Gulf Journal of Mathematics ISSN: 2309-4966
Vol 20, Issue 1 (2025) 35-51 https://doi.org/10.56947/gjom.v20i.2933

ENTROPY SOLUTION FOR A NONLINEAR DEGENERATE ELLIPTIC PROBLEM WITH DIRICHLET-TYPE BOUNDARY CONDITION AND SINGULAR TERM

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ABSTRACT. This paper establishes the existence of entropy solutions for a class of nonlinear elliptic equations governed by a degenerate coercive operator and featuring a singular right-hand side, exemplified by the model case:

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla u-\Theta(u)|^{p-2}\,(\nabla u-\Theta(u))}{(1+|u|)^{\lambda(p-1)}}\right)+|u|^{q-1}\,u=\dfrac{f}{u^{\gamma}} & \text{in } \Omega,\\ u\geq 0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

with Ω is a bounded open domain in $\mathbb{R}^N$ $(N\geq 2)$, $p>1$, $\lambda\geq 0$, $\gamma\geq 0$, $q\geq 1$, f is a non-negative function that belongs to $L^1(\Omega)$ and Θ is assumed to be continuous on the real line $\mathbb{R}$.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, with $N\geq 2$, $p>1$, $\lambda\geq 0$ and $q\geq 1$. This work focuses on the existence of entropy solutions to Dirichlet problems given by::

$$\begin{cases} -\operatorname{div}\left(\dfrac{\Psi(\nabla u-\Theta(u))}{(1+|u|)^{\lambda(p-1)}}\right)+|u|^{q-1}\,u=fh(u) & \text{in } \Omega,\\ u\geq 0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\Psi(\xi)=|\xi|^{p-2}\,\xi \quad \forall\xi\in\mathbb{R}^N,$$

Θ is a continuous function defined from $\mathbb{R}$ to $\mathbb{R}^N$, h is a continuous function on $[0,\infty)$, which may exhibit a singularity at the origin and the datum f is non-negative and belongs to $L^1(\Omega)$.

In recent years, considerable attention has been devoted to the study of partial differential equations and variational problems, especially the singular problems, has emerged as a new research field that has attracted significant interest from

Date: Received: May 6, 2025; Accepted: May 18, 2025.

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2020 *Mathematics Subject Classification.* Primary 35J60; Secondary 35J70; 35J75.

Key words and phrases. Degenerate coercivity, Entropy solutions, Existence results, Singular non-linearity.

many mathematicians. The primary motivation for exploring these problems stems from their diverse applications, particularly in areas such as image processing (e.g., noise removal, edge detection, and image restoration) also for their relation to the theory of non-Newtonian fluids, boundary layer phenomena in viscous fluids, and chemical heterogeneity(for instance, refer to [7], [10]). As example applications of problem (1.1), we present the following two models:

Model 1: Filtration in a porous medium

The filtration process of fluids in porous media is governed by the equation

$$\frac{\partial c(p)}{\partial t} = \nabla a\left[k\left(c(p)\right)\left(\nabla c(p) + e\right)\right]$$

where p is the unknown pressure, c is the volumetric moisture content, k is the hydraulic conductivity of the porous medium, a is the heterogeneity matrix, and $-e$ represents the direction of gravity.

Model 2: Fluid flow through porous media

This model is represented by the following equation:

$$\frac{\partial \rho}{\partial t} - \operatorname{div}\left(|\nabla\varphi(\rho) - K(\rho)e|^{p-2}\left(\nabla\varphi(\rho) - K(\rho)e\right)\right) = 0$$

where ρ represents the volumetric moisture content, $K(\rho)$ is the hydraulic conductivity, $\varphi(\rho)$ is the hydrostatic potential, and e is the unit vector in the vertical direction.

The main characteristics of the problems like (1.1) lie in the non-coercivity of the principal operator

$$-\operatorname{div}\left(\frac{\Psi(\nabla u - \Theta(u))}{(1+|u|)^{\lambda(p-1)}}\right)$$

when $\lambda > 0$. Additionally, the right-hand side of equation (1.1) can exhibit singularities near zero. These factors prevent the application of standard existence theorems to study problem (1.1), highlighting the need for a deeper analysis.

In the following, we provide an overview of problems that have been studied and are closely related to our problem (1.1). To begin with, let's focus on the case where $\Theta = 0$. In the coercive case (i.e. $\lambda = 0$), much research has been devoted to proving the existence of solutions to problem (1.1). For $p = 2$, $q = 1$ and $h(s) = \frac{1}{s^\gamma}$, existence and regularity results were proven in [5]. For $p > 1$, Olivia [11] have proved the existence and regularity of solution to the problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + g(u) = fh(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

f is a non-negative function and it belongs $L^m(\Omega), m \geq 1$, $h(u) = \frac{1}{u^\gamma}$ with $0 < \delta \leq 1$, while g is a continuous function such that $g(0) = 0$ and its could act as s^q at infinity with $q \geq -1$, and h is continuous function that it possibly blows up at the origin and it is bounded at infinity. In most situations, more precisely, in the non-coercive case (i.e. $\lambda > 0$), the authors in [13] investigated the existence and regularity of solutions to the following degenerate elliptic problem involving a singular nonlinearity:

$$\begin{cases} -\mathrm{div}\,(a(x,u,\nabla u)) + |u|^{s-1}\,u = fh(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

under different hypotheses on f and s.

In contrast, for the case Θ is not identically zero, $\lambda = 0$ and $h(u) = 1$, several authors have proven the existence and uniqueness of solutions of problems like (1.1) in either the framework of elliptic problem in weighted Sobolev spaces (see [6]) or the framework of parabolic problem in weighted Sobolev spaces (see [12]), depending on the assumptions. Note that our paper can be seen as a generalization of [6].

The remainder of this paper is organized as follows. In Section 2, we present the necessary preliminaries and notations. In Section 3, we specify the assumptions concerning the data of our problem, we introduce the definition of an entropy solution for problem (1.1) and we state our main result. Finally, section 4 devoted to the proof of this result which will be divided into four steps.

2. Preliminary results and notations

In this section, we review some standard definitions and properties that will be used throughout the paper. Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain with smooth boundary $\partial\Omega$, $N \geq 2$, and $p > 1$. The following Poincaré inequality, stated in [8] (Lemma 2.2), will be used throughout this work.

Lemma 2.1. *([8]) Let $p > 1$, there exists a positive constant $\mathcal{S}_p$ such that, for each $u \in W_0^{1,p}(\Omega)$,*

$$\|u\|_{L^p(\Omega)} \leq \mathcal{S}_p \|\nabla u\|_{L^p(\Omega)}.$$

Lemma 2.2. *([1]) For each $\xi, \eta \in \mathbb{R}^N$ and $p > 1$, we find*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\,\xi(\xi - \eta).$$

For clarity, we introduce the following notation, which will be used throughout the paper. For any $u : \Omega \to \mathbb{R}$ and $k \geq 0$, we write $\{|u| \leq k\}$ for the set $\{x \in \Omega : |u(x)| \leq k\}$.

For a fixed $k > 0$, let the truncation functions T_k and V_k of real variable defines as follows:

$$T_k(s) = \max\left(-k, \min(k, s)\right),$$

and

$$V_k(s) = \begin{cases} 1 & |s| \leq k, \\ \frac{2k-|s|}{k} & k < |s| < 2k, \\ 0 & |s| \geq 2k. \end{cases}$$

Moreover, we note by $\tau^+ = \max\{\tau; 0\}$ and $\tau^- = -\min\{\tau; 0\}$ the positive and the negative part of τ.

Finally, We mention that the constants C or C_i for $i = 1, 2, \cdots$ used in our paper may change from line to line and its only depend of the data of our problem but it is independent of the indices of the sequences that will be frequently introduced.

3. Statement of the problem and main result

In this work, we assume the following conditions hold:
Ω is a bounded open domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ $(N \geq 2)$, $p > 1$, and $\lambda \geq 0$.
The function $\Theta : \mathbb{R} \longrightarrow \mathbb{R}^N$ is continuous and satisfies

$$\Theta(0) = 0 \quad \text{and} \quad |\Theta(s) - \Theta(s')| \leq \beta |s - s'| \quad \text{for all } s, s' \in \mathbb{R}, \tag{3.1}$$

where β is a real constant such that $0 \leq \beta \leq \dfrac{1}{2\mathcal{S}_p}$ and $\mathcal{S}_p$ is the Poincare constant.

The function $h \in \mathcal{C}^0([0; +\infty[\,; [0; +\infty])$ is a singular sourcing that is finite outside the origin such that

$$\exists c > 0, \gamma \geq 0 \text{ such that} : h(s) \leq \frac{c}{s^\gamma} \quad \text{for all } s \in (0, \infty). \tag{3.2}$$

Finally,

$$f \text{ is a non-negative function that belongs to } L^1(\Omega). \tag{3.3}$$

At this point, we define what is meant by an entropy solution to (1.1) and we will give our principal result.

Definition 3.1. We say that a non-negative measurable function u is an entropy solution to problem (1.1) if

$$T_k(u) \in W_0^{1,p}(\Omega), \tag{3.4}$$

$$|u|^{q-1} u \in L^1(\Omega), \tag{3.5}$$

$$f h(u) T_k(u - \varphi) \in L^1(\Omega), \tag{3.6}$$

and

$$\int_\Omega \frac{\Psi(\nabla u - \Theta(u))}{(1 + |u|)^{\lambda(p-1)}} \cdot \nabla T_k(u - \varphi) dx + \int_\Omega |u|^{q-1} u T_k(u - \varphi) dx \leq \int_\Omega f h(u) T_k(u - \varphi) dx \tag{3.7}$$

for every $k > 0$ and for any $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Our main result is the following theorem:

Theorem 3.2. *Assume that (3.1)-(3.3) hold with $0 \leq \gamma < 1$ and $\lambda < 1 + \dfrac{\gamma}{p-1}$. Then there exists an entropy solution u of problem (1.1).*

4. **Proof of our result**

The proof is divided into four steps.

Step 1: Approximating problems
For $n \in \mathbb{N}$, we consider the following problem

$$\begin{cases} -\text{div}\left(\dfrac{\Psi(\nabla u_n - \Theta(u_n))}{(1+|T_n(u_n)|)^{\lambda(p-1)}}\right) + |u_n|^{q-1} u_n = f_n h_n(u_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $f_n = T_n(f)$ and

$$h_n(s) = \begin{cases} T_n(h(s)) & s \geq 0, \\ \min(n, h(0)) & s < 0. \end{cases}$$

In this approximating problem, we have truncated both the degenerate coercivity and the singularity of the right-hand side. The weak formulation of this problem is given by

$$\int_\Omega \frac{\Psi(\nabla u_n - \Theta(u_n))}{(1+|T_n(u_n)|)^{\lambda(p-1)}} \cdot \nabla\varphi \, dx + \int_\Omega |u_n|^{q-1} u_n \varphi \, dx = \int_\Omega f_n h_n(u_n)\varphi \, dx, \tag{4.2}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Lemma 4.1. *For each $n \in \mathbb{N}$, there exists a non-negative weak solution u_n to the problem (4.1) such that $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Proof. The proof relies on a standard Schauder's fixed point argument. Let $n \in \mathbb{N}$ and fix $v \in L^p(\Omega)$. We set the operator

$$T : L^p(\Omega) \to L^p(\Omega)$$

defined by $T(v) = \varpi$, where ϖ is the unique solution of problem

$$\begin{cases} -\text{div}\left(\dfrac{\Psi(\nabla \varpi - \Theta(\varpi))}{(1+|T_n(\varpi)|)^{\lambda(p-1)}}\right) + |\varpi|^{q-1} \varpi = f_n h_n(v) & \text{in } \Omega, \\ \varpi = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

whose existence is guaranteed from [9, 3]. In particular, since $f_n h_n(v)$ is bounded, we have that $\varpi \in L^\infty(\Omega)$ and there exists a positive constant C_∞ independents of v and ϖ (but possibly depending in n), such that

$$\|\varpi\|_{L^\infty(\Omega)} \leq C_\infty.$$

Now, we show that the map T has an invariant ball, it's relatively compact and continuous in $L^p(\Omega)$. Let us take ϖ as a test function in the weak formulation of (4.3), we get

$$\int_\Omega \frac{\Psi(\nabla \varpi - \Theta(\varpi))}{(1+|T_n(\varpi)|)^{\lambda(p-1)}} \cdot \nabla\varpi \, dx + \int_\Omega |\varpi|^{q+1} \, dx = \int_\Omega f_n h_n(v)\varpi \, dx,$$

using the definitions of f_n and h_n, and dropping the positive term, we get

$$\frac{1}{(1+n)^{\lambda(p-1)}}\int_\Omega \Psi(\nabla\varpi-\Theta(\varpi))\cdot\nabla\varpi\,dx \le n^2\int_\Omega |\varpi|\,dx,$$

now, we estimate the left hand-side of the former inequality by applying Lemma 2.2, as follows

$$\begin{aligned}
\int_\Omega \Psi(\nabla\varpi-\Theta(\varpi))\cdot\nabla\varpi\,dx \;&= \int_\Omega |\nabla\varpi-\Theta(\varpi)|^{p-2}\left(\nabla\varpi-\Theta(\varpi)\right)\cdot\nabla\varpi\,dx\\
&= \int_\Omega |\nabla\varpi-\Theta(\varpi)|^{p-2}\left(\nabla\varpi-\Theta(\varpi)\right)\cdot\left(\nabla\varpi-\Theta(\varpi)+\Theta(\varpi)\right)\,dx\\
&\ge \frac{1}{p}\int_\Omega |\nabla\varpi-\Theta(\varpi)|^p\,dx-\frac{1}{p}\int_\Omega |\Theta(\varpi)|^p\,dx\\
&\ge \frac{1}{p2^{p-1}}\int_\Omega |\nabla\varpi|^p\,dx-\frac{2}{p}\int_\Omega |\Theta(\varpi)|^p\,dx\\
&\ge \frac{1}{p2^p}\int_\Omega |\nabla\varpi|^p\,dx,
\end{aligned}$$

where in the above inequality, we have used

$$\begin{aligned}
\frac{1}{2^{p-1}}|\nabla\varpi|^p &= \frac{1}{2^{p-1}}|\nabla\varpi-\Theta(\varpi)+\Theta(\varpi)|^p\\
&\le \frac{1}{2^{p-1}}\left[2^{p-1}\left(|\nabla\varpi-\Theta(\varpi)|^p+|\Theta(\varpi)|^p\right)\right]\\
&= |\nabla\varpi-\Theta(\varpi)|^p+|\Theta(\varpi)|^p,
\end{aligned}$$

and

$$\begin{aligned}
\frac{2}{p}\int_\Omega |\Theta(\varpi)|^p\,dx &\le \frac{2\beta^p}{p}\int_\Omega |\varpi|^p\,dx\\
&\le \frac{1}{p2^p}\int_\Omega |\nabla\varpi|^p\,dx.
\end{aligned}$$

Hence,

$$\int_\Omega |\nabla\varpi|^p\,dx \le p2^pn^2\,(1+n)^{\lambda(p-1)}\int_\Omega |\varpi|\,dx. \tag{4.4}$$

Thus, by the Poincaré inequality, we obtain

$$\|\varpi\|_p \le C', \tag{4.5}$$

moreover, in view of (4.4), one deduces that

$$\|\varpi\|_{W_0^{1,p}(\Omega)} \le C'', \tag{4.6}$$

where the positive constants C' and C'' are independent of v and ϖ. Then, the ball B in $L^p(\Omega)$ of radius C' is invariant for the map T. Furthermore, by virtue of (4.6) and Rellich-Kondrachov theorem, we deduce that $T\left(L^p(\Omega)\right)$ is relatively compact in $L^p(\Omega)$.

For the continuity of T, let consider a sequence $v_k \in L^p(\Omega)$ that strongly converges to $v \in L^p(\Omega)$ as $k\to\infty$. If we denotes by $\varpi_k = T(v_k)$, then by (4.6), ϖ_k is bounded in $W_0^{1,p}(\Omega)$ with respect to k and there exists $\varpi \in W_0^{1,p}(\Omega)$ to which ϖ_k, up to subsequence, converges weakly in $W_0^{1,p}(\Omega)$. Additionally, since $f_n h_n(v_k) \le n^2$, it follows from the Dominated Convergence Theorem that

$$f_n h_n(v_k) \longrightarrow f_n h_n(v) \quad \text{strongly in } L^p(\Omega).$$

This is sufficient to conclude, by the uniqueness of the weak solution, that $\varpi_k = T(v_k)$ converges strongly to $\varpi = T(v)$ in $L^p(\Omega)$ i.e. T is continuous.

Finally, by applying the Schauder's fixed point theorem, there exist a fixed point u_n of the map T that belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ which is solution of problem (4.3).

To demonstrate that u_n is non-negative, we take $-u_n^-$ as a test function in the weak formulation of (4.3), we find

$$\int_{\{u_n \le 0\}} \frac{\Psi(\nabla u_n - \Theta(u_n))}{(1+|T_n(u_n)|)^{\lambda(p-1)}} \cdot \nabla u_n \, dx + \int_\Omega \left|u_n^-\right|^{q+1} \, dx = -\int_\Omega f_n h_n(u_n) u_n^- \, dx,$$

since the right-hand side is non-positive, we proceed in the same way as before to get,

$$\frac{1}{(1+n)^{\lambda(p-1)}} \int_{\{u_n \le 0\}} |\nabla u_n|^p \, dx \le 0.$$

This implies $\|u_n^-\|_{W_0^{1,p}(\Omega)} = 0$, leading us to conclude that $u_n \ge 0$ almost everywhere in Ω. Which concludes the proof of the lemma. □

Step 2: A priori estimates and basic convergences results

In this step, we establish a priori estimates for the solutions u_n of the problems defined in (4.1). These estimates will help us to prove Theorem 3.2.

Lemma 4.2. *Suppose that the hypotheses of Theorem 3.2 are satisfied and let u_n be a weak solution of (4.1). Then, for every $k > 0$,*

$$\|T_k(u_n)\|_{W_0^{1,p}(\Omega)} \le C,$$

where C is a constant independent of n. Moreover, there exists a non-negative measurable function u such that

$$u_n \to u \quad a.e. \ in \ \Omega, \tag{4.7}$$

and

$$T_k(u_n) \rightharpoonup T_k(u) \quad weakly \ in \ W_0^{1,p}(\Omega) \ as \ n \to +\infty. \tag{4.8}$$

Proof. For $k > 0$, we use $T_k(u_n)$ as a test function in the weak formulation (4.2)

$$\int_\Omega \frac{\Psi(\nabla u_n - \Theta(u_n))}{(1+|T_n(u_n)|)^{\lambda(p-1)}} \cdot \nabla T_k(u_n) \, dx + \int_\Omega |u_n|^{q-1} u_n T_k(u_n) \, dx = \int_\Omega f_n h_n(u_n) T_k(u_n) \, dx.$$

From this, (3.2) as well as the observation that u_n and $T_k(u_n)$ have the same sign, we derive the inequality

$$\int_{\{u_n \le k\}} \frac{\Psi(\nabla u_n - \Theta(u_n))}{(1+|T_n(u_n)|)^{\lambda(p-1)}} \cdot \nabla u_n \, dx \le \int_\Omega f_n (T_k(u_n))^{1-\gamma} \, dx.$$

Then, since $T_n(u_n) \le k$ on the set $\{u_n \le k\}$, we get

$$\frac{1}{(1+k)^{\lambda(p-1)}} \int_{\{u_n \le k\}} \Psi(\nabla u_n - \Theta(u_n)) \cdot \nabla u_n \, dx \le k^{1-\gamma} \int_\Omega f \, dx.$$

Consequently, by using the same arguments used in Lemma 4.1, we can conclude

$$\int_\Omega |\nabla T_k(u_n)|^p\,dx \le C_1 k^{\lambda(p-1)+1-\gamma}\,\|f\|_{L^1(\Omega)}\,, \tag{4.9}$$

where C_1 is independent of k. Since $f \in L^1(\Omega)$, it follows that $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$. Thus, there exists some $v_k \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &T_k(u_n) \rightharpoonup v_k \quad \text{weakly in } W_0^{1,p}(\Omega),\\ &T_k(u_n) \to v_k \quad \text{strongly in } L^p(\Omega), \quad \text{and a.e. in } \Omega. \end{aligned} \tag{4.10}$$

Next, we will demonstrate that u_n converges in measure to some measurable function u. Let $k \ge 0$, from (4.9), the Hölder inequality and Poincaré inequality, we have

$$\begin{aligned} k \text{ meas}\left(\{|u_n| > k\}\right) &= \int_{\{|u_n|>k\}} |T_k(u_n)|\,dx\\ &\le C_2\left(\int_\Omega |T_k(u_n)|^p\,dx\right)^{\frac{1}{p}}\\ &\le C_3\left(\int_\Omega |\nabla T_k(u_n)|^p\,dx\right)^{\frac{1}{p}}\\ &\le C_4 k^{\frac{\lambda(p-1)+1-\gamma}{p}}. \end{aligned}$$

This leads us to conclude that, for every $k > 0$,

$$\text{meas}\left(\{|u_n| > k\}\right) \le \frac{C}{k^{\frac{(p-1)(1-\lambda)+\gamma}{p}}}, \tag{4.11}$$

since,

$$\frac{(p-1)(1-\lambda)+\gamma}{p} > 0 \iff \lambda < 1 + \frac{\gamma}{p-1}, \tag{4.12}$$

then,

$$\text{meas}\left(\{|u_n| > k\}\right) \longrightarrow 0 \text{ as } k \to +\infty. \tag{4.13}$$

Furthermore, for any $\delta > 0$, we have

$$\begin{aligned} \text{meas}\left(\{|u_n - u_m| > \delta\}\right) \le{}& \text{meas}\left(\{|u_n| > k\}\right) + \text{meas}\left(\{|u_m| > k\}\right)\\ &+ \text{meas}\left(\{|T_k(u_n) - T_k(u_m)| > \delta\}\right). \end{aligned} \tag{4.14}$$

Let $\varepsilon > 0$, by using (4.10)-(4.14), we can easy find some $k(\varepsilon) > 0$ such that

$$\text{meas}\left(\{|u_n - u_m| > \delta\}\right) \le \varepsilon \quad \text{for any } n, m \ge n_0(k(\varepsilon), \delta).$$

This shows that $(u_n)_n$ is a Cauchy sequence in measure, so, its converges almost everywhere, for a subsequence, to some measurable function u. Therefore, from (4.10), we obtain

$$\begin{aligned} &T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega),\\ &T_k(u_n) \to T_k(u) \quad \text{strongly in } L^p(\Omega) \quad \text{and a.e. in } \Omega. \end{aligned} \tag{4.15}$$

$\square$

Lemma 4.3. *Let u_n be a weak solution of (4.1). Then,*

$$|u_n|^{q-1} u_n \longrightarrow |u|^{q-1} u \quad \textit{strongly in } L^1(\Omega). \tag{4.16}$$

Proof. For $\delta > 0$, using $1 - V_\delta(u_n)$ as a test function in the weak formulation (4.2) and ignoring the first term which is positive (recall that $V'_\delta(u_n) \leq 0$ when $u_n \geq 0$). This yields

$$\int_{\{u_n > \delta\}} |u_n|^{q-1} u_n \, dx \leq \left(\sup_{s \in [\delta, \infty)} h(s) \right) \int_{\{u_n > \delta\}} f \; dx.$$

Consequently, the following holds for any measurable subset E of Ω,

$$\int_E |u_n|^q \; dx \leq \delta^q (measE) + \int_{E \cap \{u_n > \delta\}} |u_n|^q \; dx \leq \delta^q (measE) + C \int_{\{u_n > \delta\}} f \; dx. \tag{4.17}$$

The fact that $f \in L^1(\Omega)$ allows us to say that, for any given $\varepsilon > 0$, there exists a δ_ε such that

$$\int_{\{u_n > \delta_\varepsilon\}} f \; dx \leq \varepsilon.$$

Then, by (4.17),

$$\int_E |u_n|^q \; dx \longrightarrow 0 \qquad \text{as } measE \to 0.$$

Therefore, $|u_n|^{q-1} u_n$ is equi-integrable, accordingly, by virtue of Vitali's theorem, it converges to $|u|^{q-1} u$ in $L^1(\Omega)$. □

Step 3: Strong convergence of truncations

Before demonstrating the strong convergence of truncations, we need to establish the following lemma, which will be helpful in subsequent arguments.

Lemma 4.4. *For every non-negative $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have*

$$f h(u) \varphi \in L^1(\Omega), \tag{4.18}$$

Proof. Let $k > 0$ and φ be a non-negative function in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Taking $V_k(u_n)\varphi$ as test function in (4.2) and recalling that $V'_k(u_n) \leq 0$ when $u_n \geq 0$, we obtain

$$\begin{aligned} \int_\Omega f_n h_n(u_n) \varphi V_k(u_n) \, dx \leq & \int_\Omega \frac{\Psi(\nabla u_n - \Theta(u_n))}{(1 + |T_n(u_n)|)^{\lambda(p-1)}} \cdot \nabla \varphi V_k(u_n) \, dx \\ & + \int_\Omega |u_n|^q \varphi V_k(u_n) dx, \end{aligned} \tag{4.19}$$

In order to estimate the left-hand side of the earlier inequality, we use (3.1) and the Young inequality, we get

$$\begin{aligned}
\int_\Omega f_n h_n(u_n)\varphi V_k(u_n)\,dx &\le \int_\Omega \Psi(\nabla u_n-\Theta(u_n))\cdot\nabla\varphi V_k(u_n)\,dx+\int_\Omega |u_n|^q\,\varphi dx\\
&\le \int_\Omega |\nabla T_{2k}(u_n)-\Theta(T_{2k}(u_n))|^{p-1}\,|\nabla\varphi|\;dx+\frac{1}{q}\int_\Omega |u_n|^q\,dx\\
&+\frac{1}{q'}\int_\Omega \varphi^{q'}\,dx\\
&\le \frac{1}{p'}\int_\Omega |\nabla T_{2k}(u_n)-\Theta(T_{2k}(u_n))|^p\,dx+\frac{1}{p}\int_\Omega |\nabla\varphi|^p\;dx\\
&+\frac{1}{q}\int_\Omega |u_n|^q\,dx+\frac{1}{q'}\int_\Omega \varphi^{q'}\,dx\\
&\le C\,\|T_{2k}(u_n)\|_{W_0^{1,p}(\Omega)}+C'\,\|\varphi\|_{W_0^{1,p}(\Omega)}+\frac{1}{q}\,\|u_n\|_q^q+\frac{1}{q'}\,\|\varphi\|_{q'}^{q'}.
\end{aligned}\tag{4.20}$$

Applying Lemma 4.2, we conclude that

$$\int_\Omega f_n h_n(u_n)\chi_{\{u_n\le k\}}\varphi\,dx\le C \tag{4.21}$$

for some constant C independent of n.

Moreover, it is straightforward to show, due to the assumptions on h and f, that

$$\int_\Omega f_n h_n(u_n)\chi_{\{u_n> k\}}\varphi\,dx\le C. \tag{4.22}$$

Thus, from (4.23) and (4.22), we find

$$\int_\Omega f_n h_n(u_n)\varphi\,dx\le C. \tag{4.23}$$

Using Fatou's Lemma in (4.23) with respect to n, we derive

$$\int_\Omega f h(u)\varphi\,dx\le C. \tag{4.24}$$

□

Now, we have all the ingredients to demonstrate the strong convergence of the truncations.

Lemma 4.5. *For $k>0$, suppose that the hypotheses of Theorem 3.2 are satisfied and let u_n be a weak solution of (4.1). Then $T_k(u_n)\to T_k(u)$ strongly in $W_0^{1,p}(\Omega)$ as $n\to\infty$, where u is given by Lemma 4.2.*

Proof. For all $k>0$, without loss of generality, we can assume that $n>k$. We choose $T_k(u_n)-T_k(u)$ as the test function in (4.1), yielding

$$\begin{aligned}
&\int_\Omega \frac{\Psi(\nabla u_n-\Theta(u_n))}{(1+|T_n(u_n)|)^{\lambda(p-1)}}\cdot\nabla\left(T_k(u_n)-T_k(u)\right)\,dx+\int_\Omega |u_n|^{q-1}u_n\left(T_k(u_n)-T_k(u)\right)\,dx\\
&=\int_\Omega f_n h_n(u_n)\left(T_k(u_n)-T_k(u)\right)\,dx.
\end{aligned}$$

From this, we derive the equality

$$\begin{aligned}
&\int_\Omega \left(\frac{\Psi(\nabla T_k(u_n) - \Theta(T_k(u_n)))}{(1+|T_k(u_n)|)^{\lambda(p-1)}} - \frac{\Psi(\nabla T_k(u) - \Theta(T_k(u_n)))}{(1+|T_k(u_n)|)^{\lambda(p-1)}} \right) \cdot \nabla \left(T_k(u_n) - T_k(u)\right) dx \\
&= - \int_\Omega \frac{\Psi(\nabla T_k(u) - \Theta(T_k(u_n)))}{(1+|T_k(u_n)|)^{\lambda(p-1)}} \cdot \nabla \left(T_k(u_n) - T_k(u)\right) dx \\
&- \int_\Omega |u_n|^{q-1} u_n \left(T_k(u_n) - T_k(u)\right) \, dx + \int_\Omega f_n h_n(u_n) \left(T_k(u_n) - T_k(u)\right) dx \\
&=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned} \tag{4.25}$$

We proceed to analyze each term $(\mathcal{I}_i)_{i=1,2,3}$ in equation (4.25) individually to determine their limits as $n \to \infty$.

For the first term $\mathcal{I}_1$, by using (3.1), we observe that

$$\begin{aligned}
\left| \frac{\Psi(\nabla T_k(u) - \Theta(T_k(u_n)))}{(1+|T_k(u_n)|)^{\lambda(p-1)}} \right|^{p'} &\le |\nabla T_k(u) - \Theta(T_k(u_n))|^p \\
&\le 2^{p-1} \left(|\nabla T_k(u)|^p + |\Theta(T_k(u_n))|^p \right) \\
&\le C \left(|\nabla T_k(u)|^p + C_k \right),
\end{aligned}$$

where the constants C and C_k does not depends on n.

By applying Lemma 4.2, we recognize that

$$\frac{\Psi(\nabla T_k(u) - \Theta(T_k(u_n)))}{(1+|T_k(u_n)|)^{\lambda(p-1)}} \in L^{p'}(\Omega),$$

and

$$\frac{\Psi(\nabla T_k(u) - \Theta(T_k(u_n)))}{(1+|T_k(u_n)|)^{\lambda(p-1)}} \longrightarrow \frac{\Psi(\nabla T_k(u) - \Theta(T_k(u)))}{(1+|T_k(u)|)^{\lambda(p-1)}} \quad \text{a.e. in } \Omega,$$

this implies, by using the Dominated Convergence Theorem, that

$$\frac{\Psi(\nabla T_k(u) - \Theta(T_k(u_n)))}{(1+|T_k(u_n)|)^{\lambda(p-1)}} \longrightarrow \frac{\Psi(\nabla T_k(u) - \Theta(T_k(u)))}{(1+|T_k(u)|)^{\lambda(p-1)}} \quad \text{strongly in } L^{p'}(\Omega).$$

Therefore, by Lemma 4.2, we find

$$\lim_{n\to\infty} \mathcal{I}_1 = 0. \tag{4.26}$$

Turning to $\mathcal{I}_2$, we use Lemma 4.3 coupled with the $^*-$weak convergence of $T_k(u_n) - T_k(u)$ to zero in $L^\infty(\Omega)$, we get

$$\lim_{n\to\infty} \mathcal{I}_2 = 0. \tag{4.27}$$

Finally, for the third term, $\mathcal{I}_3$, let $\delta > 0$ be sufficiently small such that

$$\delta \notin \{\eta > 0 : \text{meas}\left(\{u = \eta\}\right) > 0\}.$$

Splitting $\mathcal{I}_3$ over $\{u_n \le \delta\}$ and $\{u_n > \delta\}$ and utilizing condition (3.2), we have

$$\mathcal{I}_3 \le 2c\delta^{1-\gamma} \int_{\{u_n \le \delta\}} f \, dx + \int_{\{u_n > \delta\}} f_n h_n(u_n) \left(T_k(u_n) - T_k(u)\right) \, dx. \tag{4.28}$$

Regarding the first term on the right-hand side of (4.28), if $h(0) < \infty$ (i.e., $\gamma = 0$), then $\delta^{1-\gamma} \int_{\{u_n \leq \delta\}} f$ converges to zero as $n \to \infty$ and $\delta \to 0^+$. If $h(0) = \infty$ (i.e., $0 < \gamma \leq 1$), then by Dominated Convergence Theorem and (4.18), we see that $fh(u) \in L^1_{loc}(\Omega)$, which implies $\{u = 0\} \subset \{f = 0\}$ except for a set of zero Lebesgue measure. Thus, in both cases,

$$\lim_{\delta \to 0^+} \lim_{n \to \infty} \left(\delta^{1-\gamma} \int_{\{u_n \leq \delta\}} f \right) = \lim_{\delta \to 0^+} \left(\delta^{1-\gamma} \int_{\{u \leq \delta\}} f \right) = 0. \tag{4.29}$$

Turning to the second term on the left-hand side of (4.28), since, in $[\delta, \infty)$,

$$h_n(u_n) \leq \sup_{s \in [\delta, \infty)} h(s),$$

then by the Dominated Convergence Theorem, we have

$$\lim_{n \to \infty} \int_{\{u_n > \delta\}} f_n h_n(u_n) \left(T_k(u_n) - T_k(u)\right) dx = 0. \tag{4.30}$$

Putting together (4.28), (4.29) and (4.30), we conclude that

$$\lim_{\delta \to 0^+} \lim_{n \to \infty} \mathcal{I}_3 = 0. \tag{4.31}$$

Bringing together (4.26), (4.27) and (4.31) in (4.25), we deduce

$$\lim_{n \to \infty} \int_\Omega \left(\frac{\Psi(\nabla T_k(u_n) - \Theta(T_k(u_n)))}{(1 + |T_k(u_n)|)^{\lambda(p-1)}} - \frac{\Psi(\nabla T_k(u) - \Theta(T_k(u_n)))}{(1 + |T_k(u_n)|)^{\lambda(p-1)}} \right) \cdot \nabla \left(T_k(u_n) - T_k(u)\right) dx = 0. \tag{4.32}$$

Thus, by Lemma 5 in [4] and the following strict monotonicity condition

$$\left(\Psi(\xi - \Theta(s)) - \Psi(\xi' - \Theta(s))\right) \cdot (\xi - \xi') > 0 \quad \forall \xi \neq \xi',$$

we conclude that

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{4.33}$$

In particular, there exists a subsequence still labelled with n such that

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega. \tag{4.34}$$

$\square$

Step 4: Proof of Theorem 3.2

First, in Lemma 4.2, we established that for all $k > 0$, $T_k(u) \in W_0^{1,p}(\Omega)$, where u is the limit of the approximate sequence u_n, and from Lemma (4.3), it follows that $|u|^{q-1} u \in L^1(\Omega)$.

Next, we now proceed to show assertion (3.6). Let $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be fixed and set $M = \|\varphi\|_{L^\infty(\Omega)} + k$. Taking $T_k(u_n - \varphi)^+$ as a test function in (4.1), we obtain

$$\begin{aligned} \int_\Omega f_n h_n(u_n) T_k(u_n - \varphi)^+ \, dx = & \int_\Omega \frac{\Psi(\nabla u_n - \Theta(u_n))}{(1 + |T_n(u_n)|)^{\lambda(p-1)}} \cdot \nabla T_k(u_n - \varphi)^+ \, dx \\ & + \int_\Omega |u_n|^{q-1} u_n T_k(u_n - \varphi)^+ \, dx, \end{aligned}$$

which implies that

$$\int_\Omega f_n h_n(u_n) T_k(u_n-\varphi)^+\,dx \le \int_\Omega \left| \frac{\Psi(\nabla T_M(u_n)-\Theta(T_M(u_n)))}{(1+|T_M(u_n)|)^{\lambda(p-1)}} \cdot \nabla\varphi \right| dx + \int_\Omega \left| \frac{\Psi(\nabla T_M(u_n)-\Theta(T_M(u_n)))}{(1+|T_M(u_n)|)^{\lambda(p-1)}} \cdot \nabla T_M(u_n) \right| dx + C. \tag{4.35}$$

For the first term on the right-hand side of (4.35), by employing (3.1) and Young's inequality, we find

$$\int_\Omega \left| \frac{\Psi(\nabla T_M(u_n)-\Theta(T_M(u_n)))}{(1+|T_M(u_n)|)^{\lambda(p-1)}} \cdot \nabla\varphi \right| dx \le C\left(C + \|T_M(u_n)\|_{W_0^{1,p}(\Omega)} + \|\varphi\|_{W_0^{1,p}(\Omega)}\right) \le C, \tag{4.36}$$

where C is a constant independent of n by Lemma 4.2.

Similarly, for the second term in (4.35), we find

$$\int_\Omega \left| \frac{\Psi(\nabla T_M(u_n)-\Theta(T_M(u_n)))}{(1+|T_M(u_n)|)^{\lambda(p-1)}} \cdot \nabla T_M(u_n) \right| dx \le C. \tag{4.37}$$

Combining (4.35), (4.36) and (4.37), we obtain

$$\int_\Omega f_n h_n(u_n) T_k(u_n-\varphi)^+\,dx \le C.$$

Applying Fatou's lemma to the above inequality yields that $fh(u)T_k(u-\varphi)^+ \in L^1(\Omega)$. A similar argument with $T_k(u_n-\varphi)^-$ shows that $fh(u)T_k(u-\varphi)^- \in L^1(\Omega)$, so $fh(u)T_k(u-\varphi) \in L^1(\Omega)$.

To establish assertion (3.7), we now take $T_k(u-\varphi)$ as a test function in (4.1), where $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, giving us

$$\int_\Omega \frac{\Psi(\nabla u_n - \Theta(u_n))}{(1+|T_n(u_n)|)^{\lambda(p-1)}} \cdot \nabla T_k(u_n-\varphi)\,dx + \int_\Omega |u_n|^{q-1} u_n T_k(u_n-\varphi)\,dx = \int_\Omega f_n h_n(u_n) T_k(u_n-\varphi)\,dx. \tag{4.38}$$

Our objective is to take the limit as $n\to\infty$ in this expression. For the first term on the left-hand side, note that $\nabla T_k(u_n-\varphi)$ is nonzero only on the set $\{|u_n-\varphi|<k\}$. Setting $M=\|\varphi\|_{L^\infty(\Omega)}+k$, Lemma 4.5 shows that

$$T_k(u_n)\to T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega) \text{ as } n\to\infty.$$

Hence, by (4.3),

$$\frac{\Psi(\nabla T_M(u_n)-\Theta(T_M(u_n)))}{(1+|T_M(u_n)|)^{\lambda(p-1)}} \rightharpoonup \frac{\Psi(\nabla T_M(u)-\Theta(T_M(u)))}{(1+|T_M(u)|)^{\lambda(p-1)}} \quad \text{weakly in } L^{p'}(\Omega) \text{ as } n\to\infty. \tag{4.39}$$

Thus,

$$\lim_{n\to\infty}\int_\Omega \frac{\Psi(\nabla u_n-\Theta(u_n))}{(1+|T_n(u_n)|)^{\lambda(p-1)}}\cdot\nabla T_k(u_n-\varphi)\,dx=\int_\Omega \frac{\Psi(\nabla u-\Theta(u))}{(1+|u|)^{\lambda(p-1)}}\cdot\nabla T_k(u-\varphi)\,dx. \tag{4.40}$$

As for the second term on the left-hand side of (4.38), we use the strong convergence (4.16) along with the weak* convergence of $T_k(u_n-\varphi)$ to $T_k(u-\varphi)$ in $L^\infty(\Omega)$, to get

$$\lim_{n\to\infty}\int_\Omega |u_n|^{q-1}\,u_nT_k(u_n-\varphi)\,dx=\int_\Omega |u|^{q-1}\,uT_k(u-\varphi)\,dx. \tag{4.41}$$

For the right-hand side of (4.38), if $h(0)<\infty$, the limit follows by the Dominated Convergence Theorem. If $h(0)=\infty$, we decompose the right-hand side of (4.38) as

$$\int_\Omega f_nh_n(u_n)T_k(u_n-\varphi)\,dx=\int_\Omega f_nh_n(u_n)T_k(u_n-\varphi)^+\,dx+\int_\Omega f_nh_n(u_n)T_k(u_n-\varphi)^-\,dx.$$

We now pass to the limit in the first term on the right-hand side of the previous inequality; the second term can be treated similarly.

Let ε be small enough such that $\varepsilon\notin\{\eta>0:\operatorname{meas}(\{u=\eta\})>0\}$, which is at most a countable set. By taking $(u_n-\varphi)^+V_\varepsilon(u_n)$ as a test function in (4.1) and disregarding the positive and negative terms, we obtain

$$\begin{aligned}\int_{\{u_n\le\varepsilon\}} f_nh_n(u_n)(u_n-\varphi)^+\,dx\le&\int_\Omega V_\varepsilon(u_n)\frac{\Psi(\nabla u_n-\Theta(u_n))}{(1+|T_n(u_n)|)^{\lambda(p-1)}}\cdot\nabla(u_n-\varphi)^+\,dx\\&+\int_\Omega V_\varepsilon(u_n)\,|u_n|^{q-1}\,u_n(u_n-\varphi)^+\,dx.\end{aligned} \tag{4.42}$$

Using (3.1) and Lemma 4.2, it is straightforward to show that $V_\varepsilon(u_n)\dfrac{\Psi(\nabla u_n-\Theta(u_n))}{(1+|T_n(u_n)|)^{\lambda(p-1)}}$ is bounded in $L^{p'}(\Omega)$. Hence, $V_\varepsilon(u_n)\dfrac{\Psi(\nabla u_n-\Theta(u_n))}{(1+|T_n(u_n)|)^{\lambda(p-1)}}$ converges weakly to $V_\varepsilon(u)\dfrac{\Psi(\nabla u-\Theta(u))}{(1+|u|)^{\lambda(p-1)}}$in $L^{p'}(\Omega)$.

Moreover, $T_{2\varepsilon}(u_n)$ converges strongly to $T_{2\varepsilon}(u)$ in $W_0^{1,p}(\Omega)$ as $n\to\infty$. Therefore,

$$\begin{aligned}\lim_{n\to\infty}\int_{\{u_n\le\varepsilon\}} f_nh_n(u_n)(u_n-\varphi)^+\,dx\le&\int_\Omega V_\varepsilon(u)\frac{\Psi(\nabla u-\Theta(u))}{(1+|u|)^{\lambda(p-1)}}\cdot\nabla(u-\varphi)^+\,dx\\&+\int_\Omega V_\varepsilon(u)\,|u|^{q-1}\,u(u-\varphi)^+\,dx:=C_{1,\varepsilon}+C_{2,\varepsilon},\end{aligned} \tag{4.43}$$

where

$$\lim_{\varepsilon\to0^+}C_{1,\varepsilon}=\int_{\{u=0\}}\frac{\Psi(\nabla u-\Theta(u))}{(1+|u|)^{\lambda(p-1)}}\cdot\nabla(u-\varphi)^+\,dx=0, \tag{4.44}$$

and

$$\lim_{\varepsilon\to0^+}C_{2,\varepsilon}=\int_{\{u=0\}}V_\varepsilon(u)\,|u|^{q-1}\,u(u-\varphi)^+\,dx=0, \tag{4.45}$$

since $\Theta(0) = 0$ and $\Psi(0) = 0$ for almost every $x \in \Omega$.

We decompose further as follows

$$\begin{aligned}\int_\Omega f_n h_n(u_n) T_k(u_n-\varphi)^+\,dx &= \int_{\{u_n\le\varepsilon\}} f_n h_n(u_n)(u_n-\varphi)^+\,dx\\ &\quad+\int_{\{u_n>\varepsilon\}} f_n h_n(u_n) T_k(u_n-\varphi)^+\,dx\end{aligned}\tag{4.46}$$

for $k > \varepsilon$. Regarding the second term on the right-hand side of the previous inequality, we find that

$$f_n h_n(u_n) T_k(u_n-\varphi)^+ \le k \sup_{s\in(\varepsilon,\infty)} h(s)\, f,$$

invoking the Dominated Convergence Theorem, we deduce

$$\lim_{n\to\infty}\int_{\{u_n>\varepsilon\}} f_n h_n(u_n) T_k(u_n-\varphi)^+\,dx = \int_{\{u>\varepsilon\}} f h(u) T_k(u-\varphi)^+\,dx.$$

Furthermore, we have already shown that $fh(u)T_k(u-\varphi)^+ \in L^1(\Omega)$. Thus, a second application of the Dominated Convergence Theorem yields

$$\lim_{\varepsilon\to0^+}\lim_{n\to\infty}\int_{\{u_n>\varepsilon\}} f_n h_n(u_n) T_k(u_n-\varphi)^+\,dx = \int_{\{u>\varepsilon\}} f h(u) T_k(u-\varphi)^+\,dx,\tag{4.47}$$

since it follows from $fh(u)T_k(u-\varphi) \in L^1(\Omega)$ that $\{u = 0\} \subset \{f = 0\}$.

Taking the limits as $n \to \infty$ and then $\varepsilon \to 0^+$ in (4.46), by using (4.43), (4.44), (4.45) and (4.47), we obtain

$$\lim_{n\to\infty}\int_\Omega f_n h_n(u_n) T_k(u_n-\varphi)^+\,dx = \int_\Omega f h(u) T_k(u-\varphi)^+\,dx.$$

Finally, by reasoning in the same manner as before, we conclude that

$$\lim_{n\to\infty}\int_\Omega f_n h_n(u_n) T_k(u_n-\varphi)^-\,dx = \int_\Omega f h(u) T_k(u-\varphi)^-\,dx,$$

which is sufficient to pass to the limit on the right-hand side of (4.38), thus proving assertion (3.7).

So, with this last step the proof of Theorem 3.2 is concluded.

Conclusion

In summary, our paper provides a complete answer to the question of the existence of entropy solutions for a class of nonlinear degenerate elliptic problems featuring a singular term on the right-hand side and a Dirichlet-type boundary condition. This class of problems involves a p-Laplacian type operator with a degenerate coercivity factor of the form $(1+|u|)^{-\lambda(p-1)}$ as well as a convection-like term $\Theta(u)$. The right-hand side includes a function $f \in L^1(\Omega)$ and a singular function $h(u)$, which may blow up as u approaches zero.

Acknowledgement. The authors would like to thank the anonymous reviewer for providing valuable suggestions that helped improve this paper.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability Statement. The manuscript has no associated data.

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