A SIMPLE PROOF OF THE EXISTENCE OF THE DICKMAN FUNCTION

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Abstract. In this article we give a simple proof of the existence of the Dickman’s function related with smooth numbers. We only use the concept of integral of a continuous function.

1. Introduction and Main Results

Let $0 < \alpha \leq 1$ be a fixed real number and consider the number of numbers not exceeding $x$ such that their greatest prime factor does not exceed $x^{\alpha}$. These numbers are called smooth numbers. We denote the number of these numbers $N(x, \alpha)$. It is well-known [1] that

$$N(x, \alpha) = \phi(\alpha) x + O \left( \frac{x}{\log x} \right), \quad (1.1)$$

where $\phi(\alpha)$ is called Dickman’s function. This function of $\alpha$ is positive, strictly increasing and continuous on the interval $(0, 1]$. Clearly $\phi(1) = 1$. The proof of (1) use the Riemann-Stieltjes integral and the prime number theorem is not necessary (see [1]).

In this article using only the concept of integral of a continuous function we give a simple proof of the weaker result

$$N(x, \alpha) = \phi(\alpha) x + o(x).$$

In this article $p$ denotes a positive prime and $[\cdot]$ denotes the integer-part function.

We shall need the following well-known theorems.

**Theorem 1.1.** The following asymptotic formula holds

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + M + O \left( \frac{1}{\log x} \right),$$

where $M$ is Mertens’s constant.
Theorem 1.2. Let \( \pi(x) \) be the prime counting function. The following asymptotic formula holds

\[
\pi(x) = O\left(\frac{x}{\log x}\right).
\]

Theorem 1.3. Let \( f(x) \) be a continuous function on the closed interval \([a, b]\), then there exists \( c \in [a, b] \) such that

\[
\int_a^b f(x) \, dx = f(c)(b - a).
\]

Theorem 1.4. Let \( f(x) \) be a continuous function on the closed interval \([a, b]\). Given any number \( \epsilon > 0 \) there exists \( \delta \in \epsilon \) such that if \( |x' - x''| < \delta \), then \( |f(x') - f(x'')| < \epsilon \). That is, \( f(x) \) is uniformly continuous on \([a, b]\).

Now, we can prove our main theorem.

Theorem 1.5. If \( 0 < \alpha \leq 1 \) then the following asymptotic formula holds

\[
N(\alpha, x) = \phi(\alpha)x + o(x) = \phi(\alpha)x + f(x)x,
\]

where \( \lim_{x \to \infty} f(x) = 0 \) and \( \phi(\alpha) \) is positive, strictly increasing and continuous on the interval \((0, 1]\). Note that \( f(x) \) depends on \( \alpha \). Besides

\[
\phi(\alpha) = 1 - \int_{1/\alpha}^1 \phi\left(\frac{x}{1-x}\right) \frac{1}{x} \, dx,
\]

where we put \( \phi\left(\frac{x}{1-x}\right) = 1 \) if \( x \in [1/2, 1] \). Therefore if \( 1/2 < \alpha < 1 \) then \( \phi(\alpha) = 1 - \int_{1/\alpha}^1 \frac{1}{x} \, dx = 1 + \log \alpha \).

Proof. Let us consider the multiples of \( p \) not exceeding \( x \). Namely

\[
\left\{p, 1, p, 2, \ldots, p \left\lfloor \frac{x}{p} \right\rfloor\right\}
\]

The number of multiples of \( p \) not exceeding \( x \) such that \( p \) is their greatest prime factor we denote \( B(x, p) \). Therefore \( B(x, p) \leq \left\lfloor \frac{x}{p} \right\rfloor \). Let \( 1/2 < \alpha < 1 \) be, then \( x^{1-\alpha} \leq x^\alpha \). On the other hand if \( p > x^\alpha \) then we have \( \frac{x}{p} < x^{1-\alpha} \). Therefore

\[
\left\lfloor \frac{x}{p} \right\rfloor \leq \frac{x}{p} < x^{1-\alpha} \leq x^\alpha < p.
\]

That is \( \left\lfloor \frac{x}{p} \right\rfloor < p \), and consequently

\[
B(x, p) = \left\lfloor \frac{x}{p} \right\rfloor.
\]

We have (see (1.5))

\[
[x] - N(\alpha, x) = \sum_{x^\alpha < p \leq x} B(x, p) = \sum_{x^\alpha < p \leq x} \left\lfloor \frac{x}{p} \right\rfloor
\]

\[
= x \sum_{x^\alpha < p \leq x} \frac{1}{p} - \sum_{x^\alpha < p \leq x} \left( \frac{x}{p} - \left\lfloor \frac{x}{p} \right\rfloor \right).
\]
Now (see Theorem 1.2)

\[
0 \leq \sum_{x^n < p \leq x} \left( \frac{x}{p} - \left\lfloor \frac{x}{p} \right\rfloor \right) \leq \sum_{p \leq x} 1 = \pi(x) < e^{\frac{x}{\log x}}.
\]

That is

\[
\sum_{x^n < p \leq x} \left( \frac{x}{p} - \left\lfloor \frac{x}{p} \right\rfloor \right) = O\left( \frac{x}{\log x} \right). \tag{1.7}
\]

On the other hand, we have (see Theorem 1.1)

\[
\sum_{x^n < p \leq x} \frac{1}{p} = \sum_{p \leq x^n} \frac{1}{p} + \sum_{p \leq x^n} \frac{1}{p} = -\log \alpha + O\left( \frac{1}{\log x} \right). \tag{1.8}
\]

Substituting (1.7) and (1.8) into (1.6) we obtain

\[
\lfloor x \rfloor - N(\alpha, x) = -x \log \alpha + O\left( \frac{x}{\log x} \right).
\]

Hence

\[
N(\alpha, x) = (1 + \log \alpha)x + O\left( \frac{x}{\log x} \right).
\]

Therefore the theorem is true if \( \alpha \in [1/2, 1] \).

Suppose that the theorem is true if \( \alpha \in [1/j, 1] \), where \( j \) is a positive integer. That is, we have in this interval

\[
N(\alpha, x) = \phi(\alpha)x + o(x) = \left( 1 - \int_{\alpha}^{1} \phi \left( \frac{x}{1-x} \right) \frac{1}{x} dx \right) x + o(x), \tag{1.9}
\]

where the function

\[
\phi(\alpha) = 1 - \int_{\alpha}^{1} \phi \left( \frac{x}{1-x} \right) \frac{1}{x} dx \tag{1.10}
\]

is positive, strictly increasing and continuous on the interval \([1/j, 1]\).

Suppose that \( \frac{1}{j+1} \leq \alpha < \frac{1}{j} \). Therefore we have

\[
\lfloor x \rfloor - N(\alpha, x) = \sum_{x^n < p \leq x} B(x, p) = \sum_{x^n < p \leq x^{1/j}} B(x, p) + \sum_{x^{1/j} < p \leq x} B(x, p), \tag{1.11}
\]

where (see (1.9))

\[
\sum_{x^{1/j} < p \leq x} B(x, p) = \left( \int_{1/j}^{1} \phi \left( \frac{x}{1-x} \right) \frac{1}{x} dx \right) x + o(x). \tag{1.12}
\]

Note that (see (1.10)) on the interval \([\alpha, 1/j]\) the function \( \phi \left( \frac{x}{1-x} \right) \) is positive, strictly increasing and uniformly continuous (see Theorem 1.4). Note that

\[
\phi \left( \frac{1/(j+1)}{1 -(1/(j+1))} \right) = \phi(1/j).
\]

Consequently the function \( \phi \left( \frac{x}{1-x} \right) \frac{1}{x} \) is positive and uniformly continuous on the interval \([\alpha, 1/j]\). Therefore if we consider the positive number \( \epsilon \) then there exists
a partition of the interval \([\alpha, 1/j]\), namely \(\alpha = \beta_1 < \beta_2 < \cdots < \beta_k = 1/j\), such that

\[
\phi\left(\frac{\beta_{i+1}}{1 - \beta_{i+1}}\right) - \phi\left(\frac{\beta_i}{1 - \beta_i}\right) < \frac{\epsilon}{j + 1} \quad (i = 1, 2, \ldots, k - 1)
\]  

and such that if \(x', x'' \in [\beta_i, \beta_{i+1}]\) \((i = 1, 2, \ldots, k - 1)\) then we have

\[
\left| \phi\left(\frac{x'}{1 - x'}\right) \frac{1}{x'} - \phi\left(\frac{x''}{1 - x''}\right) \frac{1}{x''} \right| < \frac{\epsilon}{j + 1}
\]  

We have (see (1.11))

\[
\sum_{x^\alpha < p \leq x^{1/j}} B(x, p) = \sum_{i=1}^{k-1} \sum_{x^{\beta_i} < p \leq x^{\beta_{i+1}}} B(x, p).
\]  

The inequality \(x^{\beta_i} < p \leq x^{\beta_{i+1}}\) implies the inequality \(x^{1-\beta_{i+1}} \leq \frac{p}{x^{\beta_i}} < x^{1-\beta_i}\). Therefore we have

\[
p \leq x^{\beta_{i+1}} = \left(x^{1-\beta_{i+1}}\right)^{\frac{\beta_i}{1-\beta_{i+1}}} \leq \left(\frac{x}{p}\right)^{\frac{\beta_i}{1-\beta_{i+1}}}
\]  

and

\[
p > x^{\beta_i} = \left(x^{1-\beta_i}\right)^{\frac{\beta_i}{1-\beta_i}} > \left(\frac{x}{p}\right)^{\frac{\beta_i}{1-\beta_i}}.
\]  

Let us consider (see (1.4)) the set \(\{1, 2, 3, \ldots, \left\lfloor \frac{x}{p} \right\rfloor\}\). Let \(C(x, p)\) be the number of numbers in this set such that their greatest prime factor does not exceed \(p\). Consequently \(B(x, p) = C(x, p)\). Equations (1.2) and (1.16) give

\[
B(x, p) = C(x, p) \leq N\left(x, \frac{\beta_{i+1}}{1 - \beta_{i+1}}\right) = \phi\left(\frac{\beta_{i+1}}{1 - \beta_{i+1}}\right) \frac{x}{p} + f\left(\frac{x}{p}\right) \frac{x}{p}
\]

\[
= \phi\left(\frac{\beta_{i+1}}{1 - \beta_{i+1}}\right) \frac{x}{p} + \frac{\epsilon}{j + 1} \frac{x}{p} + f\left(\frac{x}{p}\right) \frac{x}{p}
\]

\[
\leq \phi\left(\frac{\beta_{i+1}}{1 - \beta_{i+1}}\right) \frac{x}{p} + \frac{\epsilon}{j + 1} \frac{x}{p}.
\]  

Equations (1.2) and (1.17) give

\[
B(x, p) = C(x, p) \geq N\left(x, \frac{\beta_i}{1 - \beta_i}\right) = \phi\left(\frac{\beta_i}{1 - \beta_i}\right) \frac{x}{p} + f\left(\frac{x}{p}\right) \frac{x}{p}
\]

\[
= \phi\left(\frac{\beta_i}{1 - \beta_i}\right) \frac{x}{p} - \frac{\epsilon}{j + 1} \frac{x}{p} + f\left(\frac{x}{p}\right) \frac{x}{p}
\]

\[
\geq \phi\left(\frac{\beta_i}{1 - \beta_i}\right) \frac{x}{p} - \frac{\epsilon}{j + 1} \frac{x}{p}.
\]  

Theorem 1.1 gives

\[
\sum_{x^{\beta_i} < p \leq x^{\beta_{i+1}}} \frac{1}{p} = \sum_{p \leq x^{\beta_{i+1}}} \frac{1}{p} - \sum_{p \leq x^{\beta_i}} \frac{1}{p} = (\log \beta_{i+1} - \log \beta_i) + o(1).
\]  

\(\sum_{x^{\beta_i} < p \leq x^{\beta_{i+1}}} \frac{1}{p} = \sum_{p \leq x^{\beta_{i+1}}} \frac{1}{p} - \sum_{p \leq x^{\beta_i}} \frac{1}{p} = (\log \beta_{i+1} - \log \beta_i) + o(1).
\]  

(1.20)
Equations (1.18), (1.20), (1.13) and the mean value theorem give

\[ \sum_{x^{\beta_i} < p \leq x^{\beta_{i+1}}} B(x, p) \leq \left( \phi \left( \frac{\beta_{i+1}}{1 - \beta_{i+1}} \right) + \frac{\epsilon}{j + 1} \right) x \sum_{x^{\beta_i} < p \leq x^{\beta_{i+1}}} \frac{1}{p} \]

\[ = \left( \phi \left( \frac{\beta_{i+1}}{1 - \beta_{i+1}} \right) + \frac{\epsilon}{j + 1} \right) \left( \log \beta_{i+1} - \log \beta_i \right) x + o(x) \]

\[ \leq \left( \phi \left( \frac{\beta_{i+1}}{1 - \beta_{i+1}} \right) + \frac{\epsilon}{j + 1} \right) \frac{1}{\beta_i} (\beta_{i+1} - \beta_i) x + o(x) \]

\[ \leq \left( \phi \left( \frac{\beta_i}{1 - \beta_i} \right) + \frac{2 \epsilon}{j + 1} \right) \frac{1}{\beta_i} (\beta_{i+1} - \beta_i) x + o(x) \]

\[ = \phi \left( \frac{\beta_i}{1 - \beta_i} \right) \frac{1}{\beta_i} (\beta_{i+1} - \beta_i) x + 2 \frac{\epsilon}{j + 1} \frac{1}{\beta_i} (\beta_{i+1} - \beta_i) x + o(x). \tag{1.21} \]

Equations (1.19), (1.20), (1.13) and the mean value theorem give

\[ \sum_{x^{\beta_i} < p \leq x^{\beta_{i+1}}} B(x, p) \geq \left( \phi \left( \frac{\beta_i}{1 - \beta_i} \right) - \frac{\epsilon}{j + 1} \right) x \sum_{x^{\beta_i} < p \leq x^{\beta_{i+1}}} \frac{1}{p} \]

\[ = \left( \phi \left( \frac{\beta_i}{1 - \beta_i} \right) - \frac{\epsilon}{j + 1} \right) \left( \log \beta_i - \log \beta_{i+1} \right) x + o(x) \]

\[ \geq \left( \phi \left( \frac{\beta_i}{1 - \beta_i} \right) - \frac{\epsilon}{j + 1} \right) \frac{1}{\beta_{i+1}} (\beta_{i+1} - \beta_i) x + o(x) \]

\[ \geq \left( \phi \left( \frac{\beta_{i+1}}{1 - \beta_{i+1}} \right) - 2 \frac{\epsilon}{j + 1} \right) \frac{1}{\beta_i} (\beta_{i+1} - \beta_i) x + o(x) \]

\[ = \phi \left( \frac{\beta_{i+1}}{1 - \beta_{i+1}} \right) \frac{1}{\beta_{i+1}} (\beta_{i+1} - \beta_i) x - 2 \frac{\epsilon}{j + 1} \frac{1}{\beta_{i+1}} (\beta_{i+1} - \beta_i) x + o(x). \tag{1.22} \]

Equations (1.15) and (1.21) give

\[ \sum_{x^{\beta_i} < p \leq x^{\beta_{i+1}}} B(x, p) \leq \left( \sum_{i=1}^{k-1} \phi \left( \frac{\beta_i}{1 - \beta_i} \right) \frac{1}{\beta_i} (\beta_{i+1} - \beta_i) \right) x \]

\[ + \frac{2 \epsilon}{j + 1} x \sum_{i=1}^{k-1} \frac{1}{\beta_i} (\beta_{i+1} - \beta_i) + o(x). \tag{1.23} \]

Equations (1.15) and (1.22) give

\[ \sum_{x^{\beta_i} < p \leq x^{\beta_{i+1}}} B(x, p) \geq \left( \sum_{i=1}^{k-1} \phi \left( \frac{\beta_{i+1}}{1 - \beta_{i+1}} \right) \frac{1}{\beta_{i+1}} (\beta_{i+1} - \beta_i) \right) x \]

\[ - \frac{2 \epsilon}{j + 1} x \sum_{i=1}^{k-1} \frac{1}{\beta_{i+1}} (\beta_{i+1} - \beta_i) + o(x). \tag{1.24} \]
We have \( \frac{1}{\beta_i} \leq 1/\alpha \) \((i = 1, 2, \ldots, k)\), since \( \beta_i \geq \alpha \) (see above). On the other hand \((1/\alpha) \leq j + 1\), since \( \alpha \geq 1/(j + 1) \) (see above). Therefore we have
\[
\sum_{i=1}^{k-1} \frac{1}{\beta_i} (\beta_{i+1} - \beta_i) \leq \frac{1}{\alpha} \left( \frac{1}{2} - \alpha \right) \leq \frac{1}{\alpha} \leq j + 1
\] (1.25)
and
\[
\sum_{i=1}^{k-1} \frac{1}{\beta_{i+1}} (\beta_{i+1} - \beta_i) \leq \frac{1}{\alpha} \left( \frac{1}{2} - \alpha \right) \leq \frac{1}{\alpha} \leq j + 1.
\] (1.26)

On the other hand we have (see Theorem 1.3 and (1.14))
\[
\left| \sum_{i=1}^{k-1} \phi \left( \frac{\beta_i}{1 - \beta_i} \right) \frac{1}{\beta_i} (\beta_{i+1} - \beta_i) - \int_{\alpha}^{1/j} \frac{\phi \left( \frac{x}{1-x} \right)}{x} dx \right|
\leq \sum_{i=1}^{k-1} \phi \left( \frac{\beta_i}{1 - \beta_i} \right) \frac{1}{\beta_i} \left( \phi \left( \frac{c_i}{1 - c_i} \right) \frac{1}{c_i} \right) (\beta_{i+1} - \beta_i)
\leq \frac{\epsilon}{j + 1}
\] (1.27)
where \( c_i \in [\beta_i, \beta_{i+1}] \). In the same way we obtain
\[
\left| \sum_{i=1}^{k-1} \phi \left( \frac{\beta_{i+1}}{1 - \beta_{i+1}} \right) \frac{1}{\beta_{i+1}} (\beta_{i+1} - \beta_i) - \int_{\alpha}^{1/j} \frac{\phi \left( \frac{x}{1-x} \right)}{x} dx \right|
\leq \frac{\epsilon}{j + 1}.
\] (1.28)

Equations (1.23), (1.25) and (1.27) give
\[
\sum_{x^\alpha < p \leq x^{1/j}} B(x, p) \leq \left( \int_{\alpha}^{1/j} \frac{\phi \left( \frac{x}{1-x} \right)}{x} dx \right) x + \frac{\epsilon}{j + 1} x + 2 \frac{\epsilon}{j + 1} (j + 1) x
+ o(x)
\] (1.29)

Equations (1.24), (1.26) and (1.28) give
\[
\sum_{x^\alpha < p \leq x^{1/j}} B(x, p) \geq \left( \int_{\alpha}^{1/j} \frac{\phi \left( \frac{x}{1-x} \right)}{x} dx \right) x - \frac{\epsilon}{j + 1} x - 2 \frac{\epsilon}{j + 1} (j + 1) x
+ o(x)
\] (1.30)

Now, \( \epsilon \) is arbitrarily small. Therefore equations (1.29) and (1.30) give
\[
\sum_{x^\alpha < p \leq x^{1/j}} B(x, p) = \left( \int_{\alpha}^{1/j} \frac{\phi \left( \frac{x}{1-x} \right)}{x} dx \right) x + o(x).
\] (1.31)
Equations (1.31), (1.12) and (1.11) give equations (1.9) and (1.10), where these equations are true if $\alpha \in [1/(j+1), 1]$, since the theorem is true if $\alpha \in [1/2, 1]$. Now, $\lim_{j \to \infty} (1/j) = 0$ and consequently equations (1.9) and (1.10) are true for all $\alpha > 0$. The theorem is proved. □

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