_n^m}\sum_{k=0}^{m} h_i^{1-k} \\ &\leq \frac{Ch_i^m}{h_n^m}\max_{0\leq k\leq m} h_i^{1-k} \\ &\leq \frac{Ch_i}{h_n^m}. \end{aligned}$$

(here C is a generic constant).
On the other hand, by taking $a_i = \sqrt{h_i}h_n^{-1}$, $b = \frac{2^{\frac{1}{m-2}}}{h_n}$ and using the fact that h_i is a decreasing sequence, we get

$$\begin{aligned} A_n^2 &= \sum_{i=1}^{n}\frac{h_i}{h_n^2} \\ &\geq \frac{n}{h_n}. \end{aligned}$$

In this case, if we take $\varepsilon n\sqrt{\frac{\log n}{nh_n}} \geq \varepsilon' A_n$, we get

$$\begin{aligned} \forall\varepsilon &> 0,\ P\left(\left|\sum_{i=1}^{n} Z_i\right| > \varepsilon n\sqrt{\frac{\log n}{nh_n}}\right) \leq P\left(\left|\sum_{i=1}^{n} Z_i\right| > \varepsilon' A_n\right) \\ &\leq 2\exp\left\{-\frac{\varepsilon^{'2}}{2\left(1+\frac{\varepsilon' b}{A_n}\right)}\right\} \end{aligned}$$

then for $\varepsilon' = \varepsilon \log n$, assumption [A.4] leads to

$$
\begin{aligned}
P\left(\left|\sum_{i=1}^{n} Z_i\right| > \varepsilon n \sqrt{\frac{\log n}{n h_n}}\right) &\leq 2\exp\left\{-\frac{\varepsilon^2 \log n}{2\left(1 + C\varepsilon\sqrt{\frac{\log n}{n h_n}}\right)}\right\} \\
&\leq 2\exp\left\{-\frac{\varepsilon^2 \log n}{2}\right\} \\
&\leq \frac{2}{n^{\frac{\varepsilon^2}{2}}}.
\end{aligned}
$$

If we choose $1 < p = \frac{\varepsilon^2}{2}$, we obtain

$$
\widehat{r}^{l}_{j,n}(t) - E\left(\widehat{r}^{l}_{j,n}(t)\right) = O_{a.co.}\left(\sqrt{\frac{\log n}{n h_n}}\right).
$$

□

Lemma 2.4. *The conditions* $[A2.1]$, $[A3.1]$, $[A4]$ *and* $[A5]$ *provide*

$$
E\left[\widehat{r}^{l}_{j,n}(t)\right] - g_j(t) = o(1), \;\; j = 1, 2.
$$

Proof. By conditioning on T_i, we get for $j = \overline{1,2}$

$$
\begin{aligned}
E\left[\widehat{r}^{l}_{j,n}(t)\right] &= E\left[\frac{1}{\sum\limits_{i=1}^{n} h_i^{1-l}} \sum_{i=1}^{n} \frac{Y_i^{-j}}{h_i^l} K\left(\frac{t - T_i}{h_i}\right)\right] \\
&= \frac{1}{\sum\limits_{i=1}^{n} h_i^{1-l}} \sum_{i=1}^{n} \frac{1}{h_i^l} E\left[K\left(\frac{t - T_i}{h_i}\right) r_j(T_i)\right] \\
&= \frac{1}{n\beta_{n,1-l}} \sum_{i=1}^{n} \left(\frac{h_i}{h_n}\right)^{1-l} \int \frac{1}{h_i} K\left(\frac{t-u}{h_i}\right) r_j(u) f(u)\, du. \qquad (2.2)
\end{aligned}
$$

First, by setting $z = t - u$, Bochner's theorem with $[A2.1]$, $[A3.1]$ lead to

$$
\int \frac{1}{h_i} K\left(\frac{t-u}{h_i}\right) r_j(u) f(u)\, du = \int \frac{1}{h_i} K\left(\frac{z}{h_i}\right) g_j(t - z)\, du \to g_j(t), \;\; h_i \to 0.
$$

Now using $[A4]$, $[A5]$ and by applying Lemma 2.4.1 in ([2]), we get

$$
\frac{1}{\beta_{n,1-l}} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{h_i}{h_n}\right)^{1-l} \int \frac{1}{h_i} K\left(\frac{z}{h_i}\right) g_j(t - z)\, dz \to g_j(t), \;\; n \to \infty.
$$

□

□

Corollary 2.5. *Under assumptions* [A2.2], [A3.1] *and* [A3.2], *if* $h_n = p\big(\frac{n}{\log n}\big)^{-\frac{1}{5}}, p > 0$,

$$\widehat{r}_n^l(t) - r(t) = O_{a.co}\big(\big(\frac{n}{\log n}\big)^{-\frac{2}{5}}\big).$$

Remark 2.6. According to [14] and [15], the best rate of convergence in norm L^∞ in nonparametric regression estimation is $(\frac{n}{\log n})^{-\frac{k}{2k+1}}$, supposing that the regression is k times continuously derivable at a fixed point t. The rate of convergence given in the above corollary is then optimal for the suitable choice $h_n = c\big(\frac{n}{\log n}\big)^{-\frac{1}{5}}$, $c > 0$.

Theorem 2.7. *If the assumptions* [A1], [A2.2], [A3.1], [A3.2], [A4] *and* [A5] *hold, then*

$$\widehat{r}_n^l(t) - r(t) = O(h_n^2) + O_{a.co}\Big(\sqrt{\frac{\ln n}{nh_n}}\Big).$$

Proof. Using the decomposition 2.1, the convergence of the first and the fourth stochastic terms hold, while the convergence of the remaining bias terms is studied in the following lemma.

Lemma 2.8. *The conditions* [A2.2], [A3.1], [A4] *and* [A5] *provide*

$$E\left[\widehat{r}_{j,n}^l(t)\right] - g_j(t) = O(h_n^2),\ j = 1, 2.$$

Proof. For the second and the fourth terms in decomposition (2.1), we proceed by conditioning on T_i as in (2.2), we get

$$E(\widehat{r}_{j,n}^l(t)) - g_j(t) = \frac{1}{\sum_{i=1}^n h_i^{1-l}} \sum_{i=1}^n h_i^{-l} \int K\Big(\frac{t-u}{h_i}\Big) g_j(u)du - g_j(t)$$
$$= \frac{1}{\sum_{i=1}^n h_i^{1-l}} \sum_{i=1}^n h_i^{1-l} \int (g_j(t - h_i v) - g_j(t))K(v)dv,\ v = \frac{t-u}{h_i},$$

using Taylor expansion now under assumption [A2.2] and [A3.2]

$$h_n^{-2}(E(\widehat{r}_{j,n}^l(t)) - g_j(t))$$
$$= \frac{1}{2\beta_{n,1-l}} \frac{1}{n} \sum_{i=1}^n \Big(\frac{h_i}{h_n}\Big)^{3-l} \int v^2 K(v) g_j^{''}(t - \theta h_i v)dv,\ 0 < \theta < 1.$$

On one hand, the dominated convergence theorem leads to

$$\lim_{i\to\infty} \int v^2 K(v) g_j^{''}(t - \theta h_i v)dv = g_j^{''}(t) \int v^2 K(v)dv.$$

On other hand, due to conditions [A3.1], [A4] and [A5], by applying Lemma 2.4.1 ([2]), we obtain

$$\lim_{n\to\infty} h_n^{-2}(E(\widehat{r}_{j,n}^l(t)) - g_j(t)) = \frac{\beta_{3-l}}{\beta_{1-l}} g_j^{''}(t) \int v^2 K(v)dv < \infty.$$

□

Remark now first that if $|\widehat{r}^{l}_{2,n}(t))| \leq \frac{|g_2(t)|}{2}$ then $|\widehat{r}^{l}_{2,n}(t)) - g_2(t)| \geq \frac{|g_2(t)|}{2}$. If we take $\eta = \frac{|g_2(t)|}{2}$, we obtain

$$P\Big(|\widehat{r}^{l}_{2,n}(t)| \leq \eta\Big) \leq P\Big(|\widehat{r}^{l}_{2,n}(t) - g_2(t)| \geq \eta\Big). \tag{2.3}$$

Notice that Lemma2.3 and Lemma 2.8 provide

$$\lim_{n\to\infty} \widehat{r}^{l}_{2,n}(t) = g_2(t),\ a.co.$$

Consequently, from‘(2.3), we get

$$\sum_{n\geq 0} P\Big(|\widehat{r}^{l}_{2,n}(t)| \leq \eta\Big) < \infty.$$

□

3. Numerical study

As an application of recursive relative regression estimate, we choose to predict response values of the following model

$$Y_i = r(T_i) + \epsilon_i,\ i = \overline{1,n}$$

- The regression function r is estimated here using four methods, mainly by recursive relative regression estimator (RR), non recursive relative regression estimator (RNR), recursive regression estimator (R) and Nadaraya-Watson's one (NW).
- The first model denoted $(M1)$ is simulated with the co-variate $T_i \sim N(3,1)$, $1 \leq i \leq n$, the errors follow a standard gaussian distribution, and the true regression function $r_1(t) = t+1$.
- Model 2 denoted $(M2)$ with $T_i \sim N(3,1)$, $1 \leq i \leq n$, the errors follow a standard gaussian distribution, and $r_2(t) = t\sin(\frac{t}{3})$.
- Model 3 denoted $(M3)$ with $T_i \sim U_{[1,3]}$, $1 \leq i \leq n$, the errors $\epsilon_i \sim N(0, 0.1)$ and $r_3(t) = \exp(t)$.
- The aberrant values are generated artificially by multiplying some observations by a coefficient that increases with the sample sizes.

The predicted values Y^*_s are calculated through respectively the estimators (RR), (RNR), (R) and (NW) based on $k = 100$ realizations of the random samples of size $n = 100$, 250, 800, Epanechnikov kernel, and the smoothing window $h_n = cn^{-\frac{1}{5}}$, $c > 0$. To evaluate error prediction between real response values $Y_{s,j}$ and predicted ones $Y^*_{s,j}$ at the realization j, we utilize the forthcoming empirical relative error

$$RE(Y^*) = \frac{1}{k}\sum_{j=1}^{k}\frac{1}{n}\sum_{s=1}^{n}\left(\frac{Y_{s,j} - Y^*_{s,j}}{Y_{s,j}}\right)^2.$$

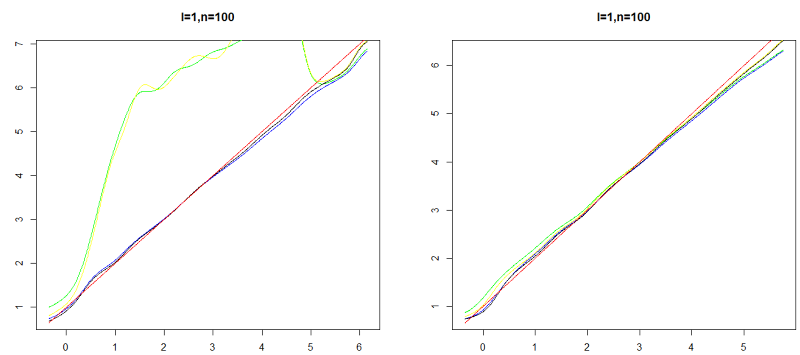

FIGURE 1. From left to right (with outliers and without) RR, RNR, R, NW-estimators, true regression of M1 (respectively blue, black, green, yellow, red).

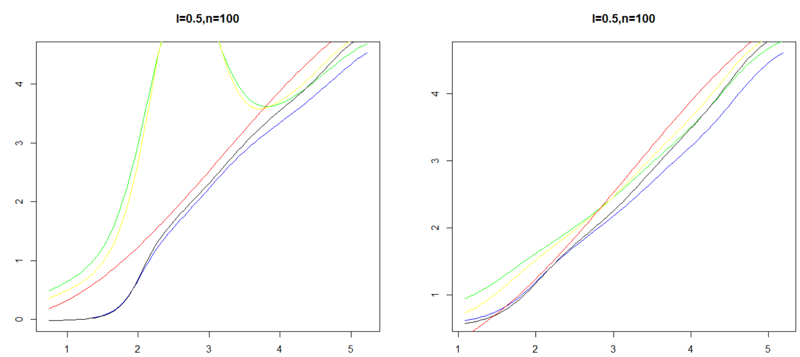

FIGURE 2. From left to right(with outliers and without) RR, RNR, R, NW-estimators, true regression of M1 (respectively blue, black, green, yellow, red).

TABLE 1. Prediction error values of $M1$.

$M1$	RR	RNR	R	NW
$n = 100$	0.001	0.0009	0.351	0.325
$n = 250$	0.001	0.0009	0.329	0.322
$n = 800$	0.0095	0.0089	0.253	0.295

According to Figures 1-4, together with Tables 1, 2, we remark the following

(1) Without outliers, all the estimates (RR, RNR, R, $N.W$) behave in the same way, they are all close to the true regression.
(2) In the presence of outliers, the two relative estimators (RR, RNR) are close to the true regression, this is not the case with the non-relative estimates (R) and (NW).

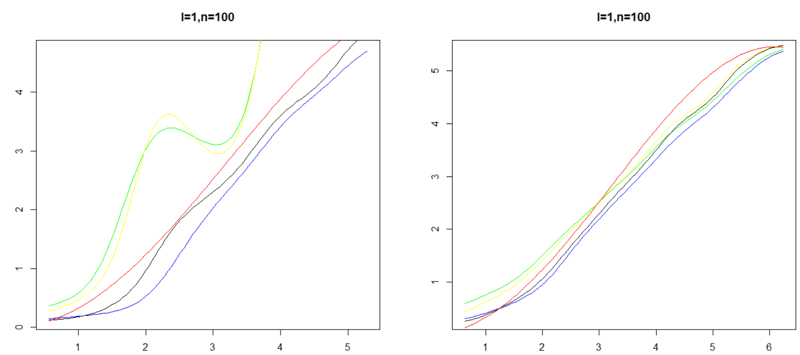

FIGURE 3. From left to right (with outliers and without) RR, RNR, R, NW-estimators, true regression of M2 (respectively blue, black, green, yellow, red).

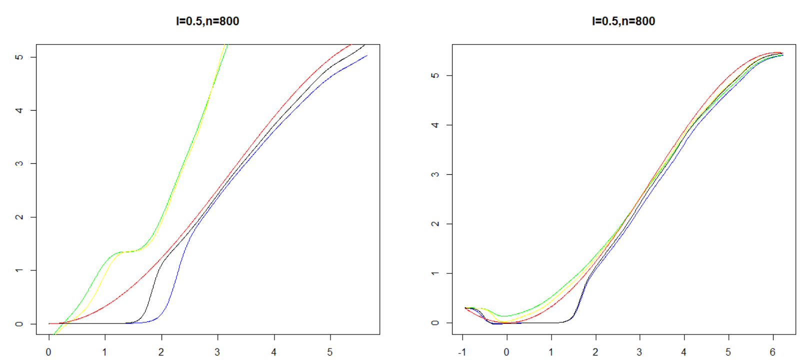

FIGURE 4. From left to right(with outliers and without) RR, RNR, R, NW-estimators, true regression of M2 (respectively blue, black, green, yellow, red).

TABLE 2. Error estimation values of model $M2$.

$M2$	RR	RNR	R	NW
$n = 100$	0.02	0.009	0.515	0.461
$n = 250$	0.013	0.007	0.443	0.394
$n = 800$	0.008	0.006	0.382	0.375

(3) The convergence of both relative estimates (RR, RNR) towards the function $r(t)$ improves when the size samples increases.
(4) Prediction errors with the two relative estimates (RR, RNR) are almost similar, they are smaller comparing with (R, NW).
(5) Prediction errors of the two relative estimators (RR, RNR) decrease when the size samples increases.

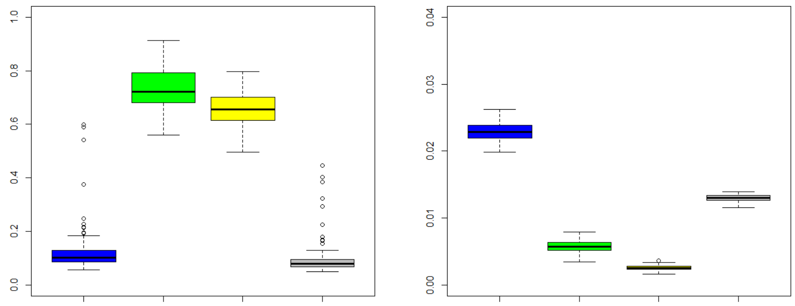

FIGURE 5. From left to right (with outliers and without) Distribution errors of M3 ($n = 200$, $k = 100$), RR-estimator(blue), RNR-estimator(gray), R-estimator (green), NW-estimator (yellow).

TABLE 3. Comparaison of computation time of RR and RNR-estimators with M3.

$M3$	$n = 100$	$n = 200$	$n = 500$	$n = 1000$
$RR, l = 0.5$	0.345	0.69	0.858	1.189
RR, $l = 1$	0.343	0.687	0.854	1.192
RNR	0.422	6.651	38.542	85.871

(6) The error distribution of the four estimators confirm the above remarks, the box-plot shows the advantage of the relative estimates (RR, RNR) in the presence of outliers in term of error prediction (see Figure 5).
(7) The computation time is reduced when calculating the recursive relative estimators comparing with the relative estimator. It is further reduced when the sample size increases (See Table 3).

One can conclude that when conducting the relative regression estimation, it is essential to be mindful of aberrant values and their potential impact on the results. By acknowledging and addressing them effectively, we can ensure that regression analysis is robust and provides accurate insights into forecasting. Both recursive and non-recursive relative estimators lead to good predicting results, recursive approach is advantageous thanks to its ability to continuously update estimation as new data becomes available. This allows for dynamic and real time adjustments. Recursive estimation tends to be more computationally efficient as it requires storing the current estimate and updating it iteratively (see (1.8)), rather than re-estimating each time new data is received.

4. Conclusion

This work shows the theoritical and pratical performance of our estimator. Indeed, when conducting the relative regression estimation, it is essential to be mindful of aberrant values and their potential impact on the results. By acknowledging and addressing them effectively, we can ensure that regression analysis is robust and provides accurate insights into forecasting. Both recursive and non-recursive relative estimators lead to good predicting results, while recursive approach is advantageous thanks to its ability to continuously update estimation as new data becomes available. This allows for dynamic and real time adjustments. Recursive estimation tends to be more computationally efficient as it requires storing the current estimate and updating it iteratively, rather than re-estimating each time new data is received.

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