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LOGARITHM SINGULAR DIRICHLET PROBLEM FOR NONLINEAR ELLIPTIC EQUATIONS WITH VARIABLE EXPONENT

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ABSTRACT. In this paper, we address certain nonlinear problems featuring a logarithm singular source term at the origin within the framework of variable exponent Sobolev spaces.

1. INTRODUCTION

We start by reviewing some of the existing literature on singular elliptic problems. Numerous studies have explored problems of this nature

$$\begin{cases} -div(|\nabla u|^{p-1}\nabla u) = \frac{f}{u^\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 \leq f \in L^m(\Omega)$, $m \geq 1$ and $\gamma > 0$ in [8], and also in [9], but in the form

$$\begin{cases} -div(|\nabla u|^{p-1}\nabla u) = H(u)\mu & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ is a positive Radon measure, and H is a function that is positive, continuous, and remains bounded everywhere except at the origin. The case where μ is a measure has been studied in [10] and [17]. In the same context, if Ω is sufficiently regular, the existence and uniqueness of a solution belonging solely to $W^{1,1}_{\text{loc}}(\Omega)$ are proven using a Kato-type argument, as shown in [19]. Lazer and McKenna, in [16], investigated the existence of solutions using the method of subsolutions and supersolutions. For a detailed review of the case when $H = 1$, we refer readers to [7]. The existence of a positive solution in the distributional sense, with the source term in $L^1(\Omega)$, was established by Boccardo and Orsina in [2]. Furthermore, a unique classical solution $u \in C^2(\Omega)\cap C(\overline{\Omega})$ were demonstrated in [6].

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For the linear case, problems of this nature have been studied in [5], where the proposed model is described as follows:

$$\begin{cases} u \in W_0^{1,2}(\Omega), \\ \int_\Omega A(x)\nabla u \cdot \nabla v dx = \langle \mu,\ h(u)v\rangle \ \forall v \in V. \end{cases} \tag{1.1}$$

Here, V denotes a function space that includes the Schwartz space $\mathcal{D}(\Omega)$, consisting of C^∞ functions with support compact in Ω.

The nonlinear case has treated in [4] whose form is

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \langle Au,\ v\rangle = \langle \mu,\ h(u)v\rangle \ \forall v \in V, \end{cases} \tag{1.2}$$

where $\mu \in W^{-1,p'}(\Omega)$, and V is a space that allows the aforementioned equality to be valid in the distributional sense.

The model under consideration in our work is as follows:

$$\begin{cases} u \in W_0^{1,p(x)}(\Omega), \\ \langle Au,\ v\rangle = \langle \mu,\ \frac{v}{u^\gamma \ln^\lambda(1+u)}\rangle \ \forall v \in V, \end{cases}$$

where μ belongs to the dual space $W^{-1,p'(x)}(\Omega)$, Ω is an open and bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, V represents the space in which the problem holds in the distributional sense and γ, λ are positive constants.

Our goal in this article is to extend previous studies to the generalized Sobolev space with a non-constant exponent $W^{1,p(\cdot)}$, and adding a logarithm singular source term. An illustrative example of the operator A is the $p(x)$-Laplacian, expressed in the form $b(x)\,|\nabla u|^{p(x)-2}\,\nabla u$ where $b \in L^\infty(\Omega)$ represents a positive function that is uniformly bounded away from 0. We present a more straightforward proof, with a particular emphasis on the nonlinear potential theory associated with variable exponents in nonlinear elliptic equations.

The paper is organized as follows. In Section 2, we present essential definitions and key results pertaining to Sobolev spaces characterized by non-constant exponents. In Section 3, we formulate our main problem along with its fundamental assumptions. In Section 4, we provide an approximation of μ and discuss some properties. In Section 5, we concentrate on proving the existence of a solution for the problem (P') presented in Section 3, under the condition that both Ω and h are bounded. In Section 6, we move to proof our main result in the situation where Ω isn't infinite and $h = \frac{1}{u^\gamma ln^\lambda(1+u)}$.

2. PRELIMINARY TOOLS

We provide an overview of the fundamental key properties and definitions of the extended Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W_0^{1,p(\cdot)}(\Omega)$, and $W^{1,p(\cdot)}(\Omega)$, where Ω represents an open domain in $\mathbb{R}^N$. More details regarding the characteristics of Lebesgue-Sobolev spaces with variable exponents, are available in [11], [13], [14], and for the general theory of Orlicz spaces, we cite [1, 18]. Let us begin by introducing the notations listed below.

$$p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x), p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x).$$

For a measurable function$p : \Omega \to \mathbb{R}$ that is bounded on Ω, one can define both the critical Sobolev exponent and the exponent conjugate to $p(\cdot)$ as follows, respectively:

$$p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1} \text{ and } p^*(\cdot) = \frac{Np(\cdot)}{N-p(\cdot)}.$$

The variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ consisting of all measurable functions $u : \Omega \to \mathbb{R}$ such that the convex modular

$$\rho_{p(\cdot)}(u,\Omega) = \int_\Omega |u|^{p(x)}\, dx$$

has a finite value. Then,

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R}, u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}dx < \infty\}.$$

When the exponent is remains, specifically if $p^+ < \infty$, a norm is specified for $L^{p(\cdot)}(\Omega)$, referred to as *the Luxembourg norm*, using expression:

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

This norm generalizes the standard L^p norm to variable exponent spaces and equips $L^{p(\cdot)}(\Omega)$ with a complete normed vector space structure. The convex modular $\rho_{p(\cdot)}(u,\Omega)$ and the Luxembourg norm are closely related and satisfy the following properties:

- If $\|u\|_{p(\cdot)} < 1$, then $\rho_{p(\cdot)}(u,\Omega) < 1$.
- If $\|u\|_{p(\cdot)} = 1$, then $\rho_{p(\cdot)}(u,\Omega) = 1$.
- If $\|u\|_{p(\cdot)} > 1$, then $\rho_{p(\cdot)}(u,\Omega) > 1$.

This framework allows $L^{p(\cdot)}(\Omega)$ to capture the behavior of functions with non-uniform integrability across Ω.

$(L^{p(\cdot)}(\Omega), \|\ \|_{L^{p(\cdot)}})$ is a Banach separable space. Additionally, if $p^- > 1$, hence $L^{p(\cdot)}$ (Ω) is uniformly convex, then reflexive, and there exists an isomorphism between its dual space and $L^{p'(\cdot)}$ (Ω), where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Moreover the generalized Hölder inequality (see [21]) is given as follow:

$$\int_\Omega |uv|dx \le (\frac{1}{p^-} + \frac{1}{(p')^-})\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \forall u \in L^{p(\cdot)}(\Omega), \forall v \in L^{p'(\cdot)}\ (\Omega).$$

At this point, we proceed to introduce Sobolev space with variable exponent,

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}\ (\Omega),\ |\nabla u| \in L^{p(\cdot)}\ (\Omega)\},$$

is structured as a Banach space using one of the equivalent norms

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

or

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \inf\{\lambda > 0,\ \int_\Omega \left(\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} + \left|\frac{u(x)}{\lambda}\right|^{p(x)}\right) dx \le 1\}.$$

The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$

$$W_0^{1,p(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}.$$

Throughout this paper, we will assume that is equipped with the norm $W_0^{1,p(\cdot)}(\Omega)$ is equipped with the norm

$$||\nabla u||_{p(x)} = \left(\int_\Omega |\nabla u|^{p(x)} dx\right)^{\frac{1}{p(x)}}. \tag{2.1}$$

Considering that $p^- > 1$, Both spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ possess the separability property, and are also reflexive Banach spaces, the space $W^{-1,p'(\cdot)}(\Omega)$ is the dual of $W_0^{1,p(\cdot)}(\Omega)$.

In $W_0^{1,p(\cdot)}(\Omega)$, the inequality established by Poincaré is satisfied. Therefore, one can find a constant $C > 0$ for which:

$$||u||_{L^{p(x)}(\Omega)} \leqslant C||\nabla u||_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega)$$

thus, $||\nabla u||_{L^{p(x)}(\Omega)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$ see [12].

In order to establish the regularity properties of variable exponent Lebesgue spaces, the following condition is considered appropriate. $p(\cdot)$ satisfies the Log-Hölder continuity condition if $p(\cdot) : \Omega \to \mathbb{R}$ is a function that is measurable in the sense that:

$$\exists C > 0 : |p(x) - p(y)| \leq \frac{C}{-ln|x-y|} \text{ for } |x-y| < \frac{1}{2}$$

$$1 < \operatorname*{ess\,inf}_{x\in\Omega} p(x) \leq \operatorname*{ess\,sup}_{x\in\Omega} p(x) < N.$$

Variable exponents Sobolev spaces with regularity results are obtained by using the Log-Hölder continuity condition, particularly every element of $W^{1,p(\cdot)}$ can be approximated by functions in $C^\infty(\overline{\Omega})$ and

$$W_0^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega).$$

Let $k > 0$, To define the truncation function $T_k : \mathbb{R} \to \mathbb{R}$, we set

$$T_k(\sigma) := \ max(-k, min(\sigma, k)),$$

and, we have

$$T_k(\sigma) = \begin{cases} \sigma & \text{if } |\sigma| \leq k, \\ k & \text{if } \sigma > k, \\ -k & \text{if } \sigma < -k. \end{cases}$$

3. Formulation of the Problem and Key Assumptions

Consider Ω an open, bounded domain of $\mathbb{R}^N (N \geq 2)$, we formulate our problem as follows

$$(P) \left\{ \begin{array}{l} u \in W_0^{1,p(x)}(\Omega), \\ \langle Au,\ v\rangle = \langle \mu,\ h(u)v\rangle \ \forall v \in V, \end{array} \right.$$

where $\mu \in W^{-1,p'(x)}(\Omega)$ and V represents the space in which the problem holds in the distributional sense. Additionally, the singular source term $h : (0, \infty) \longrightarrow$

$(0, \infty)$ is continuous and bounded. We will suppose the following conditions hold across the entire paper.
ASSUMPTION (1): Suppose that $a(x,\ \xi)$: $\Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfied the following presumptions:

$$a(x,\xi)\cdot\xi \geq \alpha|\xi|^{p(x)} \text{ a.e. } x \text{ in } \Omega, \forall \xi \in \mathbb{R}^N, \tag{3.1}$$

$$(a(x,\xi) - a(x,\xi'))\cdot(\xi - \xi') \geq 0 \ \text{ a.e. } x \text{ in } \Omega,\ \forall \xi,\ \xi' \in \mathbb{R}^N, \tag{3.2}$$

$$|a(x,\xi)| \leq \beta(\nu(x) + |\xi|^{p(x)-1}) \text{ in } x \text{ in } \Omega, \forall \xi \in \mathbb{R}^N \tag{3.3}$$

for $\nu \in L^{p'(x)}(\Omega)$, $\nu \geq 0$, $\alpha, \beta > 0$.

ASSUMPTION (2): For $u, v \in W_0^{1,p(x)}(\Omega)$ setting

$$\langle Au,\ v\rangle = \int_\Omega a(x, \nabla u)\cdot \nabla v dx \tag{3.4}$$

defines an operator from $W_0^{1,p(x)}(\Omega)$ to $W^{-1,p'(x)}(\Omega)$ which is hemicontinuous monotone and coercive.
ASSUMPTION (3): We assume that Ω is bounded and that

$$N \leq 2, \text{ or } p^- > \frac{2N}{N+2}. \tag{3.5}$$

This implies that

$$W_0^{1,p(x)}(\Omega) \subseteq W^{-1,p'(x)}(\Omega). \tag{3.6}$$

For further details about embeddings on generalized Sobolev and Lebesgue spaces, we refer the reader to [11], [13], [14], [15].

4. APPROXIMATION OF μ

Let $\hat{\mu} \in W_0^{1,p(x)}(\Omega)$ and $\varepsilon > 0$. Under the condition (3.5), it follows that $\hat{\mu} \in W^{-1,p'(x)}(\Omega)$. Moreover, the operator $u \mapsto \varepsilon Au + u$ is bounded, strictly monotone, hemicontinuous, and coercive from $W_0^{1,p(x)}(\Omega)$ to $W^{-1,p'(x)}(\Omega)$. Therefore, there exists a unique solution u_ε to

$$\begin{cases} u_\epsilon \in W_0^{1,p(x)}(\Omega), \\ \varepsilon A u_\epsilon + u_\epsilon = \hat{\mu} \text{ in } \Omega. \end{cases} \tag{4.1}$$

Later, the notation $\hat{\mu}$ for the right-hand side of the equation will be clarified. For information, we now have that $\hat{\mu}$ solves the equation $A\hat{\mu} = \mu$. Subsequently, we will demonstrate that

Proposition 4.1. *Based on the previously stated assumptions, it follows that*

$$u_\varepsilon \rightharpoonup \hat{\mu} \text{ in } W_0^{1,p(x)}(\Omega), Au_\varepsilon \rightharpoonup A\hat{\mu} \text{ in } W^{-1,p'(x)}(\Omega) \text{ when } \varepsilon \to 0. \tag{4.2}$$

Moreover, if $A\hat{\mu} \geq 0$, then $Au_\varepsilon \geq 0$ and $u_\varepsilon \leq \hat{\mu}$.

Proof. Note that by using the second equation in (4.1), $Au_\varepsilon \in W_0^{1,p(x)}(\Omega)$. Taking $u_\varepsilon - \hat{\mu}$ as a test function in the weak sense, we derive

$$\langle Au_\varepsilon,\ u_\varepsilon - \hat{\mu}\rangle = -\frac{1}{\varepsilon}\int_\Omega (\hat{\mu} - u_\varepsilon)^2 dx \leq 0. \tag{4.3}$$

Therefore,

$$\int_\Omega a(x,\ \nabla u_\varepsilon)\cdot\nabla u_\varepsilon dx \leq \int_\Omega a(x,\ \nabla u_\varepsilon)\cdot\nabla\hat{\mu}dx.$$

By utilizing (3.1) and (3.3), we arrive at the following expression

$$\alpha\int_\Omega |\nabla u_\varepsilon|^{p(x)}dx \leq \int_\Omega (\nu(x)+\beta|\nabla u_\varepsilon|^{p(x)-1})|\nabla\hat{\mu}|dx.$$

Applying the generalized Hölder inequality, we obtain

$$\alpha\int_\Omega |\nabla u_\varepsilon|^{p(x)}dx \leq \int_\Omega \nu(x)|\nabla\hat{\mu}|dx + \beta(\frac{p^-}{p^- -1}+\frac{1}{p^-})\left(\int_\Omega |\nabla u_\varepsilon|^{p(x)}dx\right)^{\frac{p(x)-1}{p(x)}}$$
$$\times\left(\int_\Omega |\nabla\hat{\mu}|^{p(x)}dx\right)^{\frac{1}{p(x)}}$$

i.e.

$$\alpha||\nabla u_\varepsilon||_{p(x)}^{p(x)} \leq C + C'||\nabla u_\varepsilon||_{p(x)}^{p(x)-1}$$

with $C=\int_\Omega \nu(x)|\nabla\hat{\mu}|\,dx$ and $C'=\beta\left(\frac{p^-}{p^- -1}+\frac{1}{p^-}\right)\|\nabla\hat{\mu}\|_{p(x)}$, both of which are independent of ε, it follows that u_ε is bounded in $W_0^{1,p(x)}(\Omega)$ regardless of ε. Thus, up to a subsequence, one can find some $u\in W_0^{1,p(x)}(\Omega)$ such that

$$u_\varepsilon \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega).$$

Furthermore, one readily observes that since $a(x,\nabla u_\varepsilon)$ is bounded in $L^{p'(x)}(\Omega)$, Au_ε remains bounded in $W^{-1,p'(x)}(\Omega)$. From (4.1), we obtain the following result:

$$u_\varepsilon - \hat{\mu} = -\varepsilon Au_\varepsilon \underset{\varepsilon\to 0}{\to} 0 \text{ in } W^{-1,p'(x)}(\Omega).$$

Since the limit in the distributional sense is unique, it follows that $u=\hat{\mu}$. Moreover, because the limit of u_ε is determined in a unique way the entire sequence u_ε converges weakly to $\hat{\mu}$.

We now assert that $Au_\varepsilon \overset{\varepsilon\to 0}{\rightharpoonup} A\hat{\mu}$ in $W^{-1,p'(x)}(\Omega)$. In fact, since Au_ε is bounded in $W^{-1,p'(x)}(\Omega)$, it follows that for a subsequence and some $\chi\in W^{-1,p'(x)}(\Omega)$,

$$Au_\varepsilon \rightharpoonup \chi \ in\ W^{-1,p'(x)}(\Omega).$$

By applying (4.3) i.e.

$$\langle Au_\varepsilon,\ u_\varepsilon-\hat{\mu}\rangle \leq 0.$$

It follows that for this subsequence,

$$\limsup_{\varepsilon\to 0}\langle Au_\varepsilon,\ u_\varepsilon\rangle \leq \langle\chi,\hat{\mu}\rangle. \tag{4.4}$$

Based on the monotonicity assumption, we infer that

$$\langle Au_\varepsilon - Av,\ u_\varepsilon - v\rangle \geq 0\ \forall v\in W_0^{1,p(x)}(\Omega).$$

As a result, this implies

$$\langle Au_\varepsilon,\ u_\varepsilon\rangle \geq \langle Au_\varepsilon,\ v\rangle + \langle Av,\ u_\varepsilon - v\rangle\ \forall v\in W_0^{1,p(x)}(\Omega).$$

Taking the lim sup, we get from (4.4)

$$\langle\chi,\hat{\mu}\rangle \geq \langle\chi,\ v\rangle + \langle Av,\hat{\mu}-v\rangle\ \forall v\in W_0^{1,p(x)}(\Omega)$$

i.e
$$\langle \chi - Av, \hat{\mu} - v\rangle \geq 0 \ \forall v \in W_0^{1,p(x)}(\Omega).$$
For $v = \hat{\mu} - tw$, where $w \in W_0^{1,p(x)}(\Omega)$ and $t > 0$, we assert that
$$\langle \chi - A(\hat{\mu} - tw), w\rangle \geq 0 \ \forall w \in W_0^{1,p(x)}(\Omega).$$
Taking the limit as $t \to 0$, it results that
$$\langle \chi - A\hat{\mu}, w\rangle \geq 0 \ \forall w \in W_0^{1,p(x)}(\Omega)$$
and hence $\chi = A\hat{\mu}$. Since the limit is unique, the entire sequence converges to $A\hat{\mu}$, which implies
$$Au_\varepsilon \rightharpoonup A\hat{\mu} \ in \ W^{-1,p'(x)}(\Omega).$$
To demonstrate the final point of the proposition, it is helpful to use the test function $(u_\varepsilon - \hat{\mu})^+$, which gives
$$\varepsilon\langle Au_\varepsilon, \ (u_\varepsilon - \hat{\mu})^+\rangle + \int_\Omega \{(u_\varepsilon - \hat{\mu})^+\}^2 dx = 0.$$
From this, we deduce
$$\int_\Omega \{(u_\varepsilon - \hat{\mu})^+\}^2 dx = -\varepsilon \int_\Omega a(x, \ \nabla u_\varepsilon) \cdot \nabla (u_\varepsilon - \hat{\mu})^+ dx.$$
By the monotonicity assumption, it follows that
$$\int_\Omega (a(x, \ \nabla u_\varepsilon) - a(x, \ \nabla \hat{\mu})) \cdot \nabla (u_\varepsilon - \hat{\mu})^+ dx \geq 0.$$
As a result, we arrive at the conclusion that
$$\int_\Omega \{(u_\varepsilon - \hat{\mu})^+\}^2 dx \leq -\varepsilon \int_\Omega a(x, \ \nabla \hat{\mu}) \cdot \nabla (u_\varepsilon - \hat{\mu})^+ dx = -\varepsilon\langle A\hat{\mu}, (u_\varepsilon - \hat{\mu})^+\rangle.$$
Finally, if $A\hat{\mu} \geq 0$, then $u_\varepsilon \leq \hat{\mu}$, and from (4.1), it follows that $Au_\varepsilon \geq 0$. Hence, the proof is concluded. □

The convergence discussed above can be strengthened under more stringent assumptions. In fact, it follows that

Proposition 4.2. *We consider two constants λ_{p^-} and λ_{p^+} in $\mathbb{R}^+$, one has*
$$if \ p^- \geq 2, (a(x, \ \xi) - a(x, \ \xi')). \ (\xi - \xi') \geq \lambda_{p^-} |\xi - \xi'|^{p(x)} \tag{4.5}$$
a.e. x in $\Omega, \forall \xi, \xi' \in \mathbb{R}^N$,
$$if \ p^+ < 2, (a(x, \ \xi) - a(x, \ \xi')). \ (\xi - \xi') \geq \lambda_{p^+} |\xi - \xi'|^2 \{|\xi| + |\xi'|\}^{p(x)-2} \tag{4.6}$$
a.e. x in $\Omega, \forall \xi, \xi' \in \mathbb{R}^N$.

Therefore, the convergences in (4.2) are strong.

Proof. The proof is analogous to the demonstration in [4] Proposition 2.2. with respect to variable Sobolev exponent spaces and using the generalized Hölder inequality in place of the standard Hölder inequality. □

Remark 4.3. It is worth noting that if (4.5) and (4.6) hold, the condition (3.2) in proposition 4.1 can be omitted.

Assume that a satisfies the strict monotonicity condition given below:

$$(a(x,\ \xi) - a(x,\ \xi')) \cdot (\xi - \xi') > 0 \text{ a.e. } x \text{ in } \Omega,\ \forall \xi \neq \xi' \in \mathbb{R}^N. \tag{4.7}$$

Thus, given that the operator A is strictly monotone, by invoking the Minty-Browder theorem let $f \in W^{-1,p'(x)}(\Omega)$, then there exists a unique solution u solving

$$\begin{cases} u \in W_0^{1,p(x)}(\Omega), \\ \langle Au,\ v\rangle = \langle f,\ v\rangle\ \forall v \in W_0^{1,p(x)}(\Omega). \end{cases} \tag{4.8}$$

It is beneficial to state the next lemma,

Lemma 4.4. *Assume that (3.1), (3.3), and (4.7) hold. For $\theta \in W^{-1,p'(x)}(\Omega)$, let $u = \mathcal{T}(\theta)$ be the solution to (4.8). As a result, $\mathcal{T}$ is a continuous operator from $W^{-1,p'(x)}(\Omega)$ under the strong topology to $W_0^{1,p(x)}(\Omega)$ in the weak topology setting.*

Proof. Assume that $\theta_n \to \theta$ in $W^{-1,p'(x)}(\Omega)$, and define $u_n = \mathcal{T}(\theta_n)$. Then, it follows that

$$\langle Au_n,\ u_n\rangle = \langle \theta_n,\ u_n\rangle.$$

From (3.1), we obtain that

$$\alpha||\nabla u_n||_{p(x)}^{p(x)} \leq ||\theta_n||_{-1,p'(x)}||\nabla u_n||_{p(x)}.$$

Hence, the sequence $(u_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega)$, then one can find $u \in W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$. Since Au_n is bounded in $W^{-1,p'(x)}(\Omega)$, it follows that, we find a subsequence and some $\chi \in W^{-1,p'(x)}(\Omega)$ in a way that

$$u_n \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega)\ Au_n \rightharpoonup \chi \text{ in } W^{-1,p'(x)}(\Omega). \tag{4.9}$$

We claim that χ is equal to Au, where u is the solution to (4.8). The arguments used earlier can be applied here as well. In fact, based on the monotonicity assumption (4.7), we obtain

$$\langle Au_n, u_n\rangle \geq \langle Au_n, v\rangle + \langle Av, u_n - v\rangle\ \forall v \in W_0^{1,p(x)}(\Omega). \tag{4.10}$$

Additionally, we have

$$\langle Au_n, u_n\rangle = \langle \theta_n, u_n\rangle \to \langle \theta, u\rangle.$$

Furthermore, given that $\langle Au_n, u\rangle = \langle \theta_n, u\rangle$ in the limit, we conclude that $\langle \theta, u\rangle = \langle \chi, u\rangle$. Taking the limit in (4.10), yields

$$\langle \chi - Av, u - v\rangle \geq 0\ \forall v \in W_0^{1,p(x)}(\Omega).$$

For $v = u - tw$ with $t > 0$, it can be easily shown that

$$\langle \chi - A(u - tw),\ w\rangle \geq 0\ \forall w \in W_0^{1,p(x)}(\Omega).$$

As $t \to 0$, we obtain

$$\langle \chi - Au, w\rangle \geq 0\ \forall w \in W_0^{1,p(x)}(\Omega).$$

Therefore, $\chi = Au$, and the result comes from the uniqueness of the limit of Au_n. □

Remark 4.5. Assuming (4.5) and(4.6) the convergence in (4.9) becomes strong.

5. CASE WHEN Ω AND h ARE BOUNDED

Assume that a is strictly monotone as stated in (4.7). Therefore, as demonstrated before, for $\mu \in W^{-1,p'(x)}(\Omega)$, a unique $\hat{\mu}$ exists such that

$$\begin{cases} \hat{\mu} \in W_0^{1,p(x)}(\Omega), \\ \int_\Omega a(x,\ \nabla\hat{\mu}) \cdot \nabla v dx = \langle \mu,\ v\rangle\ \forall v \in W_0^{1,p(x)}(\Omega). \end{cases} \tag{5.1}$$

It can be shown that

Theorem 5.1. *Suppose that (3.1), (3.3), (4.5), (4.6) hold. Let $h : \mathbb{R} \to \mathbb{R}$ be a function Lipschitz continuous with bounded values, such that*

$$(sh(s))' \in L^\infty(\mathbb{R}). \tag{5.2}$$

For $\mu \in W^{-1,p'(x)}(\Omega)$, with $p^- > \dfrac{2N}{N+2}$ and $\hat{\mu}$ formulated through (5.1). Then a solution u exists for

$$\begin{cases} u \in W_0^{1,p(x)}(\Omega), \\ \langle Au,\ v\rangle = \langle A\hat{\mu},\ h(u)v\rangle\ \forall v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{5.3}$$

Proof. Consider $\hat{\mu}_n$ as the solution $u_{\frac{1}{n}}$ to (4.1) where ε equals $\frac{1}{n}$. We can show that there exists a solution u_n for

$$\begin{cases} u_n \in W_0^{1,p(x)}(\Omega), \\ \langle Au_n,\ v\rangle = \langle A\hat{\mu}_n,\ h(u_n)v\rangle\ \forall v \in W_0^{1,p(x)}(\Omega). \end{cases} \tag{5.4}$$

By (4.1), one has

$$A\hat{\mu}_n \in W_0^{1,p(x)}(\Omega) \subseteq L^{p^*(x)}(\Omega). \tag{5.5}$$

Note that if $N = 1$ or $p(x) \geq N$ then $p^*(x) = \infty$. Else if $N \geq 2$ and $p(x) < N$, we claim that

$$p^*(x) > (p^*(x))'. \tag{5.6}$$

Indeed

$$p^- > \frac{2N}{N+2} \Leftrightarrow p^*(x) > (p^*(x))'.$$

We now select $r(x)$ in a way that

$$r(x) = \begin{cases} p(x) \text{ if } N = 1 \text{ or } p^- \geq N, \\ (p^*(x))' < r(x) < p^*(x) \text{ else.} \end{cases} \tag{5.7}$$

For $w \in L^{r(x)}(\Omega)$ there exists a unique solution $u = S_n(w)$ to

$$\begin{cases} u \in W_0^{1,p(x)}(\Omega), \\ \langle Au,\ v\rangle = \int_\Omega A\hat{\mu}_n h(w)v\ dx\ \forall v \in W_0^{1,p(x)}(\Omega). \end{cases} \tag{5.8}$$

Indeed, when $p^+ < N$, then by (5.5) $A\hat{\mu}_n h(w) \in L^{p^*(x)}(\Omega)$ and for $v \in W_0^{1,p(x)}(\Omega)$ one has

$$\begin{aligned} \left|\int_\Omega A\hat{\mu}_n h(w) v dx\right| &\leq \left(\frac{1}{(p^-)^*} + \frac{1}{(p^{-*})'}\right)||A\hat{\mu}_n h(w)||_{p^*(x)}||v||_{(p^*(x))'} \\ &\leq C||A\hat{\mu}_n||_{p^*(x)}|h|_\infty||v||_{p^*(x)} \leq C'||\nabla v||_{p(x)} \end{aligned}$$

where $C,\ C'$ are constants that do not depend on w (note that $|h|_\infty$ denotes the $L^\infty(\mathbb{R})$-norm *of* h) and n is fixed here. Now we treat the case when $r(x) = p(x)$, we have $W_0^{1,p(x)}(\Omega) \subseteq L^{q(x)}(\Omega)\ \forall q$ and the inequality above reads

$$\left|\int_\Omega A\hat{\mu}_n h(w) v dx\right| \leq (\frac{1}{p^-} + \frac{1}{(p')^-})||A\hat{\mu}_n h(w)||_{p(x)}||v||_{p'(x)}$$
$$\leq C||A\hat{\mu}_n||_{p(x)}|h|_\infty||v||_{p'(x)}\ \leq C'||\nabla v||_{p(x)}.$$

Consequently for both cases the equation in the right hand side of (5.8) gives rise to an element in $W^{-1,p'(x)}(\Omega)$ and guarantees that a unique solution u exists. Moreover, by setting $v = u$ in (5.8) we easily obtain, with certain constants C and C', under the condition $p(x) < N$ that

$$\alpha||\nabla u||_{p(x)}^{p(x)} \leq ||A\hat{\mu}_n h(w)||_{p^*(x)}||u||_{(p^*(x))'} \leq C||A\hat{\mu}_n||_{p^*(x)}|h|_\infty||u||_{r(x)} \leq C'||u||_{r(x)}$$

and when $r(x) = p(x)$

$$\alpha||\nabla u||_{p(x)}^{p(x)} \leq ||A\hat{\mu}_n h(w)||_{p'(x)}||u||_{p(x)} \leq C||A\hat{\mu}_n||_{p'(x)}|h|_\infty||u||_{r(x)} \leq C'||u||_{r(x)}$$

with C and C' are constants regardless of w. By applying the Sobolev space embedding result again, we conclude that $||\nabla u||_{p(x)}$ and $||u||_{r(x)}$ are bounded by another constant C independent of w. As a result $S_n : B \mapsto B$ with B is the ball defined as follows

$$B = \{v \in L^{r(x)}(\Omega),\ ||v||_{r(x)} \leq C\}.$$

Additionally, $S_n(B)$ is relatively compact in B. If we can show the continuity of S_n in $L^{r(x)}(\Omega)$, then by the Schauder fixed point theorem the existence of u_n fixed point of S_n will follow. When $w_p \to w$ in $L^{r(x)}(\Omega)$ then for a subsequence $w_p \to w$ almost everywhere in Ω and according to the Lebesgue theorem (remember that h is bounded and continuous and n is fixed)

$$A\hat{\mu}_n h(w_p) \to A\hat{\mu}_n h(w) \text{ in } L^{p^*(x)}(\Omega) \text{ if } p < N,\ (\text{in } L^{p'(x)}(\Omega) \text{ if } r(x) = p(x)).$$

But $L^{p^*(x)}(\Omega) \subseteq L^{(p^*(x))'}, (L^{p'(x)}(\Omega)$ if $r(x) = p(x))$, is continuously imbedded in $W^{-1,p'(x)}(\Omega)$ due to (3.6). To observe this, particularly when $p(x) < N$ for f belonging to $L^{p^*(x)}(\Omega)$, from (5.6)

$$\left|\int_\Omega f v dx\right| \leq C||f||_{(p^*(x))'}||v||_{p^*(x)} \leq C'||f||_{p^*(x)}||\nabla v||_{p(x)}$$

for some constants C, C'. Applying Lemma (4.4), we have

$$S_n(w_p) \rightharpoonup S_n(w)$$

in $W_0^{1,p(x)}(\Omega)$ and strongly in $L^{r(x)}(\Omega)$ thanks to the Sobolev embedding's compactness and the uniquely determined limit. Thus, the existence of a solution to (5.4) is established. Letting $v = u_n$ in (5.4) then employing (3.1), (3.3), we conclude that

$$\alpha\int_\Omega |\nabla u_n|^{p(x)}dx \leq \int_\Omega a(x,\nabla\hat{\mu}_n)\cdot\nabla(h(u_n)u_n)dx$$
$$\leq \int_\Omega (v(x) + \beta|\nabla\hat{\mu}_n|^{p(x)-1})C_\infty|\nabla u_n|dx.$$

(C_∞ represent the Lipschitz constant of $sh(s)$. As $\hat{\mu}_n$ is bounded in $W_0^{1,p(x)}(\Omega)$, One can easily verify that u_n is still bounded in $W_0^{1,p(x)}(\Omega)$ and for a subsequence, there is some u in $W_0^{1,p(x)}(\Omega)$ when $n \to \infty$

$$u_n \rightharpoonup u \ in \ W_0^{1,p(x)}(\Omega), \ u_n \to u \ in \ L^p(\Omega), \ u_n \to u \ a.e \ in \ \Omega. \tag{5.9}$$

It remains to take the limit in

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla v dx = \int_\Omega a(x, \nabla \hat{\mu}_n) \cdot \nabla(h(u_n)v) dx \ \forall v \in W_0^{1,p(x)}(\Omega).$$

We introduce the truncation function at the level $k > 0$, which means $T_k(w) = (-k) \vee w \wedge k$ where $\vee$ refers to the maximum value and $\wedge$ the minimum value between two numbers. Now let $v = T_k(v - u_n)$, it results that

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla T_k(v - u_n) dx = \int_\Omega a(x, \nabla \hat{\mu}_n) \cdot \nabla(h(u_n)T_k(v - u_n)) dx$$

$$\forall v \in W_0^{1,p(x)}(\Omega).$$

By the monotonicity assumption of a, we have

$$\int_\Omega a(x, \ \nabla v) \cdot \nabla T_k(v - u_n) dx \geq \int_\Omega a(x, \ \nabla \hat{\mu}_n) \cdot \nabla(h(u_n)T_k(v - u_n)) dx \tag{5.10}$$

$$\forall v \in W_0^{1,p(x)}(\Omega).$$

Evidently, $T_k(v - u_n)$ is bounded in $W_0^{1,p(x)}(\Omega)$ independent of n and converging to a subsequence to a certain $\eta \in W_0^{1,p(x)}(\Omega)$ weakly in $W_0^{1,p(x)}(\Omega)$. Based on (5.9), $\eta = T_k(v - u)$ and as a result

$$T_k(v - u_n) \rightharpoonup T_k(v - u) \text{ in } W_0^{1,p(x)}(\Omega). \tag{5.11}$$

One has

$$\nabla(h(u_n)T_k(v - u_n)) = h'(u_n)T_k(v - u_n)\nabla u_n + h(u_n)\nabla T_k(v - u_n).$$

Then

$$|\nabla(h(u_n)T_k(v - u_n))| \leq h'_\infty k|\nabla u_n| + |h|_\infty |\nabla T_k(v - u_n)|$$

recall that h'_∞ denotes the Lipschitz constant of h. Thus, $h(u_n)T_k(v - u_n)$ is also bounded in $W_0^{1,p(x)}(\Omega)$ independently of n. There exists f in $W_0^{1,p(x)}(\Omega)$ such that $h(u_n)T_k(v - u_n) \rightharpoonup f$ in $W_0^{1,p(x)}(\Omega)$, and by virtue of Lebesgue's theorem $h(u_n)T_k(v - u_n) \to h(u)(v - u)_k$ in $L^{p(x)}(\Omega)$. In the distributional sense both convergence also involved, it implies that $f = h(u_n)(v - u)_k$ and

$$h(u_n)T_k(v - u_n) \rightharpoonup h(u)T_k(v - u) \text{ in } W_0^{1,p(x)}(\Omega).$$

Since $\hat{\mu}_n \to \hat{\mu}$ in $W_0^{1,p(x)}(\Omega)$, using Proposition (4.2), it is possible to pass to the limit in (5.10), to obtain that

$$\int_\Omega a(x, \nabla v) \cdot \nabla T_k(v - u) dx \geq \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla(h(u)T_k(v - u)_k) dx \ \forall v \in W_0^{1,p(x)}(\Omega).$$

Within this inequality, we take $v = u + tw$ where $w \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, we obtain

$$\int_\Omega a(x, \nabla(u + tw)) \cdot T_k(tw)dx \geq \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla(h(u)T_k(tw))dx$$

$$\forall w \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega).$$

If t is small, we have $T_k(tw) = tw$ and then, we get

$$\int_\Omega a(x,\ \nabla(u + tw)) \cdot \nabla w dx \geq \int_\Omega a(x,\ \nabla\hat{\mu}) \cdot \nabla(h(u)w)dx$$

$$\forall w \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega).$$

Letting $t \to 0$, so

$\displaystyle\int_\Omega a(x, \nabla u) \cdot \nabla w dx \geq \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla(h(u)w)dx\ \forall w \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. By changing w into $-w$, the existence of a solution u to (5.3) follows. □

6. MAIN RESULT

In this section, we substitute h with its value, which means that $h = \frac{1}{u^\gamma ln^\lambda(1+u)}$, γ, λ are positive constants.

Theorem 6.1. *Assume that $\mu \in W^{-1,p'(x)}(\Omega)$ is a nonnegative bounded measure, A is an operator satisfying (2), (4), (13), and (14), and there exists a function $\mathcal{K}$ such that*

$$\exists \mathcal{K} : \mathbb{R}^+ \to \mathbb{R}^+ \text{ with } \frac{1}{s^\gamma \ln^\lambda(1+s)} \leq \mathcal{K}(s) \text{ and such that } (s\mathcal{K}(s)) \in W^{1,\infty}(\mathbb{R}^+). \tag{6.1}$$

Thus, a solution u exists for

$$\begin{cases} u \in W_0^{1,p(x)}(\Omega), \quad u \geq 0, \\ \langle Au, v\rangle = \langle A\hat{\mu}, \frac{v}{u^\gamma \ln^\lambda(1+u)}\rangle \quad \forall v \in C_c^1(\Omega). \end{cases} \tag{6.2}$$

Remark 6.2. In Theorem 6.1, we assume $\nu = 0$ in condition (3.3) for the sake of simplifying the calculations.

Proof. of Theorem 6.1.
We begin by selecting n sufficiently large, specifically $n \geq n_0$, where n_0 is chosen such that $h(n_0) = \frac{1}{n_0^\gamma \ln^\lambda(1+n_0)} < n_0$. Next, we introduce a function h_n as follows: □

$$h_n(s) = \begin{cases} T_n\left(\frac{1}{s^\gamma \ln^\lambda(1+s)}\right) & \text{for } 0 \leq s \leq n, \\ min\left[\left(\frac{1}{n^\gamma \ln^\lambda(1+n)}\right)(n+1-s), T_n\left(\frac{1}{s^\gamma \ln^\lambda(1+s)}\right)\right] & \text{for } n \leq s \leq n+1, \\ 0 & \text{for } s \geq n+1. \end{cases} \tag{6.3}$$

Thus, for each n, we have

$$h_n(s) \leq h_{n+1}(s) \leq h(s)\ \forall s > 0. \tag{6.4}$$

Moreover, h_n satisfies the conditions of Theorem 5.1, therefore, there exists u_n that is a solution to

$$\begin{cases} u_n \in W_0^{1,p(x)}(\Omega), \ u_n \geq 0, \\ \langle Au_n, \ v\rangle = \langle A\hat{\mu}, h_n(u_n)v\rangle \ \forall v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{6.5}$$

For $v = u_n \wedge k$ in (6.5) with $k > 0$, we obtain

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla(u_n \wedge k)dx = \langle A\hat{\mu}, h_n(u_n)(u_n \wedge k)\rangle. \tag{6.6}$$

Furthermore,

$$\begin{aligned} \alpha \int_\Omega |\nabla(u_n \wedge k)|^{p(x)} &\leq \int_\Omega a(x, \nabla(u_n \wedge k)) \cdot \nabla(u_n \wedge k)dx \\ &= \int_\Omega a(x, \nabla u_n) \cdot \nabla(u_n \wedge k)dx. \end{aligned} \tag{6.7}$$

Hence, we conclude that

$$\begin{aligned} \langle A\hat{\mu}, h_n(u_n)(u_n \wedge k)\rangle &= \langle A\hat{\mu}, \{h_n(u_n) - h_n(u_n \wedge k) + h_n(u_n \wedge k)\}(u_n \wedge k)\rangle \\ &\leq \langle A\hat{\mu}, h_n(u_n \wedge k)(u_n \wedge k)\rangle \end{aligned} \tag{6.8}$$

since $\{h_n(u_n)-h_n(u_n\wedge k)\}(u_n\wedge k) \leq 0$ and μ is a nonnegative measure. Combining (6.6), (6.8), we get by (6.1)

$$\alpha \int_\Omega |\nabla(u_n \wedge k)|^p \leq \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla\{\mathcal{K}(u_n \wedge k)(u_n \wedge k)\}dx. \tag{6.9}$$

Since the function $s\mathcal{K}(s)$ belongs to $W^{1,\infty}(\mathbb{K}^+)$, it follows that it is a Lipschitz continuous function. Therefore, there exists some function $f \in L^\infty(\mathbb{R})$ and a constant $C_\mathcal{K}$ such that

$$s\mathcal{K}(s) = \int_0^s f(\xi)d\xi + C_\mathcal{K}.$$

As a result, we have that $\mathcal{K}(u_n \wedge k)(u_n \wedge k) - C_\mathcal{K} \in W_0^{1,p}(\Omega)$. and from (6.9), we obtain

$$\begin{aligned} \alpha \int_\Omega |\nabla(u_n \wedge k)|^{p(x)} &\leq \langle \mu, \ \mathcal{K}(u_n \wedge k)(u_n \wedge k) - C_\mathcal{K} + C_\mathcal{K}\rangle \\ &= \int_\Omega a(x, \ \nabla\hat{\mu}) \cdot \nabla\{\mathcal{K}(u_n \wedge k)(u_n \wedge k)\} - C_\mathcal{K} \ + \langle \mu, C_\mathcal{K}\rangle \\ &\leq \beta||f||_\infty \int_\Omega |\nabla\hat{\mu}|^{p(x)-1}|\nabla(u_n \wedge k)|dx + \langle \mu, C_\mathcal{K}\rangle \\ &\leq \beta||f||_\infty||\nabla\hat{\mu}||_{p(x)}^{p(x)-1} \left(\int_\Omega |\nabla(u_n \wedge k)|^{p(x)}dx\right)^{\frac{1}{p(x)}} + \langle \mu, C_\mathcal{K}\rangle \end{aligned}$$

Therefore, for some constant C that is independent of n and k, we derive

$$\int_\Omega |\nabla(u_n \wedge k)|^{p(x)}dx \leq C.$$

Taking the limit as $k \to \infty$, we infer

$$\int_\Omega |\nabla u_n|^{p(x)} dx \leq C.$$

Hence, u_n is remains bounded in $W_0^{1,p(x)}(\Omega)$, and up to a subsequence, there exists some $u \in W_0^{1,p(x)}(\Omega)$ such that

$$u_n \rightharpoonup u \ in \ W_0^{1,p(x)}(\Omega), \ u_n \to u \ in \ L^{p(x)}(\Omega) \ a.e. \ in \ \Omega.$$

Our objective is to demonstrate that u is a solution to (6.2).

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla v dx = \langle A\hat{\mu}, \frac{v}{u_n^\gamma \ln^\lambda(1+u_n)}\rangle \ \forall v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega).$$

Choosing $v = T_k(v - u_n)$, we obtain that

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla T_k(v - u_n) dx = \langle A\hat{\mu}, \ \frac{T_k(v-u_n)}{u_n^\gamma \ln^\lambda(1+u_n)}\rangle \ \forall v \in W_0^{1,p(x)}(\Omega).$$

It is evident from the monotonicity of $h_n(u_n)$ and (6.4) that u_n is increasing with respect to n, which implies that

$$u_{n+1} \geq u_n \geq u_{n_0}, \ \forall n \geq n_0. \tag{6.10}$$

Additionally, $u \geq u_n$ for all $n \geq n_0$, which also implies $T_k(v - u_n) \geq T_k(v-u)_k$ for any v. Since $A\hat{\mu}$ is positive and $h_n(u_n)$, one can deduce that

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla T_k(v - u_n) dx \geq \langle A\hat{\mu}, h_n(u_n) T_k(v-u)\rangle \ \forall v \in W_0^{1,p(x)}(\Omega).$$

Moreover, by the monotonicity assumption of a, it follows that

$$\int_\Omega a(x, \ \nabla v) \cdot \nabla T_k(v - u_n) dx \geq \int_\Omega a(x, \ \nabla \hat{\mu}) \cdot \nabla \{h_n(u_n) T_k(v-u)\} dx \tag{6.11}$$

$$\forall v \in W_0^{1,p(x)}(\Omega).$$

By the definition of u_{n_0} and applying the maximum principle (see [20]), it follows that

$$-\text{div}\{a(x, \nabla u_{n_0})\} \geq 0, \ -\text{div}\{a(x, \nabla u_{n_0})\} \not\equiv 0.$$

Consequently, for every subdomain $\omega \subseteq \Omega$ and a given positive constant c_ω, the following holds:

$$u \geq u_n \geq u_{n_0} \geq c_\omega \text{ a.e. } in \ \omega. \tag{6.12}$$

Then, take $w \in C_c^1(\Omega)$ such that its support is included in ω. When $t > 0$, replacing in (6.11) v by $v = u + tw$, we conclude:

$$\int_\Omega a(x, \nabla(u + tw)) \cdot \nabla T_k(u + tw - u_n) dx \tag{6.13}$$

$$\geq \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla \{h_n(u_n) T_k(tw)\} dx.$$

As $n \to \infty$, one obtains $\nabla T_k(u + tw - u_n) \rightharpoonup \nabla T_k(tw)$ in $L^{p(x)}(\Omega)$.

Furthermore, from the definition of h_n, consequently

$$0 \leq -h_n' \leq C \ on \ [c_\omega, \ \infty)$$

where C not depending on n. Hence

$$\nabla\{h_n(u_n)T_k(tw)\} = h_n(u_n)\nabla T_k(tw) + h'_n(u_n)T_k(tw)\nabla u_n$$

is bounded independently of n in $L^{p(x)}(\Omega)$. Therefore, it weakly converges in $L^{p(x)}(\Omega)$ to a subsequence. However, the limit must necessarily be $\nabla\{\frac{T_k(tw)}{u^\gamma ln^\lambda(1+u)}\}$ due to the convergence of u_n. Recall that as $n \to \infty$, we have $h_n \to \frac{1}{u^\gamma ln^\lambda(1+u)}$. By passing to the limit in (6.13), we obtain

$$\int_\Omega a(x, \nabla(u+tw)) \cdot \nabla T_k(tw)dx \geq \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla\left(\frac{T_k(tw)}{u^\gamma ln^\lambda(1+u)}\right)dx.$$

When t is sufficiently small, $T_k(tw) = tw$. Allowing $t \to 0$, this leads to

$$\int_\Omega a(x, \nabla u) \cdot \nabla w dx \geq \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla\left(\frac{w}{u^\gamma ln^\lambda(1+u)}\right)dx.$$

This holds true for every w. Consequently, by replacing w with $-w$, we can conclude that u satisfies equation (6.2). Thus, the proof of the theorem is finalized.

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